ON A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS

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Abstract: The class of univalent harmonic functions on the unit disc satisfying the condition $\sum_{k=2}^{\infty}(k^m - \alpha k^n)(|a_k| + |b_k|) \le (1-\alpha)(1-|b_1|)$ is given. Sharp coefficient relations and distortion theorems are given for these functions. In this paper we find that many results of Özturk and Yalcin [5] are incorrect. Some of the results of this paper correct the theorems and examples of [5]. Further, sharp coefficient relations and distortion theorems are given. Results concerning the convolutions of functions satisfying the above inequalities with univalent, harmonic and convex functions in the unit disc and harmonic functions having positive real part are obtained.

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1. Introduction

Let U denote the open unit disc and S_H denote the class of all complex valued, harmonic, orientation-preserving, univalent functions f in U normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_H$ can be expressed as $f = h + \bar{g}$ where h and g belong to the linear space $H(U)$ of all analytic functions on U.

Firstly, Clunie and Sheil-Small [3] studied S_H together with some geometric subclasses of S_H. They proved that although S_H is not compact, it is normal with respect to the topology of uniform convergence on compact subsets of U. Meanwhile, the subclass S_H^0 of S_H consisting of the functions having the property that $f_{\bar{z}}(0) = 0$ is compact.

In this article we concentrate on a subclass of univalent harmonic mappings defined in Section 2. The technique employed by us is entirely different to that of Özturk and Yalcin [5].

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2. The Class $HS(m,n;\alpha)$

Let $U_r = \{z : |z| < r, 0 < r \leq 1\}$ and $U_1 = U$. A harmonic, complex-valued, orientation-preserving, univalent mapping f defined on U can be written as:

$$f(z) = h(z) + \overline{g(z)}, \tag{2.1}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \tag{2.2}$$

are analytic in U.

Denote by $HS(m,n,\alpha)$ the class of all functions of the form (2.1) that satisfy the condition:

$$\sum_{k=2}^{\infty} (k^m - \alpha k^n)(|a_k| + |b_k|) \leq (1-\alpha)(1-|b_1|), \tag{2.3}$$

where $m \in N, n \in N_0,\ m > n,\ 0 \leq \alpha < 1$ and $0 \leq |b_1| < 1$.

The class $HS(m,n,\alpha)$ with $b_1 = 0$ will be denoted by $HS^o(m,n,\alpha)$.

We note that by specializing the parameter we obtain the following subclasses which have been studied by various authors.

1. The classes $HS(1,0,\alpha) \equiv HS(\alpha)$ and $HS(2,1,\alpha) \equiv HC(\alpha)$ were studied by Özturk and Yalcin [5].

2. The classes $HS(1,0,0) \equiv HS$ and $HS(2,1,0) \equiv HC$ were studied by Avci and Zlotkiewicz [2]. If h, g, H, G, are of the form (2.2) and if $f(z) = h(z) +$

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$\overline{g(z)}$ and $F(z) = H(z) + \overline{G(z)}$, then the convolution of f and F is defined to be the function:

$$(f * F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k},$$

while the integral convolution is defined by:

$$(f \diamondsuit F)(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k + \overline{\sum_{k=1}^{\infty} \frac{b_k B_k}{k} z^k}.$$

The δ – neighborhood of f is the set:

$$N_\delta(f) = \left\{ F : \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + |b_1 - B_1| \leq \delta \right\}$$

(see [1], [6]). In this case, let us define the generalized δ-neighborhood of f to be the set:

$$N(f) = \left\{ F : \sum_{k=2}^{\infty} (k - \alpha)(|a_k - A_k| + |b_k - B_k|) + (1 - \alpha)|b_1 - B_1| \leq (1 - \alpha)\delta \right\}.$$

In the present paper we find that many results of Özturk and Yalcin [5, Theorem 3.6, 3.8] are incorrect, and we correct them. It should be noted that the examples supporting the sharpness of [5, Theorem 3.6, 3.8] are not correct and we remedy this problem. Finally, we improve Theorem 3.15 of Özturk and Yalcin [5].

