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THE DUAL SPACES OF THE SETS OF DIFFERENCE SEQUENCES OF ORDER m

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Abstract

The idea of difference sequence spaces was introduced by Kızmaz [5] and the concept was generalized by Et and Çolak [3]. Let $p = (p_k)$ be a bounded sequence of positive real numbers and $v = (v_k)$ be any fixed sequence of non-zero complex numbers. If $x = (x_k)$ is any sequence of complex numbers we write $\Delta_v^m x$ for the sequence of the m -th order differences of x and $\Delta_v^m(X) = \{x = (x_k) : \Delta_v^m x \in X\}$ for any set X of sequences. In this paper we determine the α -, β - and γ - duals of the sets $\Delta_v^m(X)$ which are defined by Et et al. [2] for $X = \ell_\infty(p)$, $c(p)$ and $c_0(p)$. This study generalizes results of Malkowsky [9] in special cases.

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Key words: Difference sequences, $\alpha-$, $\beta-$ and $\gamma-$duals.

Contents

1. Introduction, Notations and Known Results

Throughout this paper ω denotes the space of all scalar sequences and any subspace of ω is called a sequence space. Let ℓ_∞, c and c_0 be the linear space of bounded, convergent and null sequences with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k| ,$$

where $k \in \mathbb{N} = \{1, 2, 3, \dots\}$, the set of positive integers. Furthermore, let $p = (p_k)$ be bounded sequences of positive real numbers and

$$\ell_\infty(p) = \left\{ x \in \omega : \sup_k |x_k|^{p_k} < \infty \right\},$$

$$c(p) = \left\{ x \in \omega : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \, , \text{ for some } l \in \mathbb{C} \right\},$$

$$c_0(p) = \left\{ x \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0 \right\}$$

(for details see [6], [7], [11]).

Let x and y be complex sequences, and E and F be subsets of ω. We write

$$M(E, F) = \bigcap_{x\in E} x^{-1} * F = \{ a \in \omega : ax \in F \text{ for all } x \in E \} \qquad [12].$$

In particular, the sets

$$E^\alpha = M(E, l_1), \qquad E^\beta = M(E, cs) \quad \text{and} \quad E^\gamma = M(E, bs)$$

are called the $\alpha-$, $\beta-$ and $\gamma-$ duals of E, where l_1, cs and bs are the sets of all convergent, absolutely convergent and bounded series, respectively. If $E \subset F$, then $F^\eta \subset E^\eta$ for $\eta = \alpha, \beta, \gamma$. It is clear that $E^\alpha \subset (E^\alpha)^\alpha = E^{\alpha\alpha}$. If $E = E^{\alpha\alpha}$, then E is an α -space. In particular, an α -space is called a Köthe space or a perfect sequence space.

Throughout this paper X will be used to denote any one of the sequence spaces ℓ_∞, c and c_0.

Kızmaz [5] introduced the notion of difference sequence spaces as follows:

$$X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}.$$

Later on the notion was generalized by Et and Çolak in [3], namely,

$$X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\}.$$

Subsequently difference sequence spaces have been studied by Malkowsky and Parashar [8], Mursaleen [10], Çolak [1] and many others.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Et and Esi [4] generalized the above sequence spaces to the following ones

$$\Delta_v^m(X) = \{x = (x_k) : (\Delta_v^m x_k) \in X\},$$

where $\Delta_v^0 x = (v_k x_k)$, $\Delta_v^m x = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ such that $\Delta_v^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$. Recently Et et al. [2] generalized the sequence spaces $\Delta_v^m(X)$ to the sequence spaces

$$\Delta_v^m(X(p)) = \{x = (x_k) : (\Delta_v^m x_k) \in X(p)\}$$

and showed that these spaces are complete paranormed spaces paranormed by

$$g(x) = \sum_{i=1}^m |x_i v_i| + \sup_k \; |\Delta_v^m x_k|^{p_k/M},$$

where $H = \sup_k p_k$ and $M = \max(1, H)$.

