journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 3, Article 90, 7 pp.

ON CLASS $wF(p, r, q)$ OPERATORS AND QUASISIMILARITY

CHANGSEN YANG AND YULIANG ZHAO

COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE
HENAN NORMAL UNIVERSITY,
XINXIANG 453007, PEOPLE'S REPUBLIC OF CHINA
yangchangsen117@yahoo.com.cn

DEPARTMENT OF MATHEMATICS
ANYANG INSTITUTE OF TECHNOLOGY
ANYANG CITY, HENAN PROVINCE 455000
PEOPLE'S REPUBLIC OF CHINA
zhaoyuliang512@163.com

Received 17 October, 2006; accepted 15 June, 2007
Communicated by C.K. Li

ABSTRACT. Let T be a bounded linear operator on a complex Hilbert space H. In this paper, we show that if T belongs to class $wF(p, r, q)$ operators, then we have (i) $T^*X = XN^*$ whenever $TX = XN$ for some $X \in B(H)$, where N is normal and X is injective with dense range. (ii) T satisfies the property $(\beta)_\varepsilon$, i.e., T is subscalar, moreover, T is subdecomposable. (iii) Quasisimilar class $wF(p, r, q)$ operators have the same spectra and essential spectra.

Key words and phrases: Class $wF(p, r, q)$ operators, Fuglede-Putnam's theorem, Property $(\beta)_\varepsilon$, Subscalar, Subdecomposable.

2000 *Mathematics Subject Classification.* 47B20, 47A30.

1. INTRODUCTION

Let X denote a Banach space, $T \in B(X)$ is said to be generalized scalar ([3]) if there exists a continuous algebra homomorphism (called a spectral distribution of T) $\Phi : \varepsilon(\mathcal{C}) \to B(X)$ with $\Phi(1) = I$ and $\Phi(z) = T$, where $\varepsilon(\mathcal{C})$ denotes the algebra of all infinitely differentiable functions on the complex plane $\mathcal{C}$ with the topology defined by uniform convergence of such functions and their derivatives ([2]). An operator similar to the restriction of a generalized scalar (decomposable) operator to one of its closed invariant subspaces is said to be subscalar (subdecomposable). Subscalar operators are subdecomposable operators ([3]). Let H, K be complex Hilbert spaces and $B(H), B(K)$ be the algebra of all bounded linear operators in H and K respectively, $B(H, K)$ denotes the algebra of all bounded linear operators from H to K. A capital letter (such as T) means an element of $B(H)$. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for any $x \in H$. An operator T is said to be $p-$hyponormal if $(T^*T)^p \geq (TT^*)^p$, $0 < p \leq 1$.

The authors are grateful to the referee for comments which improved the paper.

266-06

Definition 1.1 ([10]). For $p > 0, r \geq 0$, and $q \geq 1$, an operator T belongs to class $wF(p,r,q)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}$$

and

$$|T|^{2(p+r)(1-\frac{1}{q})} \geq (|T|^p|T^*|^{2r}|T|^p)^{1-\frac{1}{q}}.$$

Let $T = U|T|$ be the polar decomposition of T. We define

$$\widetilde{T}_{p,r} = |T|^pU|T|^r (p + r = 1).$$

The operator $\widetilde{T}_{p,r}$ is known as the generalized Aluthge transform of T. We define $(\widetilde{T}_{p,r})^{(1)} = \widetilde{T}_{p,r}$, $(\widetilde{T}_{p,r})^{(n)} = [(\widetilde{T}_{p,r})^{(n-1)}]_{p,r}$, where $n \geq 2$.

The following Fuglede-Putnam's theorem is famous. We extend this theorem for class $wF(p,r,q)$ operators.

Theorem 1.1 (Fuglede-Putnam's Theorem [7]). *Let A and B be normal operators and X be an operator on a Hilbert space. Then the following hold and follow from each other:*
(i) (Fuglede) If $AX = XA$, then $A^*X = XA^*$.
(ii) (Putnam) If $AX = XB$, then $A^*X = XB^*$.

