ON SOME INEQUALITIES FOR p-NORMS

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Abstract: In this paper we establish several new inequalities including p-norms for functions whose absolute values aroused to the p-th power are convex functions.

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1. Introduction

Integral inequalities have become a major tool in the analysis of integral equations, so it is not surprising that many of them appear in the literature (see for example [2], [5], [3] and [1]).

One of the most important inequalities in analysis is the integral Hölder's inequality which is stated as follows (for this variant see [3, p. 106]).

Theorem A. *Let* $p, q \in \mathbb{R} \setminus \{0\}$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and let* $f, g : [a, b] \to \mathbb{R}$, $a < b$, *be such that* $|f(x)|^p$ *and* $|g(x)|^q$ *are integrable on* $[a, b]$. *If* $p, q > 0$, *then*

$$\int_a^b |f(x) g(x)| \, dx \le \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}}. \tag{1.1}$$

If $p < 0$ *and additionally* $f([a, b]) \subseteq \mathbb{R} \setminus \{0\}$, *or* $q < 0$ *and* $g([a, b]) \subseteq \mathbb{R} \setminus \{0\}$, *then the inequality in* (1.1) *is reversed.*

The Hermite-Hadamard inequalities for convex functions is also well known. This double inequality is stated as follows (see for example [3, p. 10]): Let f be a convex function on $[a, b] \subset \mathbb{R}$, where $a \neq b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) dx \le \frac{f(a) + f(b)}{2}. \tag{1.2}$$

To prove our main result we need comparison inequalities between the power

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means defined by

$$M_n^{[r]}(\boldsymbol{x};\boldsymbol{p}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \neq -\infty, 0, \infty; \\ \left(\prod_{i=1}^n x_i^{pi}\right)^{\frac{1}{P_n}}, & r = 0; \\ \min(x_1, \dots, x_n), & r = -\infty; \\ \max(x_1, \dots, x_n), & r = \infty, \end{cases}$$

where $\boldsymbol{x}$, $\boldsymbol{p}$ are positive n-tuples and $P_n = \sum_{i=1}^n p_i$. It is well known that for such means the following inequality holds:

$$M_n^{[r]}(\boldsymbol{x};\boldsymbol{p}) \leq M_n^{[s]}(\boldsymbol{x};\boldsymbol{p}) \tag{1.3}$$

whenever $r < s$ (see for example [3, p. 15]).

In this paper we also use the following result (see [5, p. 152]):

Theorem B. *Let* $\boldsymbol{\xi} \in [a,b]^n$, $0 < a < b$, *and* $\boldsymbol{p} \in [0,\infty)^n$ *be two* n*-tuples such that*

$$\sum_{i=1}^n p_i \xi_i \in [a,b], \qquad \sum_{i=1}^n p_i \xi_i \geq \xi_j, \quad j = 1, 2, \dots, n.$$

If $f : [a,b] \to \mathbb{R}$ *is such that the function* $f(x)/x$ *is decreasing, then*

$$f\left(\sum_{i=1}^n p_i \xi_i\right) \leq \sum_{i=1}^n p_i f(\xi_i). \tag{1.4}$$

If $f(x)/x$ *is increasing, then the inequality in (1.4) is reversed.*

Our goal is to establish several new inequalities for functions whose absolute values raised to some real powers are convex functions.

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2. Results

In the literature, the following definition is well known.

Let $f:[a,b]\to\mathbb{R}$ and $p\in\mathbb{R}^{+}$. The p-norm of the function f on $[a,b]$ is defined by

$$\|f\|_p=\begin{cases}\left(\int_a^b|f(x)|^p\,dx\right)^{\frac{1}{p}}, & 0<p<\infty;\\ \sup|f(x)|, & p=\infty,\end{cases}$$

and $L^p([a,b])$ is the set of all functions $f:[a,b]\to\mathbb{R}$ such that $\|f\|_p<\infty$.

Observe that if $|f|^p$ is convex (or concave) on $[a,b]$ it is also integrable on $[a,b]$, hence $0\le\|f\|_p<\infty$, that is, f belongs to $L^p([a,b])$.

