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TWO ITERATIVE SCHEMES FOR AN H-SYSTEM

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Abstract

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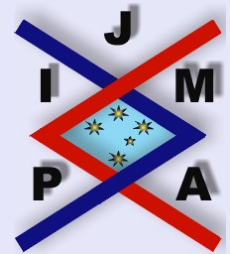
Two iterative schemes for the solution of an H-system with Dirichlet boundary data for a revolution surface are studied: a Newton imbedding type procedure, which yields the local quadratic convergence of the iteration and a more simple scheme based on the method of upper and lower solutions.

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1. Introduction

The prescribed mean curvature equation with Dirichlet condition for a vector function $X : \bar{\Omega} \rightarrow \mathbb{R}^3$ is given by the following nonlinear system of partial differential equations:

$$(1.1) \quad \begin{cases} \Delta X = 2H(X)X_u \wedge X_v & \text{in } \Omega \\ X = X_0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, \wedge denotes the exterior product in \mathbb{R}^3 , $H : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a given continuous function and X_0 is the boundary data.

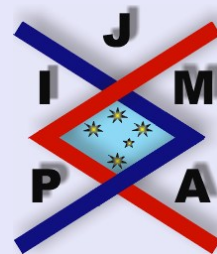
The parametric Plateau and Dirichlet problems have been studied by different authors (see [3, 4], [7] – [9]). Nonparametric and more general quasilinear equations are considered in [1, 2, 6].

We shall consider the particular case of a revolution surface

$$X(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

with $f, g \in C^2(\bar{I})$ such that $f > 0$ and $g' > 0$ over the interval $I \subset \mathbb{R}$. Without loss of generality we may assume that $I = (0, L)$, and problem (1.1) becomes

$$(1.2) \quad \begin{cases} f'' - f = -2H(f, g)fg' & \text{in } I \\ g'' = 2H(f, g)fg' & \text{in } I \\ f(0) = \alpha_0 & f(L) = \alpha_L \\ g(0) = \beta_0 & g(L) = \beta_L \end{cases}$$



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where $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a given continuous function, and $\alpha_0, \alpha_L > 0, \beta_0 < \beta_L$ are fixed real numbers.

It is easy to see that any solution of (1.2) verifies the equality

$$(f')^2 + (g')^2 = f^2 + c.$$

Hence, the *isothermal* condition

$$|X_u| - |X_v| = X_u X_v = 0$$

holds if and only if $c = 0$. In this case, H is the mean curvature of the surface parameterized by X (see [8]).

We shall study problem (1.2) for a surface with connected boundary, namely

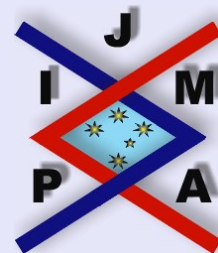
$$(1.3) \quad \begin{cases} f'' - f = -2H(f, g)fg' & \text{in } I \\ g'' = 2H(f, g)ff' & \text{in } I \\ f(L) = \alpha_L, & g(L) = \beta_L \\ f(0) = g'(0) = 0. \end{cases}$$

In particular, if H depends only on the radius f , from the equality

$$g'' = 2H(f)ff', \quad g'(0) = 0, \quad g(L) = \beta_L$$

problem (1.3) easily reduces to a single equation; indeed, if $\tilde{H}(t) = \int_0^t sH(s)ds$, the following integral formula holds for g :

$$g(t) = \beta_L - 2 \int_t^L \tilde{H}(f(s))ds.$$



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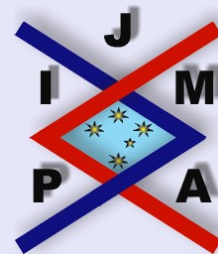
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Thus, problem (1.3) is equivalent to the equation

$$(1.4) \quad \begin{cases} f'' - f = -4H(f)f\tilde{H}(f) = -2\left(\tilde{H}^2\right)'(f) & \text{in } I \\ f = \alpha & \text{on } \partial I \end{cases}$$

with $\alpha(t) := \frac{\alpha_L}{L}t$. We remark that if $\alpha_L = 0$ then $g'(L) = 2\tilde{H}(0) = 0$, which corresponds to a surface without boundary.

The paper is organized as follows. In Section 2, a Newton imbedding type procedure for problem (1.4) is considered, which yields the local quadratic convergence of the iteration. In Section 3 we construct a more simple convergent scheme based on the existence of an ordered pair of a lower and an upper solution.