Let us define the operator $D : \Delta_v^m X(p) \to \Delta_v^m X(p)$ by

$$Dx = (0, 0, \dots, x_{m+1}, x_{m+2}, \dots),$$

where $x = (x_1, x_2, x_3, \dots)$. It is trivial that D is a bounded linear operator on $\Delta_v^m X(p)$. Furthermore the set

$$\begin{aligned} D[\Delta_v^m X(p)] &= D\Delta_v^m X(p) \\ &= \{x = (x_k) : x \in \Delta_v^m X(p), x_1 = x_2 = \dots = x_m = 0\} \end{aligned}$$

is a subspace of $\Delta_v^m X(p)$. $D\Delta_v^m X(p)$ and $X(p)$ are equivalent as topological spaces, since

$$\Delta_v^m : D\Delta_v^m X(p) \to X(p) \tag{1.1}$$

defined by $\Delta_v^m x = y = (\Delta_v^m x_k)$ is a linear homeomorphism. Let $[X(p)]'$ and $[D\Delta_v^m X(p)]'$ denote the continuous duals of $X(p)$ and $D\Delta_v^m X(p)$, respectively. It can be shown that

$$T : [D\Delta_v^m X(p)]' \to [X(p)]', \qquad f_\Delta \to f_\Delta \circ (\Delta_v^m)^{-1} = f,$$

is a linear isometry. So $[D\Delta_v^m X(p)]'$ is equivalent to $[X(p)]'$.

Lemma 1.1 ([5]). *Let (t_n) be a sequence of positive numbers increasing monotonically to infinity, then*

i) If $\sup_n \left|\sum_{i=1}^n t_i a_i\right| < \infty$, then $\sup_n \left|t_n \sum_{k=n+1}^\infty a_k\right| < \infty$,

ii) If $\sum_k t_k a_k$ is convergent, then $\lim_{n\to\infty} t_n \sum_{k=n+1}^\infty a_k = 0$.

2. Main Results

In this section we determine the $\alpha-$, $\beta-$ and $\gamma-$ duals of $\Delta_v^m X(p)$.

Theorem 2.1. *For every strictly positive sequence $p = (p_k)$, we have*

(i) $[\Delta_v^m l_\infty(p)]^\alpha = D_1^\alpha(p)$,

(ii) $[\Delta_v^m l_\infty(p)]^{\alpha\alpha} = D_1^{\alpha\alpha}(p)$

where

$$D_1^\alpha(p) = \bigcap_{N=2}^{\infty} \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} < \infty \right\}$$

and

$$D_1^{\alpha\alpha}(p) = \bigcup_{N=2}^{\infty} \left\{ a = (a_k) : \sup_{k \geq m+1} |a_k| \, |v_k| \left[\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right]^{-1} < \infty \right\}.$$

Proof.

(i) Let $a \in D_1^\alpha(p)$ and $x \in \Delta_v^m l_\infty(p)$. We choose $N > \max(1, \sup_n |\Delta_v^m a_n|^{p_n})$. Since

$$\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} > \sum_{j=1}^{m} \binom{k-j-1}{m-j} N^{1/p_j} \geq 1$$

for arbitrary $N > 1$ $(k = 2m, 2m+1, \dots)$ and $|\Delta_v^{m-j} x_j| \le M$ $(1 \le j \le m)$ for some constant M, $a \in D_1^{\alpha}(p)$ implies

$$\sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \sum_{j=1}^{m} \binom{k-j-1}{m-j} \left|\Delta_v^{m-j} x_j\right| < \infty.$$

Then

$$\begin{aligned}\sum_{k=1}^{\infty} |a_k x_k| &= \sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \left(\left| \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} \Delta_v^m x_j \right.\right. \\ &\qquad \left.\left. + \sum_{j=1}^{m} (-1)^{m-j} \binom{k-j-1}{m-j} \Delta_v^{m-j} x_j \right| \right) \\ &\le \sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \\ &\qquad + \sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \sum_{j=1}^{m} \binom{k-j-1}{m-j} \left|\Delta_v^{m-j} x_j\right| \\ &< \infty.\end{aligned}$$

Conversely let $a \notin D_1^{\alpha}(p)$. Then we have

$$\sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} = \infty$$

for some integer $N > 1$. We define the sequence x by

$$x_k = v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \qquad (k = m+1, m+2, \dots).$$

Then it is easy to see that $x \in \Delta_v^m l_\infty(p)$ and $\sum_k |a_k x_k| = \infty$. Hence $a \notin [\Delta_v^m l_\infty(p)]^\alpha$. This completes the proof of (i).

(ii) Let $a \in D_1^{\alpha\alpha}(p)$ and $x \in [\Delta_v^m l_\infty(p)]^\alpha = D_1^\alpha(p)$, by part (i). Then for some $N > 1$, we have

$$\begin{aligned} \sum_{k=m+1}^{\infty} |a_k x_k| &= \sum_{k=m+1}^{\infty} |a_k| \, |v_k| \left[\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right]^{-1} \\ &\qquad \times |x_k| \, |v_k|^{-1} \left[\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right] \\ &\leq \sup_{k \geq m+1} \left(|a_k| \, |v_k| \left[\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right]^{-1} \right) \\ &\qquad \times \sum_{k=m+1}^{\infty} |x_k| \, |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \\ &< \infty. \end{aligned}$$

Conversely let $a \notin D_1^{\alpha\alpha}(p)$. Then for all integers $N > 1$, we have

$$\sup_{k \geq m+1} |a_k| \, |v_k| \left[\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right]^{-1} = \infty.$$

We recall that

$$\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} y_j = 0 \qquad (k < m+1)$$

for arbitrary y_j.