Although p-norms are not defined for $p<0$, for the sake of the simplicity we will use the same notation $\|f\|_p$ when $p\in\mathbb{R}\setminus\{0\}$.

In order to prove our results we need the following two lemmas.

Lemma 2.1. *Let $\boldsymbol{x}$ and $\boldsymbol{p}$ be two n-tuples such that*

$$x_i>0,\ p_i\ge1,\quad i=1,2,\dots,n. \tag{2.1}$$

If $r<s<0$ or $0<r<s$, then

$$\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{1}{s}}\le\left(\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, \tag{2.2}$$

and if $r<0<s$, then

$$\left(\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}\le\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{1}{s}}.$$

If the n-tuple $\boldsymbol{x}$ is only nonnegative, then (2.2) holds whenever $0<r<s$.

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Proof. Suppose that $\boldsymbol{x}$ and $\boldsymbol{p}$ are such that the inequalities in (2.1) hold. It can be easily seen that in this case for any $q \in \mathbb{R}$

$$\sum_{i=1}^{n} p_i x_i^q \geq x_j^q > 0, \quad j = 1, 2, \ldots, n.$$

To prove the lemma we must consider three cases: (i) $r < s < 0$, (ii) $0 < r < s$ and (iii) $r < 0 < s$. In case (i) we define the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(x) = x^{\frac{s}{r}}$. Since in this case we have $(s - r)/r < 0$, the function

$$f(x)/x = x^{\frac{s}{r}-1} = x^{\frac{s-r}{r}}$$

is decreasing. Applying Theorem B on f, $\boldsymbol{\xi} = (x_1^r, \ldots, x_n^r)$ and $\boldsymbol{p}$ we obtain

$$\left(\sum_{i=1}^{n} p_i x_i^r\right)^{\frac{s}{r}} \leq \sum_{i=1}^{n} p_i (x_i^r)^{\frac{s}{r}} = \sum_{i=1}^{n} p_i x_i^s,$$

i.e.,

$$\left(\sum_{i=1}^{n} p_i x_i^r\right)^{\frac{1}{r}} \geq \left(\sum_{i=1}^{n} p_i x_i^s\right)^{\frac{1}{s}}$$

since s is negative.

In case (ii) for the same f as in (i) we have $(s - r)/r > 0$, so similarly as before from Theorem B we obtain

$$\left(\sum_{i=1}^{n} p_i x_i^r\right)^{\frac{s}{r}} \geq \sum_{i=1}^{n} p_i (x_i^r)^{\frac{s}{r}} = \sum_{i=1}^{n} p_i x_i^s,$$

and since s is positive, (2.2) immediately follows.

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And in the end, in case (iii) we have $(s-r)/r<0$, so using again Theorem B we obtain (2.2) reversed. □

Remark 1. In this paper we will use Lemma 2.1 only in a special case when all weights are equal to 1. Then for $r<s<0$ or $0<r<s$, (2.2) becomes

$$\left(\sum_{i=1}^{n} x_i^s\right)^{\frac{1}{s}} \le \left(\sum_{i=1}^{n} x_i^r\right)^{\frac{1}{r}} \tag{2.3}$$

and for $r<0<s$,

$$\left(\sum_{i=1}^{n} x_i^s\right)^{\frac{1}{s}} \ge \left(\sum_{i=1}^{n} x_i^r\right)^{\frac{1}{r}}.$$

In the rest of the paper we denote

$$C_p=\begin{cases} 2^{-\frac{1}{p}}, & p\le -1 \text{ or } p\ge 1; \\ 2, & -1<p<0; \\ 2^{-1}, & 0<p<1; \end{cases} \qquad \widetilde{C}_p=\begin{cases} 2, & p\le -1; \\ 2^{-\frac{1}{p}}, & -1<p<1,\ p\ne 0; \\ 2^{-1}, & p\ge 1. \end{cases}.$$