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2. A Newton Imbedding Type Procedure

Throughout this section we shall assume the following condition:

$$(2.1) \quad \left(\tilde{H}^2\right)''(x) \leq \frac{1 + \frac{\pi}{L} \left(\frac{\pi}{L} - \delta_0\right)}{2} \quad \text{for } -k \leq x \leq k + \alpha_L,$$

where $\delta_0 < \frac{\pi}{L}$ and k satisfies:

$$k\delta_0 > L^{1/2} \left\| 2 \left(\tilde{H}^2\right)'(\alpha) - \alpha \right\|_{L^2}.$$

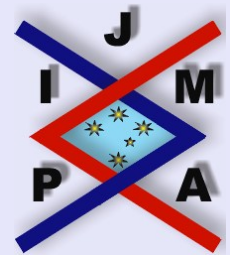
Remark 1. A straightforward application of Leray-Schauder degree theory proves that if condition (2.1) holds then there exists at least one solution of (1.4), which is unique in the set

$$\mathcal{K} = \{u \in H^2(I) : -k \leq u \leq k + \alpha_L\}.$$

Remark 2. As $\left(\tilde{H}^2\right)''(0) = 0$ and $\left\| 2 \left(\tilde{H}^2\right)'(\alpha) - \alpha \right\|_{L^2} \rightarrow 0$ for $\alpha_L \rightarrow 0$, we deduce that for any H there exists a positive number α^* such that (2.1) holds when $\alpha_L < \alpha^*$.

In order to solve equation (1.4) in an iterative manner, we shall embed it in a family of problems

$$(1.4_\lambda) \quad \begin{cases} f'' + \lambda \left[2 \left(\tilde{H}^2\right)'(f) - f \right] = 0 \\ f(0) = 0, \quad f(L) = \alpha_L. \end{cases}$$



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A simple computation shows that if S_λ is the semilinear operator given by

$$S_\lambda(f) = f'' + \lambda \left[2 \left(\tilde{H}^2 \right)' (f) - f \right],$$

then the following estimate holds for any $f, g \in \mathcal{K}$ such that $f = g$ on ∂I :

$$(2.2) \quad \|f' - g'\|_{L^2} \leq \frac{1}{\delta_0} \|S_\lambda(f) - S_\lambda(g)\|_{L^2}.$$

Hence, if f^λ is the (unique) solution of (1.4 $_\lambda$) in \mathcal{K} , we have that

$$\|(f^\lambda - \alpha)'\|_{L^2} \leq \frac{1}{\delta_0} \|S_\lambda(\alpha)\|_{L^2} \leq \frac{1}{\delta_0} \left\| 2 \left(\tilde{H}^2 \right)' (\alpha) - \alpha \right\|_{L^2}.$$

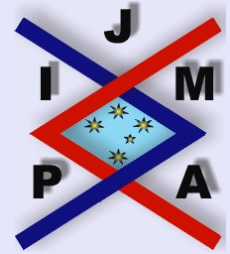
Thus, setting $k_0 = \frac{L^{1/2}}{\delta_0} \|2 \left(\tilde{H}^2 \right)' (\alpha) - \alpha\|_{L^2}$ we obtain:

$$-k_0 \leq f^\lambda \leq k_0 + \alpha_L.$$

We first present a sketch of the method: given $\lambda < 1$, and assuming that $f^\lambda \in \mathcal{K}$ is known, we shall prove the existence of a positive ε and a recursive sequence $\{f_n\}$ which converges quadratically to the unique solution of (1.4 $_{\lambda+\varepsilon}$) in \mathcal{K} . As ε can be chosen independently of f^λ and λ , starting at $f^0 = \alpha$, we deduce the existence of a sequence

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_N = 1,$$

where $f^{\lambda_k} \in \mathcal{K}$ is obtained iteratively from $f^{\lambda_{k-1}}$.