2. PRELIMINARIES

Lemma 2.1 ([9]). *If N is a normal operator on H, then we have*

$$\bigcap_{\lambda\in\mathcal{C}}(N-\lambda)\mathcal{H} = \{0\}.$$

Lemma 2.2 ([5]). *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $p > 0$. Then the following assertions hold:*

(i) $\widetilde{T}_{s,t} = |T|^sU|T|^t$ *is* $\frac{p+\min(s,t)}{s+t}$*-hyponormal for any $s > 0$ and $t > 0$ such that $\max\{s,t\} \geq p$.*

(ii) $\widetilde{T}_{s,t} = |T|^sU|T|^t$ *is hyponormal for any $s > 0$ and $t > 0$ such that $\max\{s,t\} \leq p$.*

Lemma 2.3 ([8]). *Let $T \in B(H)$, $D \in B(H)$ with $0 \leq D \leq M(T-\lambda)(T-\lambda)^*$ for all λ in $\mathcal{C}$, where M is a positive real number. Then for every $x \in D^{\frac{1}{2}}H$ there exists a bounded function $f : \mathcal{C} \to H$ such that $(T-\lambda)f(\lambda) \equiv x$.*

Lemma 2.4 ([10]). *If $T \in wF(p,r,q)$, then $\left|\widetilde{T}_{p,r}\right|^{2m} \geq |T|^{2m} \geq \left|(\widetilde{T}_{p,r})^*\right||^{2m}$, where $m = \min\left\{\frac{1}{q}, \max\left\{\frac{p}{p+r}, 1-\frac{1}{q}\right\}\right\}$, i.e., $\widetilde{T}_{p,r} = |T|^pU|T|^r$ is m-hyponormal operator.*

Lemma 2.5 ([11]). *Let $A, B \geq 0$, $\alpha_0, \beta_0 > 0$ and $-\beta_0 \leq \delta \leq \alpha_0$, $-\beta_0 \leq \bar{\delta} \leq \alpha_0$, if $0 \leq \delta \leq \alpha_0$ and $\left(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}}\right)^{\frac{\beta_0+\delta}{\alpha_0+\beta_0}} \geq B^{\beta_0+\delta}$, then*

$$\left(B^{\frac{\beta}{2}}A^{\alpha}B^{\frac{\beta}{2}}\right)^{\frac{\beta+\delta}{\alpha+\beta}} \geq B^{\beta+\delta},$$

and

$$A^{\alpha-\bar{\delta}} \geq \left(A^{\frac{\alpha}{2}}B^{\beta}A^{\frac{\alpha}{2}}\right)^{\frac{\alpha-\bar{\delta}}{\alpha+\beta}}$$

hold for each $\alpha \geq \alpha_0, \beta \geq \beta_0$ and $0 \leq \bar{\delta} \leq \alpha$.

Lemma 2.6 ([6]). *Let $A \geq 0$, $B \geq 0$, if $B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq B^2$ and $A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2$ then $A = B$.*

Lemma 2.7. *Let $A, B \geq 0$, $s, t \geq 0$, if $B^sA^{2t}B^s = B^{2s+2t}$, $A^tB^{2s}A^t = A^{2s+2t}$ then $A = B$.*

Proof. We choose $k > \max\{s, t$. Since $B^sA^{2t}B^s = B^{2s+2t}$, $A^tB^{2s}A^t = A^{2s+2t}$ it follows from Lemma 2.5 that:

$$(B^kA^{2k}B^k)^{\frac{2k+2t}{4k}} \geq B^{2k+2t},$$
$$A^{2k-2t} \geq (A^kB^{2k}A^k)^{\frac{2k-2t}{4k}},$$

and

$$(A^kB^{2k}A^k)^{\frac{2k+2s}{4k}} \geq A^{2k+2s},$$
$$B^{2k-2s} \geq (B^kA^{2k}B^k)^{\frac{2k-2s}{4k}}.$$

So

$$A^kB^{2k}A^k = A^{4k}, \qquad B^kA^{2k}B^k = B^{4k},$$

by Lemma 2.6

$$A = B.$$

□

Lemma 2.8 ([11]). *Let T be a class $wF(p,r,q)$ operator, if $\widetilde{T}_{p,r} = |T|^pU|T|^r$ is normal, then T is normal.*

The following theorem have been shown by T. Huruya in [3], here we give a simple proof.

