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SOME CONVEXITY PROPERTIES FOR A GENERAL INTEGRAL OPERATOR

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Abstract

In this paper we consider the classes of starlike functions, starlike functions of order α, convex functions, convex functions of order α and the classes of the univalent functions denoted by $SH(\beta)$, SP and $SP(\alpha,\beta)$. On these classes we study the convexity and α- order convexity for a general integral operator.

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1. Introduction

Let $U = \{z \in C, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$, the class of the holomorphic functions in U. Consider

$$A = \left\{ f \in H(U), f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in U \right\}$$

the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$. We denote by S^* the class of starlike functions that are defined as holomorphic functions in the unit disc with the properties $f(0) = f'(0) - 1 = 0$ and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U.$$

A function $f \in A$ is a starlike function by the order α, $0 \le \alpha < 1$ if f satisfies the inequality

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in U.$$

We denote this class by $S^*(\alpha)$. Also, we denote by K the class of convex functions that are defined as holomorphic functions in the unit disc with the properties $f(0) = f'(0) - 1 = 0$ and

$$\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0, \quad z \in U.$$

A function $f \in A$ is a convex function by the order α, $0 \le \alpha < 1$ if f verifies the inequality

$$\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > \alpha, \quad z \in U.$$

We denote this class by $K(\alpha)$.

In the paper [5] J. Stankiewicz and A. Wisniowska introduced the class of univalent functions, $SH(\beta)$, $\beta > 0$ defined by:

$$\left| \frac{zf'(z)}{f(z)} - 2\beta\left(\sqrt{2}-1\right) \right| < \operatorname{Re}\left\{ \sqrt{2}\frac{zf'(z)}{f(z)} \right\} + 2\beta\left(\sqrt{2}-1\right), \quad f \in S, \tag{1.1}$$

for all $z \in U$.

Also, in the paper [3] F. Ronning introduced the class of univalent functions, SP, defined by

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad f \in S, \tag{1.2}$$

for all $z \in U$. The geometric interpretation of the relation (1.2) is that the class SP is the class of all functions $f \in S$ for which the expression $zf'(z)/f(z)$, $z \in U$ takes all values in the parabolic region

$$\Omega = \{\omega : |\omega - 1| \leq \operatorname{Re}\omega\} = \left\{\omega = u + iv : v^2 \leq 2u - 1\right\}.$$

In the paper [3] F. Ronning introduced the class of univalent functions $SP(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0, 1)$, as the class of all functions $f \in S$ which have the property:

$$\left| \frac{zf'(z)}{f(z)} - (\alpha + \beta) \right| \leq \operatorname{Re}\frac{zf'(z)}{f(z)} + \alpha - \beta, \tag{1.3}$$

for all $z \in U$. Geometric interpretation: $f \in SP(\alpha, \beta)$ if and only if $zf'(z)/f(z)$, $z \in U$ takes all values in the parabolic region

$$\begin{aligned}\Omega_{\alpha,\beta} &= \{\omega : |\omega - (\alpha+\beta)| \leq \operatorname{Re}\omega + \alpha - \beta\} \\ &= \left\{\omega = u + iv : v^2 \leq 4\alpha(u-\beta)\right\}.\end{aligned}$$

We consider the integral operator F_n, defined by:

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.4}$$

and we study its properties.

Remark 1. *We observe that for $n = 1$ and $\alpha_1 = 1$ we obtain the integral operator of Alexander, $F(z) = \int_0^z \frac{f(t)}{t} dt$.*

2. Main Results

Theorem 2.1. *Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$ and*

$$\sum_{i=1}^{n} \alpha_i \leq n + 1.$$

We suppose that the functions f_i, $i = \{1, \dots, n\}$ are the starlike functions by order $\frac{1}{\alpha_i}$, $i \in \{1, \dots, n\}$, that is $f_i \in S^* \left(\frac{1}{\alpha_i}\right)$ for all $i \in \{1, \dots, n\}$. In these conditions the integral operator defined in (1.4) is convex.

