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A WEIGHTED GEOMETRIC INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. A new weighted geometric inequality is established by Klamkin's polar moment of inertia inequality and the inversion transformation, some interesting applications of this result are given, and some conjectures which verified by computer are also mentioned.

Key words and phrases: Triangle, Point, Polar moment of inertia inequality.

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1. INTRODUCTION

In 1975, M.S. Klamkin [1] established the following inequality: Let ABC be an arbitrary triangle of sides a, b, c, and let P be an arbitrary point in a space, the distances of P from the vertices A, B, C are R_1, R_2, R_3. If x, y, z are real numbers, then

$$(x + y + z)(xR_1^2 + yR_2^2 + zR_3^2) \geq yza^2 + zxb^2 + xyc^2, \tag{1.1}$$

with equality if and only if P lies in the plane of $\triangle ABC$ and $x : y : z = \overrightarrow{S}_{\triangle PBC} : \overrightarrow{S}_{\triangle PCA} : \overrightarrow{S}_{\triangle PAB}$ ($\overrightarrow{S}_{\triangle PBC}$ denote the algebra area, etc.)

Inequality (1.1) is called the polar moment of the inertia inequality. It is one of the most important inequalities for the triangle, and there exist many consequences and applications for it, see [1] – [5]. In this paper, we will apply Klamkin's inequality (1.1) and the inversion transformation to deduce a new weighted geometric inequality, then we discuss applications of our results. In addition, we also pose some conjectures.

2. MAIN RESULT

In order to prove our new results, we firstly give the following lemma.

Lemma 2.1. *Let ABC be an arbitrary triangle, and let P be an arbitrary point on the plane of the triangle ABC. If the following inequality:*

$$f(a, b, c, R_1, R_2, R_3) \geq 0 \tag{2.1}$$

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holds, then we have the dual inequality:

$$f(aR_1, bR_2, cR_3, R_2R_3, R_3R_1, R_1R_2) \geq 0. \tag{2.2}$$

Indeed, the above conclusion can be deduced by using inversion transformation, see [6] or [3], [7].

Now, we state and prove main result.

Theorem 2.2. *Let x, y, z be positive real numbers. Then for any triangle ABC and arbitrary point P in the plane of $\triangle ABC$, the following inequality holds:*

$$\frac{R_1^2}{x} + \frac{R_2^2}{y} + \frac{R_3^2}{z} \geq \frac{aR_1 + bR_2 + cR_3}{\sqrt{yz + zx + xy}}, \tag{2.3}$$

with equality if and only if $\triangle ABC$ is acute-angled, P coincides with its orthocenter and $x : y : z = \cot A : \cot B : \cot C$.

Proof. If P coincides with one of the vertices of $\triangle ABC$, for example $P \equiv A$, then $PA = 0, PB = c, PC = b$, and (2.3) becomes a trivial inequality. In this case, equality in (2.3) obviously cannot occur.

Next, assume that P does not coincide with the vertices.

If x, y, z are positive real numbers, then by the polar moment of inertia inequality (1.1) we have

$$(xR_1^2 + yR_2^2 + zR_3^2)\left(\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}\right) \geq \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}.$$

On the other hand, from the Cauchy-Schwarz inequality we get

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z},$$

with equality if and only if $x : y : z = a : b : c$.

Combining these two above inequalities, for any positive real numbers x, y, z, the following inequality holds:

$$(xR_1^2 + yR_2^2 + zR_3^2)\left(\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}\right) \geq \frac{(a+b+c)^2}{x+y+z}. \tag{2.4}$$

and equality holds if and only if $x : y : z = a : b : c$ and P is the incenter of $\triangle ABC$.

Now, applying the inversion transformation in the lemma to inequality (2.4), we obtain

$$\left[x(R_2R_3)^2 + y(R_3R_1)^2 + z(R_1R_2)^2\right]\left(\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}\right) \geq \frac{(aR_1 + bR_2 + cR_3)^2}{x+y+z},$$

or equivalently

$$\frac{(R_2R_3)^2}{yz} + \frac{(R_3R_1)^2}{zx} + \frac{(R_1R_2)^2}{xy} \geq \left(\frac{aR_1 + bR_2 + cR_3}{x+y+z}\right)^2. \tag{2.5}$$

where x, y, z are positive numbers.