Theorem 2.9 (Furuta inequality [4]). *If $A \geq B \geq 0$, then for each $r > 0$,*

(i) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *and*

(ii) $\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Theorem 2.10. *Let T be a $p-$hyponormal operator on H and let $T = U|T|$ be the polar decomposition of T, if $\widetilde{T}_{s,t} = |T|^sU|T|^t$ $(s+t=1)$ is normal, then T is normal.*

Proof. First, consider the case $\max\{s,t\} \geq p$. Let $A = |T|^{2p}$ and $B = |T^*|^{2p}$, p-hyponormality of T ensures $A \geq B \geq 0$. Applying Theorem 2.9 to $A \geq B \geq 0$, since

$$\left(1+\frac{t}{p}\right)\frac{s+t}{p+\min(s,t)} \geq \frac{s}{p}+\frac{t}{p} \quad \text{and} \quad \frac{s+t}{p+\min(s,t)} \geq 1,$$

we have

$$\begin{aligned}
(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} &= (|T|^tU^*|T|^{2s}U|T|^t)^{\frac{p+\min(s,t)}{s+t}} \\
&= (U^*U|T|^tU^*|T|^{2s}U|T|^tU^*U)^{\frac{p+\min(s,t)}{s+t}} \\
&= (U^*|T^*|^t|T|^{2s}|T^*|^tU)^{\frac{p+\min(s,t)}{s+t}} \\
&= U^*(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{p+\min(s,t)}{s+t}}U \\
&= U^*(B^{\frac{t}{2p}}A^{\frac{s}{p}}B^{\frac{t}{2p}})^{\frac{p+\min(s,t)}{s+t}}U \\
&\geq U^*B^{\frac{p+\min(s,t)}{p}}U \\
&= U^*|T^*|^{2(p+\min(s,t))}U \\
&= |T|^{2(p+\min(s,t))}.
\end{aligned}$$

Similarly, we also have

$$(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^{\frac{p+\min(s,t)}{s+t}} \leq |T|^{2(p+\min(s,t))}.$$

Therefore, we have

$$(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} \geq |T|^{2(p+\min(s,t))} \geq (\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^{\frac{p+\min(s,t)}{s+t}}.$$

If

$$\widetilde{T}_{s,t} = |T|^s U|T|^t \quad (s+t=1)$$

is normal, then

$$(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} = |T|^{2(p+\min(s,t))} = (\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^{\frac{p+\min(s,t)}{s+t}},$$

which implies

$$|T^*|^t|T|^{2s}|T^*|^t = |T^*|^{2(s+t)} \quad \text{and} \quad |T|^s|T^*|^{2t}|T|^s = |T|^{2(s+t)},$$

then $|T^*| = |T|$ by Lemma 2.7. Next, consider the case $\max\{s,t\} \leq p$. Firstly, $p-$hyponormality of T ensures $|T|^{2s} \geq |T^*|^{2s}$ and $|T|^{2t} \geq |T^*|^{2t}$ for $\max\{s,t\} \leq p$ by the Löwner-Heinz theorem. Then we have

$$\begin{aligned}
\widetilde{T}^*_{s,t}\widetilde{T}_{s,t} &= |T|^t U^*|T|^{2s}U|T|^t \geq |T|^t U^*|T^*|^{2s}U|T|^t \\
&= |T|^{2(s+t)} \\
\widetilde{T}_{s,t}\widetilde{T}^*_{s,t} &= |T|^s U|T|^{2t}U^*|T|^s \\
&\leq |T|^{2(s+t)}.
\end{aligned}$$

If $\widetilde{T}_{s,t} = |T|^s U|T|^t$ $(s+t=1)$ is normal, then

$$\widetilde{T}^*_{s,t}\widetilde{T}_{s,t} = |T|^{2((s+t)} = \widetilde{T}_{s,t}\widetilde{T}^*_{s,t},$$

which implies

$$|T^*|^t|T|^{2s}|T^*|^t = |T^*|^{2(s+t)} \quad \text{and} \quad |T|^s|T^*|^{2t}|T|^s = |T|^{2(s+t)},$$

then $|T^*| = |T|$ by Lemma 2.7. □

3. MAIN THEOREM

Theorem 3.1. *Assume that T is a class $wF(p,r,q)$ operator with $\mathrm{Ker}(T) \subset \mathrm{Ker}(T^*)$, and N is a normal operator on H and K respectively. If $X \in B(K,H)$ is injective with dense range which satisfies $TX = XN$, then $T^*X = XN^*$.*

Proof. $\mathrm{Ker}(T) \subset \mathrm{Ker}(T^*)$ implies $\mathrm{Ker}(T)$ reduces T. Also $\mathrm{Ker}(N)$ reduces N since N is normal. Using the orthogonal decompositions $H = \overline{\mathrm{Ran}(|T|)} \bigoplus \mathrm{Ker}(T)$ and $H = \overline{\mathrm{Ran}(N)} \bigoplus \mathrm{Ker}(N)$, we can represent T and N as follows.

