INEQUALITIES FOR THE GAMMA FUNCTION

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Abstract: For $x > 1$, the inequalities

$$\frac{x^{x-\gamma}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-1/2}}{e^{x-1}}$$

hold, and the constants γ and $1/2$ are the best possible, where $\gamma = 0.577215\ldots$ is the Euler-Mascheroni constant. For $0 < x < 1$, the left-hand inequality also holds, but the right-hand inequality is reversed. This improves the result given by G. D. Anderson and S. -L. Qiu (1997).

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The classical gamma function is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \tag{1}$$

which is one of the most important special functions and has many extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. The history and the development of this function are described in detail in [4]. The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed as

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\, dt, \tag{2}$$

$$\psi^{(m)}(x) = (-1)^{m+1} \int_0^\infty \frac{t^m}{1 - e^{-t}} e^{-xt}\, dt \tag{3}$$

for $x > 0$ and $m = 1, 2, \ldots$, where $\gamma = 0.577215\ldots$ is the Euler-Mascheroni constant.

In 1997, G. D. Anderson and S. -L. Qiu [3] presented the following upper and lower bounds for $\Gamma(x)$:

$$x^{(1-\gamma)-1} < \Gamma(x) < x^{x-1} \quad (x > 1). \tag{4}$$

Actually, the authors proved more. They established that the function $F(x) = \frac{\ln \Gamma(x+1)}{x \ln x}$ is strictly increasing on $(1, \infty)$ with $\lim_{x \to 1} F(x) = 1-\gamma$ and $\lim_{x \to 1} F(x) = 1$, which leads to (4).

In 1999, H. Alzer [2] showed that if $x \in (1, \infty)$, then

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma} \tag{5}$$

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is valid with the best possible constants $\alpha = (\pi^2/6 - \gamma)/2$ and $\beta = 1$. This improves the bounds given in (4). Moreover, the author showed that if $x \in (0, 1)$, then (5) holds with the best possible constants $\alpha = 1 - \gamma$ and $\beta = (\pi^2/6 - \gamma)/2$.

Here we provide an improvement of (4) as follows.

Theorem 1. *For $x > 1$, the inequalities*

$$\frac{x^{x-\gamma}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-1/2}}{e^{x-1}} \tag{6}$$

hold, and the constants γ and $1/2$ are the best possible. For $0 < x < 1$, the left-hand inequality of (6) also holds, but the right-hand inequality of (6) is reversed.

Proof. Define for $x > 0$,

$$f(x) = \frac{e^{x-1}\Gamma(x)}{x^{x-\gamma}}.$$

Differentation yields

$$\frac{xf'(x)}{f(x)} = x(\psi(x) - \ln x) + \gamma \triangleq g(x).$$

Using the representations [5, p. 153]

$$\psi(x) = -\frac{1}{2x} + \ln x - \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) e^{-xt}\,\mathrm{d}t, \tag{7}$$

$$\frac{1}{x} = \int_0^\infty e^{-xt}\,\mathrm{d}t \quad (x > 0), \tag{8}$$

and (3), we imply

$$\frac{g'(x)}{x} = \psi'(x) - \frac{1}{x} - \frac{1}{x}(\ln x - \psi(x)) = \int_0^\infty t\delta(t)e^{-xt}\,\mathrm{d}t - \int_0^\infty e^{-xt}\,\mathrm{d}t \int_0^\infty \delta(t)e^{-xt}\,\mathrm{d}t,$$

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where

$$\delta(t) = \frac{1}{1-e^{-t}} - \frac{1}{t}$$

is strictly increasing on $(0,\infty)$ with $\lim_{x\to 0}\delta(t) = \frac{1}{2}$ and $\lim_{x\to\infty}\delta(t) = 1$.

By using the convolution theorem for Laplace transforms, we have

$$\begin{aligned}\frac{g'(x)}{x} &= \int_0^\infty t\delta(t)e^{-xt}\,\mathrm{d}t - \int_0^\infty \left[\int_0^t \delta(s)\,\mathrm{d}s\right] e^{-xt}\,\mathrm{d}t \\ &= \int_0^\infty \left[\int_0^t (\delta(t)-\delta(s))\,\mathrm{d}s\right] e^{-xt}\,\mathrm{d}t > 0,\end{aligned}$$

and therefore, the function g is strictly increasing on $(0,\infty)$, and then, $g(x) < g(1) = 0$ and $f'(x) < 0$ for $0 < x < 1$, and $g(x) > g(1) = 0$ and $f'(x) > 0$ for $x > 1$. Thus, the function f is strictly decreasing on $(0,1)$, and is strictly increasing on $(1,\infty)$, and therefore, the function f takes its minimum $f(1) = 1$ at $x = 1$. Hence, the left-hand inequality of (6) is valid for $x > 0$ and $x \neq 1$.

Define for $x > 0$,

$$h(x) = \frac{e^{x-1}\Gamma(x)}{x^{x-1/2}}.$$

we have by (7),

$$\frac{h'(x)}{h(x)} = \int_0^\infty \left(\frac{1}{2} - \delta(t)\right) e^{-xt}\,\mathrm{d}t < 0.$$

This means that the function h is strictly decreasing on $(0,\infty)$, and then, $h(x) < h(1) = 1$ for $x > 1$, and $h(x) > h(1) = 1$ for $0 < x < 1$. Thus, the right-hand inequality of (6) is valid for $x > 1$, reversed for $0 < x < 1$.

Write (6) as

$$\frac{1}{2} < \frac{1 - x + x\ln x - \ln\Gamma(x)}{\ln x} < \gamma.$$

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From the asymptotic expansion [1, p. 257]

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + O(x^{-1}),$$

we conclude that

$$\lim_{x\to\infty} \frac{1 - x + x\ln x - \ln\Gamma(x)}{\ln x} = \frac{1}{2}.$$

Easy computation reveals

$$\lim_{x\to 0} \frac{1 - x + x\ln x - \ln\Gamma(x)}{\ln x} = \gamma.$$

Hence, for $x > 1$, the inequalities (6) hold, and the constants γ and $1/2$ are the best possible. The proof is complete. ∎

We remark that the upper and lower bounds of (5) and (6) cannot be compared to each other.

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