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## THE DUAL SPACES OF THE SETS OF DIFFERENCE SEQUENCES OF ORDER $m$

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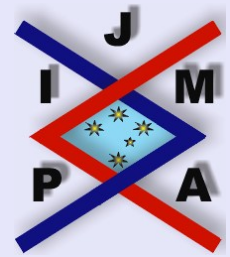


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## Abstract

The idea of difference sequence spaces was introduced by Kizmaz [5] and the concept was generalized by Et and Çolak [3]. Let  $p = (p_k)$  be a bounded sequence of positive real numbers and  $v = (v_k)$  be any fixed sequence of non-zero complex numbers. If  $x = (x_k)$  is any sequence of complex numbers we write  $\Delta_v^m x$  for the sequence of the  $m$ -th order differences of  $x$  and  $\Delta_v^m(X) = \{x = (x_k) : \Delta_v^m x \in X\}$  for any set  $X$  of sequences. In this paper we determine the  $\alpha$ -,  $\beta$ - and  $\gamma$ -duals of the sets  $\Delta_v^m(X)$  which are defined by Et et al. [2] for  $X = \ell_\infty(p)$ ,  $c(p)$  and  $c_0(p)$ . This study generalizes results of Malkowsky [9] in special cases.

*2000 Mathematics Subject Classification:* 40C05, 46A45.

*Key words:* Difference sequences,  $\alpha$ -,  $\beta$ - and  $\gamma$ -duals.

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# 1. Introduction, Notations and Known Results

Throughout this paper  $\omega$  denotes the space of all scalar sequences and any subspace of  $\omega$  is called a sequence space. Let  $\ell_\infty$ ,  $c$  and  $c_0$  be the linear space of bounded, convergent and null sequences with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|,$$

where  $k \in \mathbb{N} = \{1, 2, 3, \dots\}$ , the set of positive integers. Furthermore, let  $p = (p_k)$  be bounded sequences of positive real numbers and

$$\ell_\infty(p) = \left\{ x \in \omega : \sup_k |x_k|^{p_k} < \infty \right\},$$

$$c(p) = \left\{ x \in \omega : \lim_{k \rightarrow \infty} |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C} \right\},$$

$$c_0(p) = \left\{ x \in \omega : \lim_{k \rightarrow \infty} |x_k|^{p_k} = 0 \right\}$$

(for details see [6], [7], [11]).

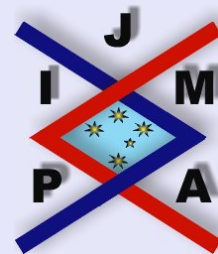
Let  $x$  and  $y$  be complex sequences, and  $E$  and  $F$  be subsets of  $\omega$ . We write

$$M(E, F) = \bigcap_{x \in E} x^{-1} * F = \{a \in \omega : ax \in F \text{ for all } x \in E\} \quad [12].$$

In particular, the sets

$$E^\alpha = M(E, l_1), \quad E^\beta = M(E, cs) \quad \text{and} \quad E^\gamma = M(E, bs)$$

are called the  $\alpha$ -,  $\beta$ - and  $\gamma$ -duals of  $E$ , where  $l_1$ ,  $cs$  and  $bs$  are the sets of all convergent, absolutely convergent and bounded series, respectively. If  $E \subset F$ ,



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then  $F^\eta \subset E^\eta$  for  $\eta = \alpha, \beta, \gamma$ . It is clear that  $E^\alpha \subset (E^\alpha)^\alpha = E^{\alpha\alpha}$ . If  $E = E^{\alpha\alpha}$ , then  $E$  is an  $\alpha$ -space. In particular, an  $\alpha$ -space is called a Köthe space or a perfect sequence space.

Throughout this paper  $X$  will be used to denote any one of the sequence spaces  $\ell_\infty, c$  and  $c_0$ .

Kızmaz [5] introduced the notion of difference sequence spaces as follows:

$$X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}.$$

Later on the notion was generalized by Et and Çolak in [3], namely,

$$X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\}.$$

Subsequently difference sequence spaces have been studied by Malkowsky and Parashar [8], Mursaleen [10], Çolak [1] and many others.