Proof. We calculate for F_n the derivatives of the first and second order. From (1.4) we obtain:

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \dots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F_n''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)}\right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}.$$

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{z f_1'(z) - f_1(z)}{z f_1(z)}\right) + \dots + \alpha_n \left(\frac{z f_n'(z) - f_n(z)}{z f_n(z)}\right),$$

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \dots + \alpha_n \left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right). \tag{2.1}$$

By multiplying the relation (2.1) with z we obtain:

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} - \alpha_1 - \cdots - \alpha_n. \tag{2.2}$$

The relation (2.2) is equivalent with

$$\frac{zF_n''(z)}{F_n'(z)} + 1 = \alpha_1 \frac{zf_1'(z)}{f_1(z)} + \cdots + \alpha_n \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - \cdots - \alpha_n + 1. \tag{2.3}$$

From (2.3) we obtain that:

$$\begin{aligned} \operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) \\ = \alpha_1 \operatorname{Re} \frac{zf_1'(z)}{f_1(z)} + \cdots + \alpha_n \operatorname{Re} \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - \cdots - \alpha_n + 1. \end{aligned} \tag{2.4}$$

But $f_i \in S^* \left(\frac{1}{\alpha_i} \right)$, for all $i \in \{1, \ldots, n\}$, so $\operatorname{Re} \frac{zf_i'(z)}{f_i(z)} > \frac{1}{\alpha_i}$, for all $i \in \{1, \ldots, n\}$. We apply this affirmation in the equality (2.4) and obtain:

$$\begin{aligned} \operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) &> \alpha_1 \frac{1}{\alpha_1} + \cdots + \alpha_n \frac{1}{\alpha_n} - \alpha_1 - \cdots - \alpha_n + 1 \\ &= n + 1 - \sum_{i=1}^{n} \alpha_i. \end{aligned} \tag{2.5}$$

But, in accordance with the hypothesis, we obtain:

$$\operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) > 0$$

so, F_n is a convex function. □

Theorem 2.2. *Let α_i, $i \in \{1, \dots, n\}$, be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$ and*

$$\sum_{i=1}^{n} \alpha_i \leq 1.$$

We suppose that the functions f_i, $i = \{1, \dots, n\}$, are the starlike functions. Then the integral operator defined in (1.4) is convex by order, $1 - \sum_{i=1}^{n} \alpha_i$.

Proof. Following the same steps as in Theorem 2.1, we obtain:

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)} - \alpha_1 - \dots - \alpha_n. \tag{2.6}$$

The relation (2.6) is equivalent with

$$\frac{zF_n''(z)}{F_n'(z)} + 1 = \alpha_1 \frac{zf_1'(z)}{f_1(z)} + \dots + \alpha_n \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - \dots - \alpha_n + 1. \tag{2.7}$$

From (2.7) we obtain that:

$$\begin{aligned} \operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) \\ = \alpha_1 \operatorname{Re} \frac{zf_1'(z)}{f_1(z)} + \dots + \alpha_n \operatorname{Re} \frac{zf_n'(z)}{f_n(z)} - \alpha_1 - \dots - \alpha_n + 1. \end{aligned} \tag{2.8}$$

But $f_i \in S^*$ for all $i \in \{1, \dots, n\}$, so $\operatorname{Re} \frac{zf_i'(z)}{f_i(z)} > 0$ for all $i \in \{1, \dots, n\}$. We apply this affirmation in the equality (2.8) and obtain that:

$$\operatorname{Re} \left(\frac{zF_n''(z)}{F_n'(z)} + 1 \right) > \alpha_1 \cdot 0 + \dots + \alpha_n \cdot 0 - \alpha_1 - \dots - \alpha_n + 1 = 1 - \sum_{i=1}^{n} \alpha_i. \tag{2.9}$$

But in accordance with the inequality (2.9), obtain that

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>1-\sum_{i=1}^{n}\alpha_i$$

so, F_n is a convex function by order $1-\sum_{i=1}^{n}\alpha_i$. □

Theorem 2.3. *Let α_i, $i\in\{1,\dots,n\}$, be real numbers with the properties $\alpha_i>0$, for $i\in\{1,\dots,n\}$ and*

$$\sum_{i=1}^{n}\alpha_i\le\frac{\sqrt{2}}{2\beta\left(\sqrt{2}-1\right)+\sqrt{2}}. \tag{2.10}$$

We suppose that $f_i\in SH(\beta)$, for $i=\{1,\dots,n\}$ and $\beta>0$. In these conditions, the integral operator defined in (1.4) is convex.