For $x \to xR_1^2, y \to yR_2^2, z \to zR_3^2$, we have:

$$\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} \geq \left(\frac{aR_1 + bR_2 + cR_3}{xR_1^2 + yR_2^2 + zR_3^2}\right)^2. \tag{2.6}$$

Take again $x \to \frac{1}{x}, y \to \frac{1}{y}, z \to \frac{1}{z}$, we get the inequality (2.3) of the theorem.

Note the conclusion in [7]: If equality in (2.1) occurs only when P is the incenter of $\triangle ABC$, then equality in (2.2) occurs only when $\triangle ABC$ is acute-angled and P is its orthocenter. According to this and the condition for which equality holds in (2.4), we know that equality in (2.3) holds if and only if $\triangle ABC$ is acute-angled, P is its orthocenter and

$$\frac{R_1}{xa} = \frac{R_2}{yb} = \frac{R_3}{cz}. \tag{2.7}$$

When P is the orthocenter of the acute triangle ABC, we have $R_1 : R_2 : R_3 = \cos A : \cos B : \cos C$. Hence, in this case, from (2.7) we have $x : y : z = \cot A : \cot B : \cot C$. Thus, there is equality in (2.3) if and only if $\triangle ABC$ is acute-angled, P coincides with its orthocenter and $x/\cot A = y/\cot B = z/\cot C$. This completes the proof of the theorem. □

Remark 1. If P does not coincide with the vertices, then inequality (2.4) is equivalent to the following result in [8]:

$$x\frac{R_2R_3}{R_1} + y\frac{R_3R_1}{R_2} + z\frac{R_1R_2}{R_3} \geq 2\sqrt{\frac{xyz}{x+y+z}}s, \tag{2.8}$$

where s is the semi-perimeter of $\triangle ABC$, x, y, z are positive real numbers. In [8], (2.8) was proved without using the polar moment of inertia inequality.

3. APPLICATIONS OF THE THEOREM

Besides the above notations, as usual, let R and r denote the radii of the circumcircle and incircle of triangle ABC, respectively, Δ denote the area, r_a, r_b, r_c denote the radii of the excircles. In addition, when point P lies in the interior of triangle ABC, let r_1, r_2, r_3 denote the distances of P to the sides BC, CA, AB.

According to the theorem and the well-known inequality for any point P in the plane

$$aR_1 + bR_2 + cR_3 \geq 4\Delta, \tag{3.1}$$

we get

Corollary 3.1. *For any point P in the plane and arbitrary positive numbers x, y, z, the following inequality holds:*

$$\frac{R_1^2}{x} + \frac{R_2^2}{y} + \frac{R_3^2}{z} \geq \frac{4\Delta}{\sqrt{yz+zx+xy}}, \tag{3.2}$$

with equality if and only if $x : y : z = \cot A : \cot B : \cot C$ and P is the orthocenter of the acute angled triangle ABC.

Remark 2. Clearly, (3.2) is equivalent with

$$xR_1^2 + yR_2^2 + zR_3^2 \geq 4\sqrt{\frac{xyz}{x+y+z}}\Delta. \tag{3.3}$$

The above inequality was first given in [9] by Xue-Zhi Yang. The author [10] obtained the following generalization:

$$x\left(\frac{a'}{a}R_1\right)^2 + y\left(\frac{b'}{b}R_2\right)^2 + z\left(\frac{c'}{c}R_3\right)^2 \geq 4\sqrt{\frac{xyz}{x+y+z}}\Delta', \tag{3.4}$$

where a', b', c' denote the sides of $\triangle A'B'C'$, Δ' denotes its area.

If, in (2.3) we put $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$, and note that $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$, then we get the result:

Corollary 3.2. *For arbitrary point P in the plane of $\triangle ABC$, the following inequality holds:*

$$\frac{aR_1^2+bR_2^2+cR_3^2}{aR_1+bR_2+cR_3}\geq\sqrt{2Rr}. \tag{3.5}$$

Equality holds if and only if the triangle ABC is equilateral and P is its center.

Remark 3. The conditions for equality that the following inequalities of Corollaries 3.4 – 3.8 have are the same as the statement of Corollary 3.2.

In the theorem, for $x=\frac{R_1}{a}, y=\frac{R_2}{b}, z=\frac{R_3}{c}$, after reductions we obtain

Corollary 3.3. *If P is an arbitrary point which does not coincide with the vertices of $\triangle ABC$, then*

$$\frac{R_2R_3}{bc}+\frac{R_3R_1}{ca}+\frac{R_1R_2}{ab}\geq 1. \tag{3.6}$$

Equality holds if and only if $\triangle ABC$ is acute-angled and P is its orthocenter.