Lemma 2.2. *Let* $f:[a,b]\to\mathbb{R}$, $a<b$. *If* $|f|^p$ *is convex on* $[a,b]$ *for some* $p>0$, *then*

$$\left|f\left(\frac{a+b}{2}\right)\right| \le (b-a)^{-\frac{1}{p}}\|f\|_p \le \left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le C_p\left(|f(a)|+|f(b)|\right),$$

and if $|f|^p$ *is concave on* $[a,b]$, *then*

$$\widetilde{C}_p\left(|f(a)|+|f(b)|\right) \le \left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le (b-a)^{-\frac{1}{p}}\|f\|_p \le \left|f\left(\frac{a+b}{2}\right)\right|.$$

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Proof. Suppose first that $|f|^p$ is convex on $[a,b]$ for some $p>0$. We have

$$\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}} \left(\frac{1}{b-a}\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

From (1.2) we obtain

$$\left|f\left(\frac{a+b}{2}\right)\right|^p \le \frac{1}{b-a}\int_a^b |f(x)|^p\,dx \le \frac{|f(a)|^p+|f(b)|^p}{2}, \tag{2.4}$$

hence

$$\left|f\left(\frac{a+b}{2}\right)\right| \le (b-a)^{-\frac{1}{p}}\|f\|_p \le \left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}}.$$

Now we must consider two cases. If $p\ge 1$ we can use (2.3) to obtain

$$(|f(a)|^p+|f(b)|^p)^{\frac{1}{p}} \le |f(a)|+|f(b)|,$$

hence

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le C_p\left(|f(a)|+|f(b)|\right), \tag{2.5}$$

where $C_p = 2^{-\frac{1}{p}}$.

In the other case, when $0<p<1$, from (1.3) we have

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}} \le \frac{|f(a)|+|f(b)|}{2},$$

so again we obtain (2.5), where $C_p = 2^{-1}$. This completes the proof for $|f|^p$ convex.

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Suppose now that $|f|^p$ is concave on $[a,b]$ for some $p>0$. In that case $-|f|^p$ is convex on $[a,b]$, hence (1.2) implies

$$\frac{|f(a)|^p+|f(b)|^p}{2}\le\frac{1}{b-a}\int_a^b|f(x)|^p\,dx\le\left|f\left(\frac{a+b}{2}\right)\right|^p.$$

If $p\ge 1$ from (1.3) we obtain

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}}\ge\frac{|f(a)|+|f(b)|}{2},$$

hence

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}}\ge\widetilde{C}_p\left(|f(a)|+|f(b)|\right),$$

where $\widetilde{C}_p=2^{-1}$.

In the other case, when $0<p<1$, from (2.3) we have

$$\left(|f(a)|^p+|f(b)|^p\right)^{\frac{1}{p}}\ge|f(a)|+|f(b)|,$$

hence

$$\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}}\ge\widetilde{C}_p\left(|f(a)|+|f(b)|\right),$$

where $\widetilde{C}_p=2^{-\frac{1}{p}}$. This completes the proof. □

Lemma 2.3. *Let* $f:[a,b]\to\mathbb{R}\setminus\{0\}$, $a<b$. *If* $|f|^p$ *is convex on* $[a,b]$ *for some* $p<0$, *then*

$$C_p\frac{|f(a)\,f(b)|}{|f(a)|+|f(b)|}\le\left(\frac{|f(a)|^p+|f(b)|^p}{2}\right)^{\frac{1}{p}}\le(b-a)^{-\frac{1}{p}}\|f\|_p\le\left|f\left(\frac{a+b}{2}\right)\right|$$

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and if $|f|^p$ *is concave on* $[a,b]$ *, then*

$$\left|f\left(\frac{a+b}{2}\right)\right| \le (b-a)^{-\frac{1}{p}} \|f\|_p \le \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}} \le \widetilde{C}_p \frac{|f(a)\,f(b)|}{|f(a)|+|f(b)|}.$$

Proof. Suppose that $|f|^p$ is convex on $[a,b]$ for some $p<0$. From (2.4), using the fact that $p<0$, we obtain

$$\left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}} \le (b-a)^{-\frac{1}{p}} \|f\|_p \le \left|f\left(\frac{a+b}{2}\right)\right|.$$

Again we consider two cases. If $-1<p<0$, then from (1.3) we have

$$\left(\frac{|f(a)|^{-1} + |f(b)|^{-1}}{2}\right)^{-1} \le \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}},$$

hence

$$C_p \frac{|f(a)\,f(b)|}{|f(a)|+|f(b)|} \le \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}},$$

where $C_p = 2$.