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Let $\lambda < 1$, and $f^\lambda \in \mathcal{K}$ be a solution of (1.4 $_\lambda$). Define the constants:

$$R = \frac{k - k_0}{L^{1/2}},$$

$$k_1 = \left\| f^\lambda - 2 \left(\tilde{H}^2 \right)' (f^\lambda) \right\|_2,$$

$$k_2 = \sup_{-k \leq \xi \leq k + \alpha_L} \left| \left(\tilde{H}^2 \right)''' (\xi) \right|$$

and define a sequence $\{f_n\}$ in the following way:

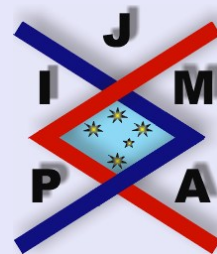
- $f_0 := f^\lambda$.
- f_{n+1} the unique element of $H^2(I)$ that solves the linear problem

$$\begin{cases} f_{n+1}'' = (\lambda + \varepsilon) \left[\left(1 - 2 \left(\tilde{H}^2 \right)'' (f_n) \right) (f_{n+1} - f_n) + f_n - 2 \left(\tilde{H}^2 \right)' (f_n) \right] \\ f_{n+1}(0) = 0, \quad f_{n+1}(L) = \alpha_L. \end{cases}$$

We shall prove that if ε is small enough, then $\{f_n\}$ is well defined. More precisely:

Theorem 2.1. *Assume that (2.1) holds, and that H is twice continuously differentiable on $[-k, k + \alpha_L]$. Then $\{f_n\}$ is well defined and converges quadratically for the H^1 -norm to the unique solution of (1.4 $_{\lambda+\varepsilon}$) in \mathcal{K} for any $\varepsilon \leq 1 - \lambda$ satisfying:*

$$\varepsilon \left(1 + \frac{RL^{3/2}k_2}{\pi\delta_0} \right) < \frac{\delta_0 R}{k_1}.$$



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Proof. As $f_0 \in \mathcal{K}$, f_1 is well defined, and by an estimate analogous to (2.2) we obtain:

$$\begin{aligned} \|f'_1 - f'_0\|_{L^2} &\leq \frac{1}{\delta_0} \left\| (f_1 - f_0)'' - (\lambda + \varepsilon) \left(1 - 2 \left(\tilde{H}^2 \right)'' (f_0) \right) (f_1 - f_0) \right\|_{L^2} \\ &= \frac{\varepsilon}{\delta_0} \left\| f_0 - 2 \left(\tilde{H}^2 \right)' (f_0) \right\|_{L^2} \\ &= \frac{\varepsilon}{\delta_0} k_1 \leq R. \end{aligned}$$

We shall assume as inductive hypothesis that f_k is well defined for $k = 1, \dots, n$, and that $\|f'_k - f'_0\|_{L^2} \leq R$. Thus, $f_n \in \mathcal{K}$ and f_{n+1} is well defined. Moreover, for $k = 1, \dots, n$ we have that

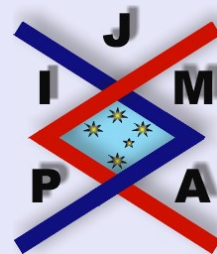
$$\begin{aligned} (f_{k+1} - f_k)'' - (\lambda + \varepsilon) \left(1 - 2 \left(\tilde{H}^2 \right)'' (f_k) \right) (f_{k+1} - f_k) \\ = -(\lambda + \varepsilon) \left(\tilde{H}^2 \right)''' (\xi_k) (f_k - f_{k-1})^2 \end{aligned}$$

for some mean value $\xi_k(x)$, and hence

$$\|(f_{k+1} - f_k)'\|_{L^2} \leq \frac{k_2}{\delta_0} \|(f_k - f_{k-1})^2\|_{L^2} \leq \frac{k_2 L^{3/2}}{\delta_0 \pi} \|(f_k - f_{k-1})'\|_{L^2}^2.$$

By induction,

$$\begin{aligned} \|(f_{k+1} - f_k)'\|_{L^2} &\leq \left(\frac{k_2 L^{3/2}}{\delta_0 \pi} \|(f_1 - f_0)'\|_{L^2} \right)^{2^k - 1} \|(f_1 - f_0)'\|_{L^2} \\ &\leq A^{2^k - 1} \|(f_1 - f_0)'\|_{L^2}, \end{aligned}$$



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where

$$A = \frac{\varepsilon k_1 k_2 L^{3/2}}{\delta_0^2 \pi} < 1.$$

Hence,

$$\begin{aligned} \|(f_{n+1} - f_0)'\|_{L^2} &\leq \sum_{k=0}^n \|(f_{k+1} - f_k)'\|_{L^2} \\ &\leq \frac{1}{1-A} \|(f_1 - f_0)'\|_{L^2} \\ &\leq \frac{\varepsilon k_1}{\delta_0(1-A)}. \end{aligned}$$