Let  $v = (v_k)$  be any fixed sequence of non-zero complex numbers. Et and Esi [4] generalized the above sequence spaces to the following ones

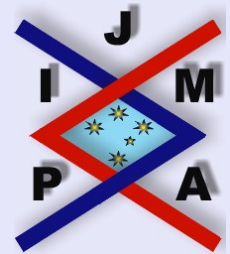
$$\Delta_v^m(X) = \{x = (x_k) : (\Delta_v^m x_k) \in X\},$$

where  $\Delta_v^0 x = (v_k x_k)$ ,  $\Delta_v^m x = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$  such that  $\Delta_v^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ . Recently Et et al. [2] generalized the sequence spaces  $\Delta_v^m(X)$  to the sequence spaces

$$\Delta_v^m(X(p)) = \{x = (x_k) : (\Delta_v^m x_k) \in X(p)\}$$

and showed that these spaces are complete paranormed spaces paranormed by

$$g(x) = \sum_{i=1}^m |x_i v_i| + \sup_k |\Delta_v^m x_k|^{p_k/M},$$



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where  $H = \sup_k p_k$  and  $M = \max(1, H)$ .

Let us define the operator  $D : \Delta_v^m X(p) \rightarrow \Delta_v^m X(p)$  by

$$Dx = (0, 0, \dots, x_{m+1}, x_{m+2}, \dots),$$

where  $x = (x_1, x_2, x_3, \dots)$ . It is trivial that  $D$  is a bounded linear operator on  $\Delta_v^m X(p)$ . Furthermore the set

$$\begin{aligned} D[\Delta_v^m X(p)] &= D\Delta_v^m X(p) \\ &= \{x = (x_k) : x \in \Delta_v^m X(p), x_1 = x_2 = \dots = x_m = 0\} \end{aligned}$$

is a subspace of  $\Delta_v^m X(p)$ .  $D\Delta_v^m X(p)$  and  $X(p)$  are equivalent as topological spaces, since

$$(1.1) \quad \Delta_v^m : D\Delta_v^m X(p) \rightarrow X(p)$$

defined by  $\Delta_v^m x = y = (\Delta_v^m x_k)$  is a linear homeomorphism. Let  $[X(p)]'$  and  $[D\Delta_v^m X(p)]'$  denote the continuous duals of  $X(p)$  and  $D\Delta_v^m X(p)$ , respectively. It can be shown that

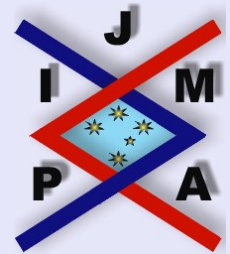
$$T : [D\Delta_v^m X(p)]' \rightarrow [X(p)]', \quad f_\Delta \rightarrow f_\Delta \circ (\Delta_v^m)^{-1} = f,$$

is a linear isometry. So  $[D\Delta_v^m X(p)]'$  is equivalent to  $[X(p)]'$ .

**Lemma 1.1 ([5]).** *Let  $(t_n)$  be a sequence of positive numbers increasing monotonically to infinity, then*

i) *If  $\sup_n |\sum_{i=1}^n t_i a_i| < \infty$ , then  $\sup_n |t_n \sum_{k=n+1}^\infty a_k| < \infty$ ,*

ii) *If  $\sum_k t_k a_k$  is convergent, then  $\lim_{n \rightarrow \infty} t_n \sum_{k=n+1}^\infty a_k = 0$ .*



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## 2. Main Results

In this section we determine the  $\alpha$ -,  $\beta$ - and  $\gamma$ - duals of  $\Delta_v^m X(p)$ .