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where T_1 is an injective class $wF(p,r,q)$ operator on $\overline{\mathrm{Ran}(|T|)}$ and N_1 is injective normal on $\overline{\mathrm{Ran}(N)}$. The assumption $TX = XN$ asserts that X maps $\mathrm{Ran}(N)$ to $\mathrm{Ran}(T) \subset \overline{\mathrm{Ran}(|T|)}$ and $\mathrm{Ker}(N)$ to $\mathrm{Ker}(T)$, hence X is of the form:

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

where $X_1 \in B(\overline{\mathrm{Ran}(N)}, \overline{\mathrm{Ran}(|T|)})$, $X_2 \in B(\mathrm{Ker}(N), \mathrm{Ker}(T))$. Since $TX = XN$, we have that $T_1X_1 = X_1N_1$. Since X is injective with dense range, X_1 is also injective with dense range. Put $W_1 = |T_1|^p X_1$, then W_1 is also injective with dense range and satisfies $\widetilde{(T_1)}_{p,r} W_1 = W_1 N$. Put $W_n = \left|(\widetilde{T}_1)^{(n)}_{p,r}\right|^p W_{(n-1)}$, then W_n is also injective with dense range and satisfies $(\widetilde{T}_1)^{(n)}_{p,r} W_n = W_n N$. From Lemma 2.2 and Lemma 2.4, if there is an integer α_0 such that $(\widetilde{T}_1)^{(\alpha_0)}_{p,r}$ is a hyponormal operator, then $(\widetilde{T}_1)^{(n)}_{p,r}$ is a hyponormal operator for $n \geq \alpha_0$. It follows from Lemma 2.3 that there exists a bounded function $f : \mathcal{C} \to H$ such that

$$\left(\left(\left(\widetilde{T}_1\right)^{(n)}_{p,r}\right)^* - \lambda\right) f(\lambda) \equiv x, \text{ for every}$$
$$x \in \left(\left(\left(\widetilde{T}_1\right)^{(n)}_{p,r}\right)^* \left(\widetilde{T}_1\right)^{(n)}_{p,r} - \left(\widetilde{T}_1\right)^{(n)}_{p,r} \left(\left(\widetilde{T}_1\right)^{(n)}_{p,r}\right)^*\right)^{\frac{1}{2}} H.$$

Hence

$$\begin{aligned} W_n^* x &= W_n^* \left(\left(\left(\widetilde{T}_1\right)^{(n)}_{p,r}\right)^* - \lambda\right) f(\lambda) \\ &= (N_1^* - \lambda) W_n^* f(\lambda) \in \operatorname{Ran}(N_1^* - \lambda) \quad \text{for all } \lambda \in \mathcal{C} \end{aligned}$$

By Lemma 2.1, we have $W_n^* x = 0$, and hence $x = 0$ because W_n^* is injective. This implies that $(\widetilde{T}_1)^{(n)}_{p,r}$ is normal. By Lemma 2.8 and Theorem 2.10, T_1 is nomal and therefore $T = T_1 \bigoplus 0$ is also normal. The assertion is immediate from Fuglede-Putnam's theorem. □

Let X be a $Banach$ space, U be an open subset of $\mathcal{C}$. $\varepsilon(U, X)$ denotes the *Fréchet* space of all $X-$valued $\mathcal{C}^\infty-$functions, i.e., infinitely differentiable functions on U ([3]). $T \in B(X)$ is said to satisfy property $(\beta)_\varepsilon$ if for each open subset U of $\mathcal{C}$, the map

$$T_z : \varepsilon(U, X) \to \varepsilon(U, X), \quad f \mapsto (T - z) f$$

is a topological monomorphism, i.e., $T_z f_n \to 0\ (n \to \infty)$ in $\varepsilon(U, X)$ implies $f_n \to 0\ (n \to \infty)$ in $\varepsilon(U, X)$ ([3]).