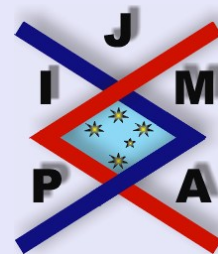
By hypothesis, we conclude that $\|(f_{n+1} - f_0)'\|_{L^2} \leq R$. Thus, f_n is well defined for every n , and the inequality

$$\|(f_{n+1} - f_n)'\|_{L^2} \leq A^{2^n - 1} \|(f_1 - f_0)'\|_{L^2}$$

holds, proving that $\{f_n\}$ is a Cauchy sequence in $H^1(I)$. Furthermore, if $f = \lim_{n \rightarrow \infty} f_n$, it is immediate that $f_n \rightarrow f$ in $H^2(I)$, and $f \in \mathcal{K}$ solves $(1.4)_{\lambda+\varepsilon}$. \square

Remark 3. A uniform choice of ε can be obtained if we set

$$k_1 = L^{1/2} \sup_{-k_0 \leq x \leq k_0 + \alpha_L} \left| x - 2 \left(\tilde{H}^2 \right)'(x) \right|.$$



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3. Upper and Lower Solutions for Problem (1.4)

In this section we define a convergent sequence based on the existence of an upper solution of the problem: namely, a nonnegative function β such that

$$(3.1) \quad \beta'' - \beta \leq -2 \left(\tilde{H}^2 \right)' (\beta), \quad \beta(L) \geq \alpha_L.$$

We remark that it suffices to consider this assumption, since 0 is a lower solution of (1.4).

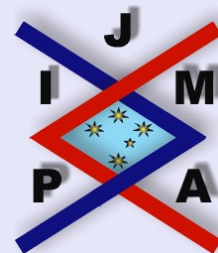
Theorem 3.1. *Assume that $\beta \geq 0$ satisfies (3.1) and that H is continuously differentiable for $0 \leq x \leq \|\beta\|_\infty$. Set*

$$C = 1 - 2 \min_{0 \leq x \leq \|\beta\|_\infty} \left(\tilde{H}^2 \right)'' (x)$$

and define the sequences $\{f_n^\pm\}$ given by:

- $f_0^- \equiv 0 \quad f_0^+ = \beta$
- $\{f_{n+1}^\pm\}$ the unique solution of the linear problem

$$\begin{cases} (f_{n+1}^\pm)'' - C f_{n+1}^\pm = (1 - C) f_n^\pm - 2 \left(\tilde{H}^2 \right)' (f_n^\pm) \\ f_{n+1}^\pm = \alpha \quad \text{on} \quad \partial I. \end{cases}$$



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Then $\{f_n^-\}$ (respectively $\{f_n^+\}$) is nondecreasing (nonincreasing), and converges pointwise to a solution of (1.2). Moreover, the respective limits f^\pm satisfy: $0 \leq f^- \leq f^+ \leq \beta$.

Proof. Let us first note that $C \geq 0$ (in fact, $C \geq 1$), which implies that both sequences are well defined. Furthermore, from the choice of C , it is immediate that the function $\psi(x) = (1 - C)x - 2 \left(\tilde{H}^2 \right)'(x)$ is nonincreasing for $0 \leq x \leq \|\beta\|_\infty$. By definition,

$$(f_1^+)'' - C f_1^+ = (1 - C)\beta - 2 \left(\tilde{H}^2 \right)'(\beta) \geq \beta'' - C\beta$$

and using the maximum principle it follows that $f_1^+ \leq \beta$. On the other hand,

$$(f_1^+)'' - C f_1^+ = \psi(\beta) \leq \psi(0) = 0$$

and as $f_1^+ \geq 0$ on ∂I we deduce that $f_1^+ \geq 0$ over I . Assume as inductive hypothesis that

$$0 \leq f_n^+ \leq f_{n-1}^+.$$

Then

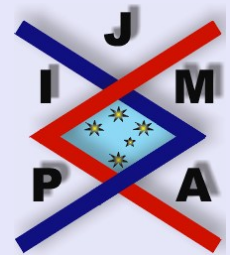
$$(f_{n+1}^+)'' - C f_{n+1}^+ = \psi(f_n^+) \geq \psi(f_{n-1}^+) = (f_n^+)'' - C f_n^+$$