**Theorem 2.1.** For every strictly positive sequence  $p = (p_k)$ , we have

- (i)  $[\Delta_v^m l_\infty(p)]^\alpha = D_1^\alpha(p)$ ,
- (ii)  $[\Delta_v^m l_\infty(p)]^{\alpha\alpha} = D_1^{\alpha\alpha}(p)$

where

$$D_1^\alpha(p) = \bigcap_{N=2}^{\infty} \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} < \infty \right\}$$

and

$$D_1^{\alpha\alpha}(p) = \bigcup_{N=2}^{\infty} \left\{ a = (a_k) : \sup_{k \geq m+1} |a_k| |v_k| \left[ \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right]^{-1} < \infty \right\}.$$

*Proof.*

- (i) Let  $a \in D_1^\alpha(p)$  and  $x \in \Delta_v^m l_\infty(p)$ . We choose  $N > \max(1, \sup_n | \Delta_v^m a_n |^{p_n})$ .  
Since

$$\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} > \sum_{j=1}^m \binom{k-j-1}{m-j} N^{1/p_j} \geq 1$$



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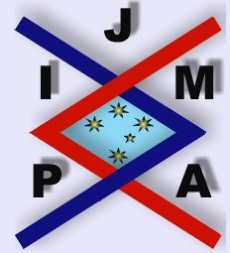


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for arbitrary  $N > 1$  ( $k = 2m, 2m + 1, \dots$ ) and  $|\Delta_v^{m-j} x_j| \leq M$  ( $1 \leq j \leq m$ ) for some constant  $M$ ,  $a \in D_1^\alpha(p)$  implies

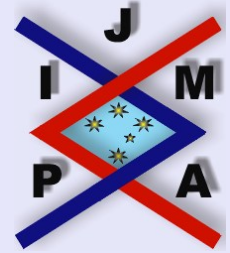
$$\sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^m \binom{k-j-1}{m-j} |\Delta_v^{m-j} x_j| < \infty.$$

Then

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k x_k| &= \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \left( \left| \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} \Delta_v^m x_j \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^m (-1)^{m-j} \binom{k-j-1}{m-j} \Delta_v^{m-j} x_j \right| \right) \\ &\leq \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \\ &\quad + \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^m \binom{k-j-1}{m-j} |\Delta_v^{m-j} x_j| \\ &< \infty. \end{aligned}$$

Conversely let  $a \notin D_1^\alpha(p)$ . Then we have

$$\sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} = \infty$$



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for some integer  $N > 1$ . We define the sequence  $x$  by

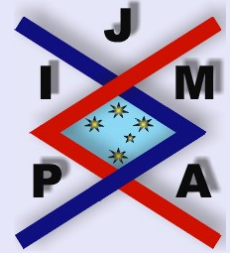
$$x_k = v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \quad (k = m+1, m+2, \dots).$$

Then it is easy to see that  $x \in \Delta_v^m l_\infty(p)$  and  $\sum_k |a_k x_k| = \infty$ . Hence  $a \notin [\Delta_v^m l_\infty(p)]^\alpha$ . This completes the proof of (i).

(ii) Let  $a \in D_1^{\alpha\alpha}(p)$  and  $x \in [\Delta_v^m l_\infty(p)]^\alpha = D_1^\alpha(p)$ , by part (i). Then for some  $N > 1$ , we have

$$\begin{aligned} \sum_{k=m+1}^{\infty} |a_k x_k| &= \sum_{k=m+1}^{\infty} |a_k| |v_k| \left[ \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right]^{-1} \\ &\quad \times |x_k| |v_k|^{-1} \left[ \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right] \\ &\leq \sup_{k \geq m+1} \left( |a_k| |v_k| \left[ \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right]^{-1} \right) \\ &\quad \times \sum_{k=m+1}^{\infty} |x_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \\ &< \infty. \end{aligned}$$





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Conversely let  $a \notin D_1^{\alpha\alpha}(p)$ . Then for all integers  $N > 1$ , we have

$$\sup_{k \geq m+1} |a_k| |v_k| \left[ \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right]^{-1} = \infty.$$

We recall that

$$\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} y_j = 0 \quad (k < m+1)$$

for arbitrary  $y_j$ .