Hence there is a strictly increasing sequence $(k(s))$ of integers $k(s) \geq m+1$ such that

$$|a_{k(s)}| \, |v_{k(s)}| \left[\sum_{j=1}^{k(s)-m} \binom{k(s)-j-1}{m-1} s^{1/p_j} \right]^{-1} > s^{m+1}$$
$$(s = m+1, m+2, \dots).$$

We define the sequence x by

$$x_k = \begin{cases} |a_{k(s)}|^{-1}, & (k = k(s)) \\ 0, & (k \neq k(s)) \;\; (k = m+1, m+2, \dots) \end{cases}$$

Then for all integers $N > m+1$, we have

$$\sum_{k=1}^{\infty} |x_k| \, |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} < \sum_{s=m+1}^{\infty} s^{-(m+1)} < \infty.$$

Hence $x \in [\Delta_v^m l_\infty(p)]^\alpha$ and $\sum_{k=1}^\infty |a_k x_k| = \sum_{N=1}^\infty 1 = \infty$. Hence $a \notin [\Delta_v^m l_\infty(p)]^{\alpha\alpha}$. The proof is completed.

□

Theorem 2.2. *For every strictly positive sequence $p = (p_k)$, we have*

(i) $[\Delta_v^m c_0(p)]^\alpha = M_0^\alpha(p)$,

(ii) $[\Delta_v^m c_0(p)]^{\alpha\alpha} = M_0^{\alpha\alpha}(p)$

where

$$M_0^\alpha(p) = \bigcup_{N=2}^\infty \left\{ a \in \omega : \sum_{k=1}^\infty |a_k| \, |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} < \infty \right\}$$

and

$$M_0^{\alpha\alpha}(p) = \bigcap_{N=2}^\infty \left\{ a \in \omega : \sup_{k \geq m+1} |a_k| \, |v_k| \left[\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} \right]^{-1} < \infty \right\}.$$

Proof.

(i) Let $a \in M_0^\alpha(p)$ and $x \in \Delta_v^m c_0(p)$. Then there is an integer k_0 such that $\sup_{k > k_0} |\Delta_v^m x_k|^{p_k} \leq N^{-1}$, where N is the number in $M_0^\alpha(p)$. We put

$$M = \max_{1 \leq k \leq k_0} |\Delta_v^m x_k|^{p_k}, \qquad n = \min_{1 \leq k \leq k_0} p_k, \quad L = (M+1)N$$

and define the sequence y by $y_k = x_k \cdot L^{-1/n}$ $(k = 1, 2, \dots)$. Then it is easy to see that $\sup_k |\Delta_v^m y_k|^{p_k} \le N^{-1}$.

Since

$$\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} > \sum_{j=1}^{m} \binom{k-j-1}{m-j} N^{-1/p_j}$$

for arbitrary $N > 1$ $(k = 2m, 2m+1, \dots)$, $a \in M_0^\alpha(p)$ implies

$$\sum_{k=1}^{\infty} |a_k|\ |v_k|\ \sum_{j=1}^{m} \binom{k-j-1}{m-j} |\Delta_v^{m-j} y_j| < \infty.$$

Then

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_k x_k| &= L^{1/n} \sum_{k=1}^{\infty} |a_k y_k| \\
&\le L^{1/n} \sum_{k=1}^{\infty} |a_k|\ |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} \\
&\qquad + L^{1/n} \sum_{k=1}^{\infty} |a_k|\ |v_k|^{-1} \sum_{j=1}^{m} \binom{k-j-1}{m-j} |\Delta_v^{m-j} y_j| \\
&< \infty.
\end{aligned}$$

So we have $a \in [\Delta_v^m c_0(p)]^\alpha$. Therefore $M_0^\alpha(p) \subset [\Delta_v^m c_0(p)]^\alpha$.