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3. Main Results

First, let us give the interrelation between the classes $HS(m,n,\alpha_1)$ and $HS(m,n,\alpha_2)$ where $0 \le \alpha_1 \le \alpha_2 < 1$.

Theorem 3.1. *$HS(m,n,\alpha_2) \subseteq HS(m,n,\alpha_1)$ where $0 \le \alpha_1 \le \alpha_2 < 1$. Consequently $HS^o(m,n,\alpha_2) \subseteq HS^o(m,n,\alpha_1)$. In particular $HS(m,n,\alpha) \subseteq HS(m,n,0)$ and$HS^o(m,n,\alpha) \subseteq HS^o(m,n,0)$.*

Proof. Let $f \in HS(m,n,\alpha_2)$. Thus we have:

$$\sum_{k=2}^{\infty} \frac{k^m - \alpha_2 k^n}{1-\alpha_2}(|a_k| + |b_k|) \le (1 - |b_1|). \tag{3.1}$$

Now, using (3.1),

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{k^m - \alpha_1 k^n}{1-\alpha_1}(|a_k| + |b_k|) &\le \sum_{k=2}^{\infty} \frac{k^m - \alpha_2 k^n}{1-\alpha_2}(|a_k| + |b_k|)\\ &\le (1 - |b_1|).\end{aligned}$$

Thus $f \in HS(m,n,\alpha_1)$.

This completes the proof of Theorem 3.1. □

Theorem 3.2. *$HS(m,n,\alpha) \subseteq HS(\alpha)$, $\forall m \in N$, $\forall n \in N_0$, $HS(m,n,\alpha) \subseteq HC(\alpha)$, $\forall m \in N - \{1\}$, $\forall n \in N_0$,where $0 \le \alpha < 1$.*

Proof. Let $f \in HS(m,n,\alpha)$. Then

$$\sum_{k=2}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha}(|a_k| + |b_k|) \le (1 - |b_1|). \tag{3.2}$$

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Now using (3.2),

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{k-\alpha}{1-\alpha}(|a_k|+|b_k|) &\le \sum_{k=2}^{\infty}\frac{k^n(k-\alpha)}{1-\alpha}(|a_k|+|b_k|)\\ &= \sum_{k=2}^{\infty}\frac{k^{n+1}-\alpha k^n}{1-\alpha}(|a_k|+|b_k|)\\ &\le \sum_{k=2}^{\infty}\frac{k^{m}-\alpha k^n}{1-\alpha}(|a_k|+|b_k|) \qquad (\text{since}\, m>n)\\ &\le (1-|b_1|).\end{aligned}$$

Thus $f\in HS(\alpha)$ and we have $HS(m,n,\alpha)\subseteq HS(\alpha)$.

We have to show that $HS(m,n,\alpha)\subseteq HC(\alpha)$. By (3.2),

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{k(k-\alpha)}{1-\alpha}(|a_k|+|b_k|) &\le \sum_{k=2}^{\infty}\frac{k^{m}-\alpha k^n}{1-\alpha}(|a_k|+|b_k|) \qquad (\text{since},\ m\ge 2)\\ &\le (1-|b_1|).\end{aligned}$$

Thus $f\in HC(\alpha)$. So we have $HS(m,n,\alpha)\subseteq HC(\alpha)$. □

Theorem 3.3. *The class $HS(m,n,\alpha)$ consists of univalent sense preserving harmonic mappings.*

Proof. If $z_1\ne z_2$ then:

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|\\ &= 1-\left|\frac{\sum_{k=1}^{\infty}b_k(z_1^k-z_2^k)}{z_1-z_2+\sum_{k=2}^{\infty}a_k(z_1^k-z_2^k)}\right|\end{aligned}$$

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$$> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \geq 1 - \frac{\sum_{k=1}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha}|b_k|}{1 - \sum_{k=2}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha}|a_k|} \geq 0,$$

which proves univalence.