Hence there is a strictly increasing sequence  $(k(s))$  of integers  $k(s) \geq m+1$  such that

$$|a_{k(s)}| |v_{k(s)}| \left[ \sum_{j=1}^{k(s)-m} \binom{k(s)-j-1}{m-1} s^{1/p_j} \right]^{-1} > s^{m+1} \\ (s = m+1, m+2, \dots).$$

We define the sequence  $x$  by

$$x_k = \begin{cases} |a_{k(s)}|^{-1}, & (k = k(s)) \\ 0, & (k \neq k(s)) \quad (k = m+1, m+2, \dots) \end{cases}$$

Then for all integers  $N > m+1$ , we have

$$\sum_{k=1}^{\infty} |x_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} < \sum_{s=m+1}^{\infty} s^{-(m+1)} < \infty.$$

Hence  $x \in [\Delta_v^m l_\infty(p)]^\alpha$  and  $\sum_{k=1}^\infty |a_k x_k| = \sum_{N=1}^\infty 1 = \infty$ . Hence  $a \notin [\Delta_v^m l_\infty(p)]^{\alpha\alpha}$ . The proof is completed.  $\square$

**Theorem 2.2.** For every strictly positive sequence  $p = (p_k)$ , we have

(i)  $[\Delta_v^m c_0(p)]^\alpha = M_0^\alpha(p)$ ,

(ii)  $[\Delta_v^m c_0(p)]^{\alpha\alpha} = M_0^{\alpha\alpha}(p)$

where

$$M_0^\alpha(p) = \bigcup_{N=2}^\infty \left\{ a \in \omega : \sum_{k=1}^\infty |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} < \infty \right\}$$

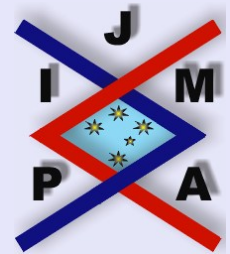
and

$$M_0^{\alpha\alpha}(p) = \bigcap_{N=2}^\infty \left\{ a \in \omega : \sup_{k \geq m+1} |a_k| |v_k| \left[ \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} \right]^{-1} < \infty \right\}.$$

*Proof.*

(i) Let  $a \in M_0^\alpha(p)$  and  $x \in \Delta_v^m c_0(p)$ . Then there is an integer  $k_0$  such that  $\sup_{k > k_0} |\Delta_v^m x_k|^{p_k} \leq N^{-1}$ , where  $N$  is the number in  $M_0^\alpha(p)$ . We put

$$M = \max_{1 \leq k \leq k_0} |\Delta_v^m x_k|^{p_k}, \quad n = \min_{1 \leq k \leq k_0} p_k, \quad L = (M + 1)N$$



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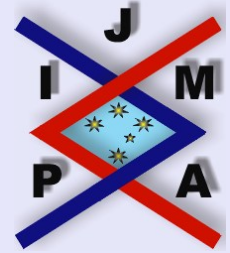


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and define the sequence  $y$  by  $y_k = x_k \cdot L^{-1/n}$  ( $k = 1, 2, \dots$ ). Then it is easy to see that  $\sup_k |\Delta_v^m y_k|^{p_k} \leq N^{-1}$ .

Since

$$\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} > \sum_{j=1}^m \binom{k-j-1}{m-j} N^{-1/p_j}$$

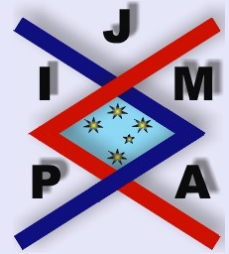
for arbitrary  $N > 1$  ( $k = 2m, 2m + 1, \dots$ ),  $a \in M_0^\alpha(p)$  implies

$$\sum_{k=1}^{\infty} |a_k| |v_k| \sum_{j=1}^m \binom{k-j-1}{m-j} |\Delta_v^{m-j} y_j| < \infty.$$

Then

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k x_k| &= L^{1/n} \sum_{k=1}^{\infty} |a_k y_k| \\ &\leq L^{1/n} \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} \\ &\quad + L^{1/n} \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^m \binom{k-j-1}{m-j} |\Delta_v^{m-j} y_j| \\ &< \infty. \end{aligned}$$

So we have  $a \in [\Delta_v^m c_0(p)]^\alpha$ . Therefore  $M_0^\alpha(p) \subset [\Delta_v^m c_0(p)]^\alpha$ .