Proof. Following the same steps as in Theorem 2.1, we obtain that:

$$\frac{zF_n''(z)}{F_n'(z)}+1=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i+1. \tag{2.11}$$

We multiply the relation (2.11) with $\sqrt{2}$ and obtain:

$$\sqrt{2}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)=\sum_{i=1}^{n}\sqrt{2}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}. \tag{2.12}$$

The equality (2.12) is equivalent with:

$$\sqrt{2}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)=\sum_{i=1}^{n}\left(\alpha_i\sqrt{2}\frac{zf_i'(z)}{f_i(z)}+2\alpha_i\beta\left(\sqrt{2}-1\right)\right)$$
$$-\sum_{i=1}^{n}2\alpha_i\beta\left(\sqrt{2}-1\right)-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}.$$

We calculate the real part from both terms of the above equality and obtain:

$$\sqrt{2}\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)=\sum_{i=1}^{n}\left(\alpha_i\left(\operatorname{Re}\left\{\sqrt{2}\frac{zf_i'(z)}{f_i(z)}\right\}+2\beta\left(\sqrt{2}-1\right)\right)\right)$$
$$-\sum_{i=1}^{n}2\alpha_i\beta\left(\sqrt{2}-1\right)-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}.$$

Because $f_i \in SH(\beta)$ for $i=\{1,\ldots,n\}$, we apply in the above relation the inequality (1.1) and obtain:

$$\sqrt{2}\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-2\beta\left(\sqrt{2}-1\right)\right|$$
$$-\sum_{i=1}^{n}2\alpha_i\beta\left(\sqrt{2}-1\right)-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}.$$

Because $\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-2\beta\left(\sqrt{2}-1\right)\right|>0$, for all $i\in\{1,\ldots,n\}$, we obtain that

$$\sqrt{2}\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>-\sum_{i=1}^{n}2\alpha_i\beta\left(\sqrt{2}-1\right)-\sqrt{2}\sum_{i=1}^{n}\alpha_i+\sqrt{2}. \tag{2.13}$$

Using the hypothesis (2.10), we have:

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>0, \tag{2.14}$$

so, F_n is a convex function. □

Corollary 2.4. *Let α be real numbers with the properties $0<\alpha\le\frac{\sqrt{2}}{2\beta\left(\sqrt{2}-1\right)+\sqrt{2}}$, $\beta>0$. We suppose that the functions $f\in SH(\beta)$. In these conditions the integral operator, $F(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}dt$ is convex.*

Proof. In Theorem 2.3, we consider $n=1$, $\alpha_1=\alpha$ and $f_1=f$. □

Theorem 2.5. *Let $\alpha_i, i\in\{1,\dots,n\}$ be real numbers with the properties $\alpha_i>0$ for $i\in\{1,\dots,n\}$,*

$$\sum_{i=1}^{n}\alpha_i<1 \tag{2.15}$$

and $1-\sum_{i=1}^{n}\alpha_i\in[0,1)$. We consider the functions f_i, $f_i\in SP$ for $i=\{1,\dots,n\}$. In these conditions, the integral operator defined in (1.4) is convex by $1-\sum_{i=1}^{n}\alpha_i$ order.

Proof. Following the same steps as in Theorem 2.1, we have:

$$\frac{zF_n''(z)}{F_n'(z)}+1=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i+1. \tag{2.16}$$

We calculate the real part from both terms of the above equality and obtain:

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)=\sum_{i=1}^{n}\alpha_i\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)-\sum_{i=1}^{n}\alpha_i+1. \tag{2.17}$$

Because $f_i \in SP$ for $i=\{1,\dots,n\}$ we apply in the above relation the inequality (1.2) and obtain:

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|-\sum_{i=1}^{n}\alpha_i+1. \tag{2.18}$$

Because $\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|>0$, for all $i\in\{1,\dots,n\}$, we get

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>1-\sum_{i=1}^{n}\alpha_i. \tag{2.19}$$

Using the hypothesis, we obtain that F_n is a convex function by $1-\sum_{i=1}^{n}\alpha_i$ order. □

Remark 2. *If* $\sum_{i=1}^{n}\alpha_i=1$ *then*

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)>0, \tag{2.20}$$

so, F_n *is a convex function.*

Corollary 2.6. *Let* γ *be a real number with the property* $0 < \gamma < 1$. *We suppose that* $f \in SP$. *In these conditions the integral operator* $F(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\gamma} dt$ *is convex of* $1 - \gamma$ *order.*

Proof. In Theorem 2.5, we consider $n = 1$, $\alpha_1 = \gamma$ and $f_1 = f$. □

Theorem 2.7. *We suppose that* $f \in SP$. *In this condition, the integral operator of Alexander, defined by*

$$F_1(z) = \int_0^z \frac{f(t)}{t} dt, \tag{2.21}$$

is convex.