Conversely, let $a \notin M_0^\alpha(p)$. Then we can determine a strictly increasing sequence $(k(s))$ of integers such that $k(1) = 1$ and

$$M(s) = \sum_{k=k(s)}^{k(s+1)-1} |a_k|\,|v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} (s+1)^{-1/p_j} > 1 \quad (s = 1, 2, \dots).$$

We define the sequence x by

$$x_k = v_k^{-1} \left(\sum_{l=1}^{s-1} \sum_{j=k(l)}^{k(l+1)-1} \binom{k-j-1}{m-1} (l+1)^{-1/p_j} + \sum_{j=k(s)}^{k-m} \binom{k-j-1}{m-1} (s+1)^{-1/p_j} \right)$$
$$(k(s) \le k \le k(s+1) - 1; s = 1, 2, \dots).$$

Then it is easy to see that

$$|\Delta_v^m x_k|^{p_k} = \frac{1}{s+1} \qquad (k(s) \le k \le k(s+1) - 1; s = 1, 2, \dots)$$

hence $x \in \Delta_v^m c_0(p)$, and $\sum_{k=1}^\infty |a_k x_k| \ge \sum_{s=1}^\infty = \infty$, i.e. $a \notin [\Delta_v^m c_0(p)]^\alpha$.

(ii) Omitted.

□

Theorem 2.3. *For every strictly positive sequence $p = (p_k)$, we have*

$$[\Delta_v^m c(p)]^\alpha = M^\alpha(p)$$
$$= M_0^\alpha(p) \cap \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} < \infty \right\}.$$

Proof. Let $a \in M^\alpha(p)$ and $x \in \Delta_v^m c(p)$. Then there is a complex number l such that $|\Delta_v^m x_k - l|^{p_k} \to 0$ $(k \to \infty)$. We define $y = (y_k)$ by

$$y_k = x_k + v_k^{-1} l(-1)^{m+1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} \qquad (k = 1, 2, \dots).$$

Then $y \in \Delta_v^m c_0(p)$ and

$$\sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} |\Delta_v^m y_j|$$
$$+ \sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \sum_{j=1}^{m} \binom{k-j-1}{m-j} |\Delta_v^{m-j} y_j|$$
$$+ |l| \sum_{k=1}^{\infty} |a_k| \, |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} < \infty$$

by Theorem 2.2(i) and since $a \in M^\alpha(p)$.

Now let $a \in [\Delta_v^m c(p)]^\alpha \subset [\Delta_v^m c_0(p)]^\alpha = M_0^\alpha(p)$ by Theorem 2.2(i). Since the sequence x defined by

$$x_k = (-1)^m v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} \qquad (k = 1, 2, \dots)$$

is in $\Delta_v^m c(p)$, we have

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} < \infty.$$

□

Theorem 2.4. *For every strictly positive sequence $p = (p_k)$, we have*

(i) $[D\Delta_v^m \ell_\infty(p)]^\beta = M_\infty^\beta(p)$,

(ii) $[D\Delta_v^m \ell_\infty(p)]^\gamma = M_\infty^\gamma(p)$

where

$$M_\infty^\beta(p) = \bigcap_{N>1} \left\{ a \in \omega : \sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \text{ converges and } \sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} < \infty \right\},$$

$$M_\infty^\gamma(p) = \bigcap_{N>1} \Bigg\{ a \in \omega : \sup_n \Big| \sum_{k=1}^{n} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \Big| < \infty,$$

$$\sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} < \infty \Bigg\}$$

and $b_k = \sum_{j=k+1}^{\infty} v_j^{-1} a_j \ \ (k = 1, 2, \dots)$.

Proof.

(i) If $x \in D\Delta_v^m \ell_\infty(p)$ then there exists a unique $y = (y_k) \in \ell_\infty(p)$ such that

$$x_k = v_k^{-1} \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} y_j$$

for sufficiently large k, for instance $k > m$ by (1.1). Then there is an integer $N > \max\{1, \sup_k |\Delta_v^m x_k|^{p_k}\}$. Let $a \in M_\infty^\beta(p)$, and suppose that $\binom{-1}{-1} = 1$. Then we may write

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \left(v_k^{-1} \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} y_j \right)$$

$$= (-1)^m \sum_{k=1}^{n-m} b_{k+m-1} \sum_{j=1}^{k} \binom{k+m-j-2}{m-2} y_j$$

$$- b_n \sum_{j=1}^{n-m} (-1)^m \binom{n-j-1}{m-1} y_j.$$

Since

$$\sum_{k=1}^{\infty}|b_{k+m-1}|\sum_{j=1}^{k}\binom{k+m-j-2}{m-2}N^{1/p_j}<\infty,$$

the series

$$\sum_{k=1}^{\infty}b_{k+m-1}\sum_{j=1}^{k}\binom{k+m-j-2}{m-2}y_j$$

is absolutely convergent. Moreover by Lemma 1.1(ii), the convergence of

$$\sum_{k=1}^{\infty}a_k v_k^{-1}\sum_{j=1}^{k-m}\binom{k-j-1}{m-1}N^{1/p_j}$$

implies

$$\lim_{n\to\infty}b_n\sum_{j=1}^{n-m}\binom{n-j-1}{m-1}N^{1/p_j}=0.$$

Hence $\sum_{k=1}^{\infty}a_k x_k$ is convergent for all $x\in D\Delta_v^m\ell_\infty(p)$, so $a\in[D\Delta_v^m\ell_\infty(p)]^\beta$.