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Conversely, let  $a \notin M_0^\alpha(p)$ . Then we can determine a strictly increasing sequence  $(k(s))$  of integers such that  $k(1) = 1$  and

$$M(s) = \sum_{k=k(s)}^{k(s+1)-1} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} (s+1)^{-1/p_j} > 1 \quad (s = 1, 2, \dots).$$

We define the sequence  $x$  by

$$x_k = v_k^{-1} \left( \sum_{l=1}^{s-1} \sum_{j=k(l)}^{k(l+1)-1} \binom{k-j-1}{m-1} (l+1)^{-1/p_j} + \sum_{j=k(s)}^{k-m} \binom{k-j-1}{m-1} (s+1)^{-1/p_j} \right) \\ (k(s) \leq k \leq k(s+1) - 1; s = 1, 2, \dots).$$

Then it is easy to see that

$$|\Delta_v^m x_k|^{p_k} = \frac{1}{s+1} \quad (k(s) \leq k \leq k(s+1) - 1; s = 1, 2, \dots)$$

hence  $x \in \Delta_v^m c_0(p)$ , and  $\sum_{k=1}^{\infty} |a_k x_k| \geq \sum_{s=1}^{\infty} = \infty$ , i.e.  $a \notin [\Delta_v^m c_0(p)]^\alpha$ .

(ii) Omitted.

□

**Theorem 2.3.** For every strictly positive sequence  $p = (p_k)$ , we have

$$\begin{aligned} [\Delta_v^m c(p)]^\alpha &= M^\alpha(p) \\ &= M_0^\alpha(p) \cap \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} < \infty \right\}. \end{aligned}$$

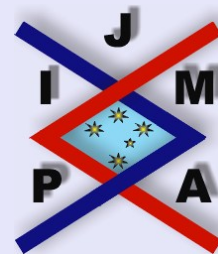
*Proof.* Let  $a \in M^\alpha(p)$  and  $x \in \Delta_v^m c(p)$ . Then there is a complex number  $l$  such that  $|\Delta_v^m x_k - l|^{p_k} \rightarrow 0$  ( $k \rightarrow \infty$ ). We define  $y = (y_k)$  by

$$y_k = x_k + v_k^{-1} l (-1)^{m+1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} \quad (k = 1, 2, \dots).$$

Then  $y \in \Delta_v^m c_0(p)$  and

$$\begin{aligned} \sum_{k=1}^{\infty} |a_k x_k| &\leq \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} |\Delta_v^m y_j| \\ &\quad + \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^m \binom{k-j-1}{m-j} |\Delta_v^{m-j} y_j| \\ &\quad + |l| \sum_{k=1}^{\infty} |a_k| |v_k|^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} < \infty \end{aligned}$$

by Theorem 2.2(i) and since  $a \in M^\alpha(p)$ .



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Now let  $a \in [\Delta_v^m c(p)]^\alpha \subset [\Delta_v^m c_0(p)]^\alpha = M_0^\alpha(p)$  by Theorem 2.2(i). Since the sequence  $x$  defined by

$$x_k = (-1)^m v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} \quad (k = 1, 2, \dots)$$

is in  $\Delta_v^m c(p)$ , we have

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} < \infty.$$

□

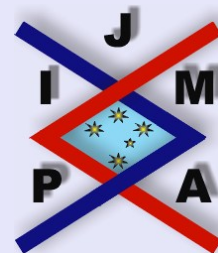
**Theorem 2.4.** For every strictly positive sequence  $p = (p_k)$ , we have

$$(i) [D\Delta_v^m \ell_\infty(p)]^\beta = M_\infty^\beta(p),$$

$$(ii) [D\Delta_v^m \ell_\infty(p)]^\gamma = M_\infty^\gamma(p)$$

where

$$M_\infty^\beta(p) = \bigcap_{N>1} \left\{ a \in \omega : \sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \text{ converges and } \sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} < \infty \right\},$$