Note that f is sense preserving in U because

$$\begin{aligned} \left|h^1(z)\right| &\geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \\ &> 1 - \sum_{k=2}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha}|a_k| \\ &\geq \sum_{k=1}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha}|b_k| \\ &> \sum_{k=1}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha}|b_k||z|^{k-1} \\ &\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g^1(z)|. \end{aligned}$$

□

Theorem 3.4. *If $f \in HS(m,n,\alpha)$ then*

$$|f(z)| \leq |z|(1 + |b_1|) + \frac{1-\alpha}{2^m - \alpha 2^n}(1 - |b_1|)|z|^2$$

and

$$|f(z)| \geq (1 - |b_1|)\left(|z| - \frac{1-\alpha}{2^m - \alpha 2^n}|z|^2\right).$$

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Equalities are attained by the functions:

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{1-\alpha}{2^m - \alpha 2^n}(1-|b_1|)z^2 \tag{3.3}$$

and

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{1-\alpha}{2^m - \alpha 2^n}(1-|b_1|)\bar{z}^2 \tag{3.4}$$

for properly chosen real θ.

Proof. We have

$$\begin{aligned}|f(z)| &\le |z|(1+|b_1|) + |z|^2 \sum_{k=2}^{\infty}(|a_k|+|b_k|)\\ &\le |z|(1+|b_1|)|z|^2 \frac{1-\alpha}{2^m-\alpha 2^n}\sum_{k=2}^{\infty}\frac{k^m-\alpha k^n}{1-\alpha}(|a_k|+|b_k|)\\ &\le |z|(1+|b_1|) + |z|^2 \frac{1-\alpha}{2^m-\alpha 2^n}(1-|b_1|)\end{aligned}$$

and

$$\begin{aligned}|f(z)| &\ge (1-|b_1|)|z| - \sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^k \ge (1-|b_1|)|z| - |z|^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)\\ &\ge (1-|b_1|)|z| - |z|^2\frac{1-\alpha}{2^m-\alpha 2^n}\sum_{k=2}^{\infty}\frac{k^m-\alpha k^n}{1-\alpha}(|a_k|+|b_k|)\\ &\ge (1-|b_1|)|z| - |z|^2\frac{1-\alpha}{2^m-\alpha 2^n}(1-|b_1|)\\ &= (1-|b_1|)\left(|z| - |z|^2\frac{(1-\alpha)}{2^m-\alpha 2^n}\right).\end{aligned}$$

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It can be easily seen that the function $f(z)$ defined by (3.3) and (3.4) is extremal for Theorem 3.4.

Thus the class $HS(m,n,\alpha)$ is uniformly bounded, and hence it is normal by Montel's Theorem. □

Remark 1.

(i) For $m=1$, $n=0$, $HS(1,0,\alpha)=HS(\alpha)$. The above theorem reduces to:

$$|f(z)| \leq |z|(1+|b_1|)+\frac{1-\alpha}{2-\alpha}(1-|b_1|)|z|^2, \tag{3.5}$$

$$|f(z)| \geq (1-|b_1|)\left(|z|-\frac{1-\alpha}{2-\alpha}|z|^2\right). \tag{3.6}$$

This result is different from that of Özturk and Yalcin [5, Theorem 3.6]. Also, our result gives a better estimate than that of [5] because

$$\begin{aligned} |f(z)| &\leq |z|(1+|b_1|)+\frac{1-\alpha}{2-\alpha}(1-|b_1|)|z|^2 \\ &\leq |z|(1+|b_1|)+\frac{(1-\alpha^2)}{2}(1-|b_1|)|z|^2 \end{aligned}$$

and

$$|f(z)| \geq (1-|b_1|\left(|z|-\frac{1-\alpha}{2-\alpha}|z|^2\right) \geq (1-|b_1|)\left(|z|-\frac{(1-\alpha^2)}{2}|z|^2\right).$$