Conversely let $a\in[D\Delta_v^m\ell_\infty(p)]^\beta$. Then $\sum_{k=1}^{\infty}a_k x_k$ is convergent for each $x\in D\Delta_v^m\ell_\infty(p)$. If we take the sequence $x=(x_k)$ defined by

$$x_k=\begin{cases}0, & k\le m\\ v_k^{-1}\sum_{j=1}^{k-m}\binom{k-j-1}{m-1}N^{1/p_j}, & k>m\end{cases}$$

then we have

$$\sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} = \sum_{k=1}^{\infty} a_k x_k < \infty.$$

Thus the series $\sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j}$ is convergent. This implies that

$$\lim_{n\to\infty} b_n \sum_{j=1}^{n-m} \binom{n-j-1}{m-1} N^{1/p_j} = 0$$

by Lemma 1.1(ii).

Now let $a \in [D\Delta_v^m \ell_\infty(p)]^\beta - M_\infty^\beta(p)$. Then $\sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j}$ is divergent, that is,

$$\sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} = \infty.$$

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0, & k \leq m \\ v_k^{-1} \sum_{i=1}^{k-1} \operatorname{sgn} b_i \sum_{j=1}^{i-m+1} \binom{i-j-1}{m-2} N^{1/p_j}, & k > m \end{cases}$$

where $a_k > 0$ for all k or $a_k < 0$ for all k. It is trivial that $x = (x_k) \in D\Delta_v^m \ell_\infty(p)$. Then we may write for $n > m$

$$\sum_{k=1}^{n} a_k x_k = -\sum_{k=1}^{m} b_{k-1} \Delta_v x_{k-1} - \sum_{k=1}^{n-m} b_{k+m-1} \Delta_v x_{k+m-1} - b_n x_n v_n.$$

Since $(b_n x_n v_n) \in c_0$, now letting $n \to \infty$we get

$$\sum_{k=1}^{\infty} a_k x_k = -\sum_{k=1}^{\infty} b_{k+m-1}\Delta_v x_{k+m-1}$$
$$= \sum_{k=1}^{\infty} |b_{k+m-1}| \sum_{j=1}^{k} \binom{k+m-j-2}{m-2} N^{1/p_j} = \infty.$$

This is a contradiction to $a \in [D\Delta_v^m \ell_\infty(p)]^\beta$. Hence $a \in M_\infty^\beta(p)$.

(ii) Can be proved by the same way as above, using Lemma 1.1(i).

□

Lemma 2.5. $[D\Delta_v^m \ell_\infty(p)]^\eta = [D\Delta_v^m c(p)]^\eta$ *for* $\eta = \beta$ *or* γ.

The proof is obvious and is thus omitted.

Theorem 2.6. *Let* c_0^+ *denote the set of all positive null sequences.*

(a) We put

$$M_3^\beta(p) = \Bigg\{ a \in \omega : \sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} u_j \quad \text{converges and}$$
$$\sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} u_j < \infty, \forall u \in c_0^+ \Bigg\}.$$

Then $[D\Delta_v^m c_0(p)]^\beta = M_3^\beta(p)$.

(b) We put

$$M_4^{\gamma}(p) = \left\{ a \in \omega : \sup_n \left| \sum_{k=1}^{n} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} u_j \right| < \infty, \right.$$
$$\left. \sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} u_j < \infty, \forall u \in c_0^{+} \right\}.$$

Then $[D\Delta_v^m c_0(p)]^{\gamma} = M_4^{\gamma}(p)$.

Proof. (a) and (b) can be proved in the same manner as Theorem 2.4, using Lemma 1.1(i) and (ii). □

Lemma 2.7.

i) $[\Delta_v^m \ell_\infty(p)]^{\eta} = [D\Delta_v^m \ell_\infty(p)]^{\eta}$,

ii) $[\Delta_v^m c(p)]^{\eta} = [D\Delta_v^m c(p)]^{\eta}$,

iii) $[\Delta_v^m c_0(p)]^{\eta} = [D\Delta_v^m c_0(p)]^{\eta}$

for $\eta = \beta$ *or* γ.

The proof is omitted.

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