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$$M_{\infty}^{\gamma}(p) = \bigcap_{N>1} \left\{ a \in \omega : \sup_n \left| \sum_{k=1}^n a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \right| < \infty, \right. \\ \left. \sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} < \infty \right\}$$

and  $b_k = \sum_{j=k+1}^{\infty} v_j^{-1} a_j$  ( $k = 1, 2, \dots$ ).

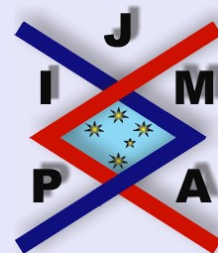
*Proof.*

(i) If  $x \in D\Delta_v^m \ell_{\infty}(p)$  then there exists a unique  $y = (y_k) \in \ell_{\infty}(p)$  such that

$$x_k = v_k^{-1} \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} y_j$$

for sufficiently large  $k$ , for instance  $k > m$  by (1.1). Then there is an integer  $N > \max\{1, \sup_k |\Delta_v^m x_k|^{p_k}\}$ . Let  $a \in M_{\infty}^{\beta}(p)$ , and suppose that  $\binom{-1}{-1} = 1$ . Then we may write

$$\sum_{k=1}^n a_k x_k = \sum_{k=1}^n a_k \left( v_k^{-1} \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} y_j \right) \\ = (-1)^m \sum_{k=1}^{n-m} b_{k+m-1} \sum_{j=1}^k \binom{k+m-j-2}{m-2} y_j \\ - b_n \sum_{j=1}^{n-m} (-1)^m \binom{n-j-1}{m-1} y_j.$$



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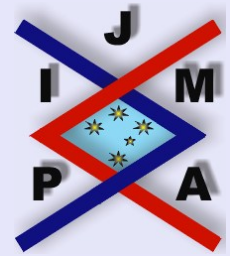


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Since

$$\sum_{k=1}^{\infty} |b_{k+m-1}| \sum_{j=1}^k \binom{k+m-j-2}{m-2} N^{1/p_j} < \infty,$$

the series

$$\sum_{k=1}^{\infty} b_{k+m-1} \sum_{j=1}^k \binom{k+m-j-2}{m-2} y_j$$

is absolutely convergent. Moreover by Lemma 1.1(ii), the convergence of

$$\sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j}$$

implies

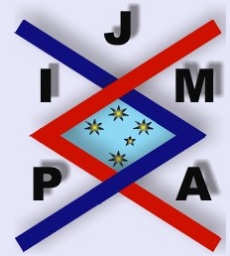
$$\lim_{n \rightarrow \infty} b_n \sum_{j=1}^{n-m} \binom{n-j-1}{m-1} N^{1/p_j} = 0.$$

Hence  $\sum_{k=1}^{\infty} a_k x_k$  is convergent for all  $x \in D\Delta_v^m \ell_{\infty}(p)$ , so  $a \in [D\Delta_v^m \ell_{\infty}(p)]^{\beta}$ .

Conversely let  $a \in [D\Delta_v^m \ell_{\infty}(p)]^{\beta}$ . Then  $\sum_{k=1}^{\infty} a_k x_k$  is convergent for each  $x \in D\Delta_v^m \ell_{\infty}(p)$ . If we take the sequence  $x = (x_k)$  defined by

$$x_k = \begin{cases} 0, & k \leq m \\ v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j}, & k > m \end{cases}$$





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then we have

$$\sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} = \sum_{k=1}^{\infty} a_k x_k < \infty.$$

Thus the series  $\sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j}$  is convergent. This implies that

$$\lim_{n \rightarrow \infty} b_n \sum_{j=1}^{n-m} \binom{n-j-1}{m-1} N^{1/p_j} = 0$$

by Lemma 1.1(ii).