Although, Özturk and Yalcin [5] state that the result is sharp for the function

$$f_\theta(z)=z+|b_1|e^{i\theta}\bar{z}+\frac{(1-|b_1|)}{2}(1-\alpha^2)\bar{z}^2,$$

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it can be easily seen that the function $f_\theta(z)$ does not satisfy the coefficient condition for the class $HS(\alpha)$ defined by them. Hence, the function $f_\theta(z)$ does not belong in $HS(\alpha)$. Therefore the results of Özturk and Yalcin are incorrect. The correct results are mentioned in (3.5) and (3.6) and these results are sharp for functions in (3.3) and (3.4) with $m = 1$, $n = 0$.

(ii) For $m = 2$, $n = 1$, $HS(2, 1, \alpha) = HC(\alpha)$. Theorem 3.4 reduces to

$$|f(z)| \leq |z|(1 + |b_1|) + \frac{1-\alpha}{4-2\alpha}(1 - |b_1|)|z|^2,$$

and

$$|f(z)| \geq (1 - |b_1|)\left(|z| - \frac{1-\alpha}{4-2\alpha}|z|^2\right).$$

This result is different from the result of Özturk and Yalcin [5, Theorem 3.8], and it can be easily seen that our result gives a better estimate. Also, it can be easily verified that the sharp result for [5, Theorem 3.8] given by the function

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{3-\alpha-2\alpha^2}{2\alpha}\bar{z}^2$$

does not belong to $HC(\alpha)$. Hence the results of Özturk and Yalcin [5] are incorrect. The correct result is obtained by Theorem 3.4 by putting $m = 2$, $n = 1$.

Theorem 3.5. *The extreme points of $HS^o(m, n, \alpha)$ are functions of the form $z + a_k z^k$ or $z + \overline{b_l z^l}$ with*

$$|a_k| = \frac{1-\alpha}{k^m - \alpha k^n}, \quad |b_l| = \frac{1-\alpha}{l^m - \alpha l^n}, \qquad 0 \leq \alpha < 1.$$

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Proof. Suppose that

$$f(z) = z + \sum_{k=2}^{\infty} \left(a_k z^k + \overline{b_k z^k} \right)$$

is such that

$$\sum_{k=2}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha} (|a_k| + |b_k|) < 1, \qquad a_k > 0.$$

Then, if $\lambda > 0$ is small enough we can replace a_k by $a_k - \lambda$, $a_k + \lambda$ and we obtain two functions that satisfy the same condition, for which one obtains $f(z) = \frac{1}{2}[f_1(z) + f_2(z)]$. Hence f is not a possible extreme point of $HS^o(m, n, \alpha)$.

Now let $f \in HS^o(m,\, n,\, \alpha)$ be such that

$$\sum_{k=2}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha} (|a_k| + |b_k|) = 1, \qquad a_k \neq 0,\ b_l \neq 0. \tag{3.7}$$

If $\lambda > 0$ is small enough and if μ, τ with $|\mu| = |\tau| = 1$ are properly chosen complex numbers, then leaving all but $a_k,\ b_l$ coefficients of $f(z)$ unchanged and replacing $a_k,\ b_l$ by

$$a_k + \lambda \frac{1-\alpha}{k^m - \alpha k^n} \mu, \qquad b_l - \lambda \frac{1-\alpha}{l^m - \alpha l^n} \tau,$$

$$a_k - \lambda \frac{1-\alpha}{k^m - \alpha k^n} \mu, \qquad b_l + \lambda \frac{1-\alpha}{l^m - \alpha l^n} \tau,$$

we obtain functions $f_1(z),\ f_2(z)$ that satisfy (3.2) such that $f(z) = \frac{1}{2}[f_1(z) + f_2(z)]$.

In this case f cannot be an extreme point. Thus for $|a_k| = \frac{1-\alpha}{k^m - \alpha k^n}$, $|b_l| = \frac{1-\alpha}{l^m - \alpha l^n}$, $f(z) = z + a_k z^k$ or $f(z) = z + \overline{b_l z^l}$ are extreme points of $HS^o(m, n, \alpha)$. $\square$

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Remark 2.