Proof. We have:

$$\operatorname{Re}\left(\frac{zF_1''(z)}{F_1'(z)} + 1\right) = \operatorname{Re}\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right| > 0. \tag{2.22}$$

So, the relation (2.22) implies that the Alexander operator is convex. □

Remark 3. *Theorem 2.7 can be obtained from Corollary 2.6, for* $\gamma = 1$.

Theorem 2.8. *Let* $\alpha_i, i \in \{1, \ldots, n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1, \ldots, n\}$,

$$\sum_{i=1}^{n} \alpha_i < \frac{1}{\alpha - \beta + 1}, \qquad \alpha > 0, \beta \in [0, 1) \tag{2.23}$$

and $(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1\in(0,1)$. *We suppose that* $f_i\in SP(\alpha,\beta)$, *for* $i=\{1,\dots,n\}$. *In these conditions, the integral operator defined in (1.4) is convex by* $(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1$ *order.*

Proof. Following the same steps as in Theorem 2.1, we obtain that:

$$\frac{zF_n''(z)}{F_n'(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}+\alpha-\beta\right)+(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i. \tag{2.24}$$

and

$$\frac{zF_n''(z)}{F_n'(z)}+1=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}+\alpha-\beta\right)+(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1. \tag{2.25}$$

We calculate the real part from both terms of the above equality and get:

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)=\operatorname{Re}\left\{\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}+\alpha-\beta\right)\right\}+(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1. \tag{2.26}$$

Because $f_i\in SP(\alpha,\beta)$ for $i=\{1,\dots,n\}$ we apply in the above relation the inequality (1.3) and obtain:

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)\geq\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-(\alpha+\beta)\right|+(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1. \tag{2.27}$$

Since $\alpha_i \left| \frac{z f_i'(z)}{f_i(z)} - (\alpha+\beta) \right| > 0$, for all $i \in \{1,\dots,n\}$, using the inequality (1.3), we have

$$\text{(2.28)} \qquad \operatorname{Re}\left(\frac{z F_n''(z)}{F_n'(z)} + 1 \right) \geq (\beta - \alpha - 1) \sum_{i=1}^{n} \alpha_i + 1 > 0.$$

From (2.28), since $(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i + 1 \in (0,1)$, we obtain that the integral operator defined in (1.4) is convex by $(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i + 1$ order. □

Corollary 2.9. *Let γ be a real number with the property $0 < \gamma < \frac{1}{\alpha-\beta+1}$, $\alpha > 0$, $\beta \in [0,1)$. We suppose that $f \in SP(\alpha,\beta)$. In these conditions, the integral operator $F(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\gamma} dt$ is convex.*

Proof. In Theorem 2.8, we consider $n = 1$, $\alpha_1 = \gamma$ and $f_1 = f$.

For $\alpha = \beta \in (0,1)$ we obtain the class $S(\alpha,\alpha)$ that is characterized by the property

$$\text{(2.29)} \qquad \left| \frac{z f'(z)}{f(z)} - 2\alpha \right| \leq \operatorname{Re} \frac{z f'(z)}{f(z)}.$$

□

Corollary 2.10. *Let α_i, $i \in \{1,\dots,n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1,\dots,n\}$ and*

$$\text{(2.30)} \qquad 1 - \sum_{i=1}^{n} \alpha_i \in [0,1).$$

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We consider the functions f_i, $f_i \in SP(\alpha,\alpha)$, $i = \{1,\dots,n\}$, $\alpha \in (0,1)$. In these conditions, the integral operator defined in (1.4) is convex by $1-\sum_{i=1}^{n}\alpha_i$ order.

Proof. From (1.4) we obtain

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^{n}\alpha_i, \tag{2.31}$$

which is equivalent with

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right) = \sum_{i=1}^{n}\alpha_i\operatorname{Re}\frac{zf_i'(z)}{f_i(z)} - \sum_{i=1}^{n}\alpha_i+1. \tag{2.32}$$

From (2.31) and (2.32), we have:

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right) > \sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-2\alpha\right| + 1 - \sum_{i=1}^{n}\alpha_i. \tag{2.33}$$

Since $\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-2\alpha\right| > 0$, for all $i \in \{1,\dots,n\}$, from (2.33), we get:

$$\operatorname{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right) > 1-\sum_{i=1}^{n}\alpha_i. \tag{2.34}$$

Now, from (2.34) we obtain that the operator defined in (1.4) is convex by $1-\sum_{i=1}^{n}\alpha_i$ order. □

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