In the other case, when $p \le -1$, from (2.3) we have

$$\left(|f(a)|^{-1} + |f(b)|^{-1}\right)^{-1} \le \left(|f(a)|^p + |f(b)|^p\right)^{\frac{1}{p}},$$

hence

$$C_p \frac{|f(a)\,f(b)|}{|f(a)|+|f(b)|} \le \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}},$$

where $C_p = 2^{-\frac{1}{p}}$.

In the other case, when $|f|^p$ is concave on $[a,b]$ for some $p<0$, the proof is similar. □

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Theorem 2.4. *Let $p,q>0$ and let $f,g:[a,b]\to\mathbb{R}$, $a<b$, be such that*

$$m\left(|g(a)|+|g(b)|\right)\le|f(a)|+|f(b)|\le M\left(|g(a)|+|g(b)|\right) \tag{2.6}$$

for some $0<m\le M$.
If $|f|^p$ and $|g|^q$ are convex on $[a,b]$, then

$$\|f\|_p+\|g\|_q\le\left[\frac{M}{M+1}C_p(b-a)^{\frac{1}{p}}+\frac{1}{m+1}C_q(b-a)^{\frac{1}{q}}\right]K(f,g), \tag{2.7}$$

where

$$K(f,g)=|f(a)|+|f(b)|+|g(a)|+|g(b)|.$$

If $|f|^p$ and $|g|^q$ are concave on $[a,b]$, then

$$\|f\|_p+\|g\|_q\ge\left[\frac{m}{m+1}\widetilde{C}_p(b-a)^{\frac{1}{p}}+\frac{1}{M+1}\widetilde{C}_q(b-a)^{\frac{1}{q}}\right]K(f,g). \tag{2.8}$$

Proof. Suppose that $|f|^p$ and $|g|^q$ are convex on $[a,b]$ for some fixed $p,q>0$. From Lemma 2.2 we have that

$$\begin{aligned}\|f\|_p+\|g\|_q&\\ &\le\left(\frac{b-a}{2}\right)^{\frac{1}{p}}\left(|f(a)|^p+|f(b)|^p\right)^{\frac{1}{p}}+\left(\frac{b-a}{2}\right)^{\frac{1}{q}}\left(|g(a)|^q+|g(b)|^q\right)^{\frac{1}{q}}\\ &\le C_p(b-a)^{\frac{1}{p}}\left(|f(a)|+|f(b)|\right)+C_q(b-a)^{\frac{1}{q}}\left(|g(a)|+|g(b)|\right).\end{aligned} \tag{2.9}$$

Using (2.6) we can write

$$|f(a)|+|f(b)|\le M\left(|f(a)|+|f(b)|+|g(a)|+|g(b)|\right)-M\left(|f(a)|+|f(b)|\right),$$

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i.e.,

$$(2.10)\qquad |f(a)|+|f(b)| \le \frac{M}{M+1}\left(|f(a)|+|f(b)|+|g(a)|+|g(b)|\right) = \frac{M}{M+1}K(f,g),$$

and analogously

$$(2.11)\qquad |g(a)|+|g(b)| \le \frac{1}{m+1}K(f,g).$$

Combining (2.10) and (2.11) with (2.9) we obtain (2.7).

Suppose now that $|f|^p$ and $|g|^q$ are concave on $[a,b]$ for some fixed $p,q>0$. From Lemma 2.2 we have that

$$\|f\|_p+\|g\|_q \ge \widetilde{C}_p(b-a)^{\frac{1}{p}}\left(|f(a)|+|f(b)|\right)+\widetilde{C}_q(b-a)^{\frac{1}{q}}\left(|g(a)|+|g(b)|\right).$$

Using again (2.6) we can write

$$|f(a)|+|f(b)| \ge m\left(|f(a)|+|f(b)|+|g(a)|+|g(b)|\right)-m\left(|f(a)|+|f(b)|\right),$$

i.e.,

$$|f(a)|+|f(b)| \ge \frac{m}{m+1}K(f,g),$$

and analogously

$$|g(a)|+|g(b)| \ge \frac{1}{M+1}K(f,g),$$

from which (2.8) easily follows. □

Remark 2. A similar type of condition as in (2.6) was used in [1, Theorem 1.1] where a variant of the reversed Minkowski's integral inequality for $p>1$ was proved.