1. If $m = 1$, $n = 0$ the extreme points of the class $HS^o(\alpha)$ are obtained.
2. If $m = 2$, $n = 1$ the extreme points of the class $HC^o(\alpha)$ are obtained.

Let K_H^o denote the class of harmonic univalent functions of the form (2.1) with $b_1 = 0$ that map U onto convex domains. It is known [3, Theorem 5.10] that the sharp inequalities $|A_k| \leq \frac{k+1}{2}$, $|B_k| \leq \frac{k-1}{2}$ are true. These results will be used in the next theorem.

Theorem 3.6. *Suppose that*

$$F(z) = z + \sum_{k=2}^{\infty} \left(A_k z^k + \overline{B_k z^k} \right)$$

belongs to K_H^o*. If* $f \in HS^o(m, n, \alpha)$ *then* $f * F \in HS^o(m-1, n-1; \alpha)$ *if* $n \geq 1$ *and* $f \diamondsuit F \in HS^o(m, n; \alpha)$*.*

Proof. Since $f \in HS^o(m, n; \alpha)$, then

$$\sum_{k=2}^{\infty} (k^m - \alpha k^n)(|a_k| + |b_k|) \leq 1 - \alpha. \tag{3.8}$$

Using (3.8), we have

$$\begin{aligned}\sum_{k=2}^{\infty} (k^{m-1} - \alpha k^{n-1})(|a_k A_k| + |b_k B_k|) &= \sum_{k=2}^{\infty} (k^m - \alpha k^n) \left(|a_k| \left| \frac{A_k}{k} \right| + |b_k| \left| \frac{B_k}{k} \right| \right) \\ &\leq \sum_{k=2}^{\infty} (k^m - \alpha k^n)(|a_k| + |b_k|) \leq 1 - \alpha.\end{aligned}$$

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It follows that $f * F \in HS^o(m-1, n-1; \alpha)$. Next, again using (3.8),

$$\sum_{k=2}^{\infty}(k^m - \alpha k^n)\left(\left|\frac{a_k A_k}{k}\right| + \left|\frac{b_k B_k}{k}\right|\right)$$
$$\leq \sum_{k=2}^{\infty}(k^m - \alpha k^n)\left(|a_k|\left|\frac{A_k}{k}\right| + |b_k|\left|\frac{B_k}{k}\right|\right)$$
$$\leq \sum_{k=2}^{\infty}(k^m - \alpha k^n)(|a_k| + |b_k|) \leq 1 - \alpha.$$

Thus we have $f \diamondsuit F \in HS^o(m, n; \alpha)$. □

Let S denote the class of analytic univalent functions of the form $F(z) = z + \sum_{k=2}^{\infty} A_k z^k$. It is well known that the sharp inequality $|A_k| \leq k$ is true. It is needed in next theorem.

Theorem 3.7. *If* $f \in HS^o(m, n; \alpha)$ *and* $F \in S$ *then for* $| \in | \leq 1$, $f * (F + \in \bar{F}) \in HS^o(m-1, n-1; \alpha)$ *if* $n \geq 1$.

Proof. Since $f \in HS^o(m, n; \alpha)$, we have

$$\sum_{k=2}^{\infty}(k^m - \alpha k^n)(|a_k| + |b_k|) \leq 1 - \alpha. \tag{3.9}$$

Now, using (3.9)

$$\sum_{k=2}^{\infty}(k^{m-1} - \alpha k^{n-1})(|a_k A_k| + |b_k B_k|) \leq \sum_{k=2}^{\infty}(k^m - \alpha k^n)(|a_k| + |b_k|)$$
$$\leq 1 - \alpha.$$

It follows that $f * (F + \in \bar{F}) \in HS^o(m-1, n-1; \alpha)$ if $n \geq 1$. □

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Let P_H^o denote the class of functions F complex and harmonic in U, $f = h + \bar{g}$ such that $\operatorname{Re}\ f(z) > 0,\ \ z \in U$ and

$$H(z) = 1 + \sum_{k=1}^{\infty} A_k z^k, \qquad G(z) = \sum_{k=2}^{\infty} B_k z^k.$$

It is known [4, Theorem 3] that the sharp inequalities $|A_k| \le k+1,\ \ |B_k| \le k-1$ are true.