Inequality (3.6) was first proved by T. Hayashi (see [11] or [3]), who gave its two generalizations in [12].

Indeed, assume P does not coincide with the vertices, put $x\to\frac{R_1}{xa}, y\to\frac{R_2}{yb}, z\to\frac{R_3}{zc}$ in (2.2), then we get a weighted generalized form of Hayashi inequality:

$$\frac{R_2R_3}{yzbc}+\frac{R_3R_1}{zxca}+\frac{R_1R_2}{xyab}\geq\left(\frac{aR_1+bR_2+cR_3}{xaR_1+ybR_2+zcR_3}\right)^2. \tag{3.7}$$

For $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$, we have

$$(R_2R_3+R_3R_1+R_1R_2)(R_1+R_2+R_3)^2\geq(aR_1+bR_2+cR_3)^2. \tag{3.8}$$

Applying the inversion transformation of the lemma to the above inequality, then dividing both sides by $R_1R_2R_3$, we get the following result.

Corollary 3.4. *If P is an arbitrary point which does not coincide with the vertices of $\triangle ABC$, then*

$$(R_2R_3+R_3R_1+R_1R_2)^2\left(\frac{1}{R_2R_3}+\frac{1}{R_3R_1}+\frac{1}{R_1R_2}\right)\geq 4s^2. \tag{3.9}$$

It is not difficult to see that the above inequality is stronger than the following result which the author obtained many years ago:

$$\sqrt{\frac{R_2R_3}{R_1}}+\sqrt{\frac{R_3R_1}{R_2}}+\sqrt{\frac{R_1R_2}{R_3}}\geq\sqrt{2\sqrt{3}s}. \tag{3.10}$$

Now, let P be an interior point of the triangle ABC. Then we have the well known inequalities (see [13]):

$$aR_1\geq br_3+cr_2, bR_2\geq cr_1+ar_3, cR_3\geq ar_2+br_1.$$

Summing them up, we note that $a+b+c=2s$ and by the identity $ar_1+br_2+cr_3=2rs$, we easily get

$$aR_1+bR_2+cR_3\geq 2s(r_1+r_2+r_3)-2rs. \tag{3.11}$$

Multiplying both sides by 2 then adding inequality (3.1) and using $\Delta=rs$,

$$3(aR_1+bR_2+cR_3)\geq 4s(r_1+r_2+r_3),$$

that is

$$\frac{aR_1+bR_2+cR_3}{r_1+r_2+r_3}\geq\frac{4}{3}s. \tag{3.12}$$

According to this and the equivalent form (2.5) of inequality (2.3), we immediately get the result:

Corollary 3.5. *Let P be an interior point of the triangle ABC. Then*

$$\frac{(R_2R_3)^2}{r_2r_3}+\frac{(R_3R_1)^2}{r_3r_1}+\frac{(R_1R_2)^2}{r_1r_2}\geq\frac{16}{9}s^2. \tag{3.13}$$

From inequalities (3.8) and (3.12) we infer that

$$(R_2R_3+R_3R_1+R_1R_2)(R_1+R_2+R_3)^2\geq\frac{16}{9}s^2(r_1+r_2+r_3)^2,$$

Noting again that $3(R_2R_3+R_3R_1+R_1R_2)\leq(R_1+R_2+R_3)^2$, we get the following inequality:

Corollary 3.6. *Let P be an interior point of triangle ABC, then*

$$\frac{(R_1+R_2+R_3)^2}{r_1+r_2+r_3}\geq\frac{4}{\sqrt{3}}s. \tag{3.14}$$

Letting $x=r_a, y=r_b, z=r_c$ in (2.3) and noting that identity $r_br_c+r_cr_a+r_ar_b=s^2$, we have

$$\frac{R_1^2}{r_a}+\frac{R_2^2}{r_b}+\frac{R_3^2}{r_c}\geq\frac{1}{s}(aR_1+bR_2+cR_3). \tag{3.15}$$

This inequality and (3.12) lead us to the following inequality:

Corollary 3.7. *Let P be an interior point of the triangle ABC, then*

$$\frac{R_1^2}{r_a}+\frac{R_2^2}{r_b}+\frac{R_3^2}{r_c}\geq\frac{4}{3}(r_1+r_2+r_3). \tag{3.16}$$