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Theorem 2.5. *Let $p, q < 0$ and let $f, g : [a, b] \to \mathbb{R} \setminus \{0\}$, $a < b$, be such that*

$$m \frac{|g(a) g(b)|}{|g(a)| + |g(b)|} \leq \frac{|f(a) f(b)|}{|f(a)| + |f(b)|} \leq M \frac{|g(a) g(b)|}{|g(a)| + |g(b)|}$$

for some $0 < m \leq M$.
If $|f|^p$ and $|g|^q$ are concave on $[a, b]$, then

$$\|f\|_p + \|g\|_q \leq \left[\frac{M}{M+1} \widetilde{C}_p (b-a)^{\frac{1}{p}} + \frac{1}{m+1} \widetilde{C}_q (b-a)^{\frac{1}{q}} \right] H(f, g),$$

where

$$H(f, g) = \frac{|f(a) f(b)|}{|f(a)| + |f(b)|} + \frac{|g(a) g(b)|}{|g(a)| + |g(b)|}.$$

If $|f|^p$ and $|g|^q$ are convex on $[a, b]$, then

$$\|f\|_p + \|g\|_q \geq \left[\frac{m}{m+1} C_p (b-a)^{\frac{1}{p}} + \frac{1}{M+1} C_q (b-a)^{\frac{1}{q}} \right] H(f, g).$$

Proof. Similar to that of Theorem 2.4. □

Theorem 2.6. *Let $f, g : [a, b] \to \mathbb{R}$, $a < b$, be such that $|f|^p$ and $|g|^q$ are convex on $[a, b]$ for some fixed $p, q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} \left| \int_a^b f(x) g(x) \, dx \right| &\leq \frac{b-a}{2} \left(|f(a)|^p + |f(b)|^p \right)^{\frac{1}{p}} \left(|g(a)|^q + |g(b)|^q \right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2} \left[M(f, g) + N(f, g) \right], \end{aligned}$$

where

$$M(f, g) = |f(a)| \, |g(a)| + |f(b)| \, |g(b)|, \qquad N(f, g) = |f(a)| \, |g(b)| + |f(b)| \, |g(a)|.$$

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Proof. First note that since $|f|^p$ and $|g|^q$ are convex on $[a,b]$ we have $f \in L^p([a,b])$ and $g \in L^q([a,b])$, and since $\frac{1}{p}+\frac{1}{q}=1$ we know that $fg \in L^1([a,b])$, that is, fg is integrable on $[a,b]$. Using Hölder's integral inequality (1.1) we obtain

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leq \int_a^b |f(x)\,g(x)|\,dx \leq \|f\|_p \|g\|_q .$$

From Lemma 2.3 we have that

$$\|f\|_p \leq \left(\frac{b-a}{2}\right)^{\frac{1}{p}} \left(|f(a)|^p + |f(b)|^p\right)^{\frac{1}{p}} \leq \left(\frac{b-a}{2}\right)^{\frac{1}{p}} \left(|f(a)| + |f(b)|\right)$$

and

$$\|g\|_q \leq \left(\frac{b-a}{2}\right)^{\frac{1}{q}} \left(|g(a)|^q + |g(b)|^q\right)^{\frac{1}{q}} \leq \left(\frac{b-a}{2}\right)^{\frac{1}{q}} \left(|g(a)| + |g(b)|\right),$$

hence

$$\begin{aligned}\left|\int_a^b f(x)\,g(x)\,dx\right| &\leq \frac{b-a}{2}\left(|f(a)|^p + |f(b)|^p\right)^{\frac{1}{p}}\left(|g(a)|^q + |g(b)|^q\right)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2}\left(|f(a)| + |f(b)|\right)\left(|g(a)| + |g(b)|\right) \\ &= \frac{b-a}{2}\left[M(f,g) + N(f,g)\right].\end{aligned}$$

□

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