and

$$(f_{n+1}^+)'' - C f_{n+1}^+ = \psi(f_n^+) \leq \psi(0) = 0$$

which implies that $0 \leq f_{n+1}^+ \leq f_n^+$. Thus, $\{f_n^+\}$ is nonincreasing and converges pointwise to a function $f^+ \geq 0$. By standard a priori estimates, we have that

$$\begin{aligned} \|f_{n+1}^+ - \alpha\|_{H^2} &\leq c \|(f_{n+1}^+ - \alpha)'' - C(f_{n+1}^+ - \alpha)\|_{L^2} \\ &= c \|\psi(f_n^+) - \alpha\|_{L^2} \leq M \end{aligned}$$



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for some constant M . By compactness, $\{f_n^+\}$ admits a convergent subsequence in $C^1(\bar{I})$, proving that $f^+ \in C^1(\bar{I})$. Furthermore,

$$(f_{n+1}^+)'(x) - (f_{n+1}^+)'(0) = \int_0^x C f_{n+1}^+ + \psi(f_n^+),$$

and by dominated convergence we conclude that

$$(f^+)'(x) - (f^+)'(0) = \int_0^x C f^+ + \psi(f^+) = \int_0^x f^+ - 2 \left(\tilde{H}^2 \right)'(f^+).$$

Thus, the result follows. The proof is analogous for $\{f_n^-\}$. □

As a simple consequence we have:

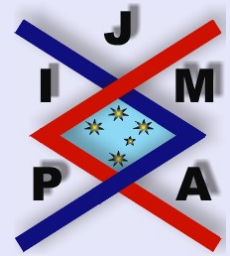
Corollary 3.2. *Assume there exists a number $k \geq \alpha_L$ such that*

$$\tilde{H}(k)H(k) \leq \frac{1}{4}.$$

Then $\beta \equiv k$ is an upper solution, and the schemes defined in the previous theorem converge.

Example 3.1. *For $H(x) = rx^n$, we have that $\tilde{H}(x) = \frac{r}{n+2}x^{n+2}$, and the conditions of the previous corollary hold for*

$$\alpha_L \leq k \leq \left(\frac{n+2}{4r^2} \right)^{\frac{1}{2n+2}}.$$



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However, it is possible to find a sharper bound for α_L , if we consider the parabola

$$\beta(x) = \alpha_L \left[1 - \left(\frac{x-L}{L} \right)^2 \right].$$

Indeed, in this case we have that β is an upper solution if and only if

$$-\frac{2\alpha_L}{L^2} - \beta \leq -\frac{4r^2}{n+2} \beta^{2n+3}$$

or equivalently

$$\phi(\beta) \leq \frac{2\alpha_L}{L^2}$$

for $\phi(x) = x \left(\frac{4r^2}{n+2} x^{2n+2} - 1 \right)$. Note that for $0 \leq x \leq \alpha_L$ we have:

$$\phi(x) \leq \max\{0, \phi(\alpha_L)\}.$$

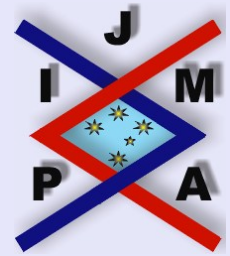
Thus, it suffices to assume that

$$0 < \alpha_L^{2n+2} \leq \frac{n+2}{4r^2} \left(1 + \frac{2}{L^2} \right).$$

Remark 4. In the previous example, equation (1.4) is superlinear, namely:

$$f'' + \frac{4r^2}{n+2} f^{2n+3} - f = 0, \quad f(0) = 0, \quad f(L) = \alpha_L.$$

It can be proved (see e.g. [5]) that this problem admits infinitely many solutions. More precisely, there exists $k_0 \in \mathbb{N}$ such that for any $j > k_0$ the problem has at least two solutions crossing the line $\alpha(t) = \frac{\alpha_L}{L}t$ exactly j times in $(0, L)$.



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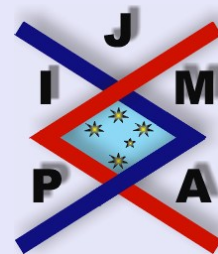
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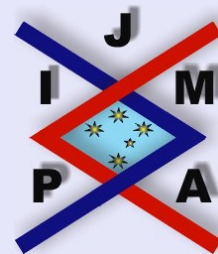
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