Adding (3.1) and (3.11) then dividing both sides by 2, we have

$$aR_1+bR_2+cR_3\geq s(r_1+r_2+r_3+r). \tag{3.17}$$

From this and (3.15), we again get the following inequality which is similar to (3.16):

Corollary 3.8. *Let P be an interior point of the triangle ABC. Then*

$$\frac{R_1^2}{r_a}+\frac{R_2^2}{r_b}+\frac{R_3^2}{r_c}\geq r_1+r_2+r_3+r. \tag{3.18}$$

When P locates the interior of the triangle ABC, let D,E,F be the feet of the perpendicular from P to the sides BC,CA,AB respectively. Take $x=ar_1, y=br_2, z=cr_3$ in the equivalent form (2.6) of inequality (2.3), then

$$\frac{1}{bcr_2r_3}+\frac{1}{car_3r_1}+\frac{1}{abr_1r_2}\geq\left(\frac{aR_1+bR_2+cR_3}{ar_1R_1+br_2R_2+cr_3R_3}\right)^2,$$

Using $ar_1+br_2+cr_3=2\Delta$ and the well known identity (see [7]):

$$ar_1R_1^2+br_2R_2^2+cr_3R_3^2=8R^2\Delta_p, \tag{3.19}$$

(where Δ_p is the area of the pedal triangle DEF), we get

$$abcr_1r_2r_3(aR_1+bR_2+cR_3)^2\leq 64\Delta R^4\Delta_p^2.$$

Let s_p, r_p denote the semi-perimeter of the triangle DEF and the radius of the incircle respectively. Note that $\Delta_p=r_ps_p, aR_1+bR_2+cR_3=4Rs_p$. From the above inequality we obtain the following inequality which was established by the author in [14]:

Corollary 3.9. *Let P be an interior point of the triangle ABC. Then*

$$\frac{r_1 r_2 r_3}{r_p^2} \leq 2R. \tag{3.20}$$

Equality holds if and only if P is the orthocenter of the triangle ABC.

It is well known that there are few inequalities relating a triangle and two points. Several years ago, the author conjectured that the following inequality holds:

$$\frac{R_1^2}{d_1} + \frac{R_2^2}{d_2} + \frac{R_3^2}{d_3} \geq 4(r_1 + r_2 + r_3), \tag{3.21}$$

where d_1, d_2, d_3 denote the distances from an interior point Q to the sides of $\triangle ABC$.

Inequality (3.21) is very interesting and the author has been trying to prove it. In what follows, we will prove a stronger result. To do so, we need a corollary of the following conclusion (see [15]):

Let Q be an interior point of $\triangle ABC$, t_1, t_2, t_3 denote the bisector of $\angle BQC, \angle CQA, \angle AQB$ respectively and $\triangle A'B'C'$ be an arbitrary triangle. Then

$$t_2 t_3 \sin A' + t_3 t_1 \sin B' + t_1 t_2 \sin C' \leq \frac{1}{2}\Delta, \tag{3.22}$$

with equality if and only if $\triangle A'B'C' \sim \triangle ABC$, and Q is the circumcentre of $\triangle ABC$.

In (3.22), letting $\triangle ABC$ be equilateral, we immediately get

$$t_2 t_3 + t_3 t_1 + t_1 t_2 \leq \frac{1}{\sqrt{3}}\Delta. \tag{3.23}$$

From this and the simple inequality $s^2 \geq 3\sqrt{3}\Delta$, we have

$$t_2 t_3 + t_3 t_1 + t_1 t_2 \leq \frac{1}{9}s^2. \tag{3.24}$$

According to inequality (2.3) of the theorem and (3.24), we can see that

$$\frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq \frac{3}{s}(aR_1 + bR_2 + cR_3). \tag{3.25}$$

By using inequality (3.12), we obtain the following stronger version of inequality (3.21).

Corollary 3.10. *Let P and Q be two interior points of $\triangle ABC$, then*

$$\frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq 4(r_1 + r_2 + r_3), \tag{3.26}$$

with equality if and only if $\triangle ABC$ is equilateral and P, Q are both its center.

Analogously, from inequality (3.17) and inequality (3.25) we get:

Corollary 3.11. *Let P and Q be two interior points of $\triangle ABC$, then*

$$\frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq 3(r_1 + r_2 + r_3 + r). \tag{3.27}$$

with equality if and only if $\triangle ABC$ is equilateral and P, Q are both its center.

4. SOME CONJECTURES

In this section, we will state some conjectures in relation to our results.