Now let  $a \in [D\Delta_v^m \ell_{\infty}(p)]^{\beta} - M_{\infty}^{\beta}(p)$ . Then  $\sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j}$  is divergent, that is,

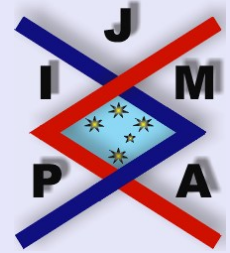
$$\sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} = \infty.$$

We define the sequence  $x = (x_k)$  by

$$x_k = \begin{cases} 0, & k \leq m \\ v_k^{-1} \sum_{i=1}^{k-1} \operatorname{sgn} b_i \sum_{j=1}^{i-m+1} \binom{i-j-1}{m-2} N^{1/p_j}, & k > m \end{cases}$$

where  $a_k > 0$  for all  $k$  or  $a_k < 0$  for all  $k$ . It is trivial that  $x = (x_k) \in D\Delta_v^m \ell_{\infty}(p)$ . Then we may write for  $n > m$

$$\sum_{k=1}^n a_k x_k = - \sum_{k=1}^m b_{k-1} \Delta_v x_{k-1} - \sum_{k=1}^{n-m} b_{k+m-1} \Delta_v x_{k+m-1} - b_n x_n v_n.$$



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Since  $(b_n x_n v_n) \in c_0$ , now letting  $n \rightarrow \infty$  we get

$$\begin{aligned} \sum_{k=1}^{\infty} a_k x_k &= - \sum_{k=1}^{\infty} b_{k+m-1} \Delta_v x_{k+m-1} \\ &= \sum_{k=1}^{\infty} |b_{k+m-1}| \sum_{j=1}^k \binom{k+m-j-2}{m-2} N^{1/p_j} = \infty. \end{aligned}$$

This is a contradiction to  $a \in [D\Delta_v^m \ell_{\infty}(p)]^{\beta}$ . Hence  $a \in M_{\infty}^{\beta}(p)$ .

(ii) Can be proved by the same way as above, using Lemma 1.1(i).

□

**Lemma 2.5.**  $[D\Delta_v^m \ell_{\infty}(p)]^{\eta} = [D\Delta_v^m c(p)]^{\eta}$  for  $\eta = \beta$  or  $\gamma$ .

The proof is obvious and is thus omitted.

**Theorem 2.6.** Let  $c_0^+$  denote the set of all positive null sequences.

(a) We put

$$M_3^{\beta}(p) = \left\{ a \in \omega : \sum_{k=1}^{\infty} a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} u_j \text{ converges and } \sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} u_j < \infty, \forall u \in c_0^+ \right\}.$$

Then  $[D\Delta_v^m c_0(p)]^{\beta} = M_3^{\beta}(p)$ .

(b) We put

$$M_4^\gamma(p) = \left\{ a \in \omega : \sup_n \left| \sum_{k=1}^n a_k v_k^{-1} \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} u_j \right| < \infty, \right. \\ \left. \sum_{k=1}^{\infty} |b_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} u_j < \infty, \forall u \in c_0^+ \right\}.$$

Then  $[D\Delta_v^m c_0(p)]^\gamma = M_4^\gamma(p)$ .

*Proof.* (a) and (b) can be proved in the same manner as Theorem 2.4, using Lemma 1.1(i) and (ii).  $\square$

### Lemma 2.7.

- i)  $[\Delta_v^m \ell_\infty(p)]^\eta = [D\Delta_v^m \ell_\infty(p)]^\eta,$
- ii)  $[\Delta_v^m c(p)]^\eta = [D\Delta_v^m c(p)]^\eta,$
- iii)  $[\Delta_v^m c_0(p)]^\eta = [D\Delta_v^m c_0(p)]^\eta$

for  $\eta = \beta$  or  $\gamma$ .

The proof is omitted.



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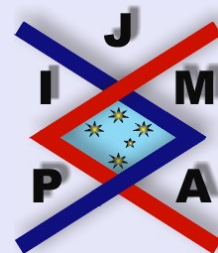
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