Theorem 3.8. *Suppose that*

$$F(z) = 1 + \sum_{k=1}^{\infty} \left(A_k z^k + \overline{B_k z^k} \right)$$

belong to P_H^o. *Then* $f \in HS^o(m, n; \alpha)$ *and for* $\frac{3}{2} \le |A_1| \le 2$, $\frac{1}{A_1} f * F \in HS^o(m-1, n-1, \alpha)$ *if* $n \ge 1$ *and* $\frac{1}{A_1} f \diamondsuit F \in HS^o(m, n; \alpha)$

Proof. Since $f \in HS^o(m, n; \alpha)$, then we have

$$\sum_{k=2}^{\infty} (k^m - \alpha k^n)(|a_k| + |b_k|) \le 1 - \alpha. \tag{3.10}$$

Now, using (3.10),

$$\begin{aligned}
\sum_{k=2}^{\infty} (k^{m-1} - \alpha k^{n-1}) &\left(\left| \frac{a_k A_k}{A_1} \right| + \left| \frac{b_k B_k}{A_1} \right| \right) \\
&\le \sum_{k=2}^{\infty} (k^m - \alpha k^n) \left(\frac{|a_k|}{|A_1|} \frac{k+1}{k} + \frac{|b_k|}{|A_1|} \frac{k-1}{k} \right) \\
&\le \sum_{k=2}^{\infty} (k^m - \alpha k^n)(|a_k| + |b_k|) \le 1 - \alpha.
\end{aligned}$$

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Thus $\frac{1}{A_1} f * F \in HS^o(m-1, n-1, \alpha)$ if $n \geq 1$. Similarly, we can show that $\frac{1}{A_1} f \diamondsuit F \in HS^o(m, n; \alpha)$. $\square$

Theorem 3.9. *Let*

$$f(z) = z + \overline{b_1 z} + \sum_{k=2}^{\infty} \left(a_k z^k + \overline{b_k z^k} \right)$$

be a member of $HS(m, n, \alpha)$. *If* $\delta \leq (\frac{2^n-1}{2^n})(1-|b_1|)$, *then* $N(f) \subset HS(\alpha)$, *provided that* $n \geq 1$.

Proof. Let $f \in HS(m, n; \alpha)$ and

$$F(z) = z + \overline{B_1 z} + \sum_{k=2}^{\infty} \left(A_k z^k + \overline{B_k z^k} \right)$$

belong to $N(f)$. We have

$$\begin{aligned}
(1-\alpha)|B_1| &+ \sum_{k=2}^{\infty} (k-\alpha)(|A_k| + |B_k|) \\
&\leq (1-\alpha)|B_1 - b_1| + (1-\alpha)|b_1| \\
&\quad + \sum_{k=2}^{\infty} (k-\alpha)|(|A_k - a_k| + |B_k - b_k|) + \sum_{k=2}^{\infty} (k-\alpha)(|a_k| + |b_k|) \\
&\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{1}{2^n} \sum_{k=2}^{\infty} (k^{n+1} - \alpha k^n)(|a_k| + |b_k|) \\
&\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{1}{2^n}(1-\alpha)(1-|b_1|) \leq (1-\alpha),
\end{aligned}$$

if $\delta \leq \left(\frac{2^n-1}{2^n}\right)(1-|b_1|)$. Thus $F(z) \in HS(\alpha)$. $\square$

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Remark 3. For $f \in HS(2, 1, \alpha) \equiv HC(\alpha)$, our result is different from the result given by Özturk and Yalcin [5, Theorem 3.15]. It can be easily seen that our result improves it.

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