Inequality (3.8) is equivalent to

$$R_2R_3 + R_3R_1 + R_1R_2 \geq \left(\frac{aR_1 + bR_2 + cR_3}{R_1 + R_2 + R_3}\right)^2. \tag{4.1}$$

With this one and the well known inequality:

$$R_2R_3 + R_3R_1 + R_1R_2 \geq 4(w_2w_3 + w_3w_1 + w_1w_2) \tag{4.2}$$

in mind, we pose the following

Conjecture 4.1. *Let P be an arbitrary interior point of the triangle ABC, then*

$$\left(\frac{aR_1 + bR_2 + cR_3}{R_1 + R_2 + R_3}\right)^2 \geq 4(w_2w_3 + w_3w_1 + w_1w_2). \tag{4.3}$$

Considering Corollary 3.5, the author posed these two conjectures:

Conjecture 4.2. *Let P be an arbitrary interior point of the triangle ABC, then*

$$\frac{(R_2R_3)^2}{w_2w_3} + \frac{(R_3R_1)^2}{w_3w_1} + \frac{(R_1R_2)^2}{w_1w_2} \geq \frac{4}{3}(a^2 + b^2 + c^2). \tag{4.4}$$

Conjecture 4.3. *Let P be an arbitrary interior point of the triangle ABC, then*

$$\frac{(R_2R_3)^2}{r_2r_3} + \frac{(R_3R_1)^2}{r_3r_1} + \frac{(R_1R_2)^2}{r_1r_2} \geq 4(R_1^2 + R_2^2 + R_3^2). \tag{4.5}$$

From the inequality of Corollary 3.6, we surmise that the following stronger inequality holds:

Conjecture 4.4. *Let P be an arbitrary interior point of the triangle ABC, then*

$$\frac{R_2R_3 + R_3R_1 + R_1R_2}{r_1 + r_2 + r_3} \geq \frac{4}{3\sqrt{3}}s. \tag{4.6}$$

On the other hand, for the acute-angled triangle, we pose the following:

Conjecture 4.5. *Let $\triangle ABC$ be acute-angled and P an arbitrary point in its interior, then*

$$\frac{(R_1 + R_2 + R_3)^2}{w_1 + w_2 + w_3} \geq 6R. \tag{4.7}$$

Two years ago, Xue-Zhi Yang proved the following inequality (private communication):

$$\frac{(R_1 + R_2 + R_3)^2}{r_1 + r_2 + r_3} \geq 2\sqrt{a^2 + b^2 + c^2}. \tag{4.8}$$

which is stronger than (3.14). Here, we further put forward the following

Conjecture 4.6. *Let P be an arbitrary interior point of the triangle ABC, then*

$$\frac{(R_1 + R_2 + R_3)^2}{w_1 + w_2 + w_3} \geq 2\sqrt{a^2 + b^2 + c^2}. \tag{4.9}$$

In [14], the author pointed out the following phenomenon (the so-called $r - w$ phenomenon): If the inequality holds for r_1, r_2, r_3 (this inequality can also include R_1, R_2, R_3 and other geometric elements), then after changing r_1, r_2, r_3 into w_1, w_2, w_3 respectively, the stronger inequality often holds or often holds for the acute triangle. Conjecture 4.6 was proposed based on this kind of phenomenon. Analogously, we pose the following four conjectures:

Conjecture 4.7. *Let $\triangle ABC$ be acute-angled and P an arbitrary point in its interior. Then*

$$\frac{aR_1 + bR_2 + cR_3}{w_1 + w_2 + w_3} \geq \frac{4}{3}s. \tag{4.10}$$

Conjecture 4.8. *Let $\triangle ABC$ be acute-angled and P an arbitrary point in its interior. Then*

$$\frac{aR_1 + bR_2 + cR_3}{w_1 + w_2 + w_3 + r} \geq 2s. \tag{4.11}$$

Conjecture 4.9. *Let P and Q be two interior points of the $\triangle ABC$. Then*

$$\frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq 4(w_1 + w_2 + w_3). \tag{4.12}$$

Conjecture 4.10. *Let P and Q be two interior points of the $\triangle ABC$. Then*

$$\frac{R_1^2}{t_1} + \frac{R_2^2}{t_2} + \frac{R_3^2}{t_3} \geq 3(w_1 + w_2 + w_3 + r). \tag{4.13}$$

Remark 4. If Conjectures 4.7 and 4.8 are proved, then we can prove that Conjectures 4.9 and 4.10 are valid for the acute triangle ABC.

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