Lemma 3.2 ([1])**.** *Let $T \in B(X)$. T is subscalar if and only if T satisfies property $(\beta)_\varepsilon$.*

Lemma 3.3. *Let $T \in B(X)$. T satisfies property $(\beta)_\varepsilon$ if and only if $\widetilde{T}_{p,r}$ satisfies property $(\beta)_\varepsilon$.*

Proof. First, we suppose that T satisfies property $(\beta)_\varepsilon$, U is an open subset of $\mathcal{C}$, $f_n \in \varepsilon(U, X)$ and

$$(\widetilde{T}_{p,r} - z) f_n \to 0 \quad (n \to \infty), \tag{3.1}$$

in $\varepsilon(U, X)$, then

$$(T - z) U |T|^r f_n = U |T|^r (\widetilde{T}_{p,r} - z) f_n \to 0 \quad (n \to \infty).$$

Since T satisfies property $(\beta)_\varepsilon$, we have $U|T|^r f_n \to 0\ (n \to \infty)$. and therefore

$$\widetilde{T}_{p,r} f_n \to 0 \quad (n \to \infty). \tag{3.2}$$

(3.1) and (3.2) imply that

$$z f_n = \widetilde{T}_{p,r} f_n - (\widetilde{T}_{p,r} - z) f_n \to 0 \quad (n \to \infty) \tag{3.3}$$

in $\varepsilon(U, X)$. Notice that $T = 0$ is a subscalar operator and hence satisfies property $(\beta)_\varepsilon$ by Lemma 3.2. Now we have

$$f_n \to 0 \quad (n \to \infty). \tag{3.4}$$

(3.1) and (3.4) imply that $\widetilde{T}_{p,r}$ satisfies property $(\beta)_\varepsilon$. Next we suppose that $\widetilde{T}_{p,r}$ satisfies property $(\beta)_\varepsilon$, U is an open subset of $\mathcal{C}$, $f_n \in \varepsilon(U,X)$ and

$$(T-z)f_n \to 0 \quad (n\to\infty), \tag{3.5}$$

in $\varepsilon(U,X)$. Then

$$(\widetilde{T}_{p,r}-z)|T|^p f_n = |T|^p(T-z)f_n \to 0 \quad (n\to\infty).$$

Since $\widetilde{T}_{p,r}$ satisfies property $(\beta)_\varepsilon$, we have $|T|^p f_n \to 0$ $(n\to\infty)$, and therefore

$$Tf_n \to 0 \quad (n\to\infty). \tag{3.6}$$

(3.5) and (3.6) imply

$$zf_n = Tf_n - (T-z)f_n \to 0 \quad (n\to\infty).$$

So $f_n \to 0$ $(n\to\infty)$. Hence T satisfies property $(\beta)_\varepsilon$. □

Lemma 3.4 ([1]). *Suppose that T is a $p-$hyponormal operator, then T is subscalar.*

Theorem 3.5. *Let $T \in wF(p,r,q)$ and $p+r=1$, then T is subdecomposable.*

Proof. If $T \in wF(p,r,q)$, then $\widetilde{T}_{p,r}$ is a m-hyponormal operator by Lemma 2.4, and it follows from Lemma 3.4 that $\widetilde{T}_{p,r}$ is subscalar. So we have T is subscalar by Lemma 3.2 and Lemma 3.3. It is well known that subscalar operators are subdecomposable operators ([3]). Hence T is subdecomposable. □

Recall that an operator $X \in B(H)$ is called a quasiaffinity if X is injective and has dense range. For $T_1, T_2 \in B(H)$, if there exist quasiaffinities $X \in B(H_2,H_1)$ and $Y \in B(H_1,H_2)$ such that $T_1X = XT_2$ and $YT_1 = T_2Y$ then we say that T_1 and T_2 are quasisimilar.

Lemma 3.6 ([2]). *Let $S \in B(H)$ be subdecomposable, $T \in B(H)$. If $X \in B(K,H)$ is injective with dense range which satisfies $XT = SX$, then $\sigma(S) \subset \sigma(T)$; if T and S are quasisimilar, then $\sigma_e(S) \subseteq \sigma_e(T)$.*

Theorem 3.7. *Let $T_1, T_2 \in wF(p,r,q)$. If T_1 and T_2 are quasisimilar then $\sigma(T_1) = \sigma(T_2)$ and $\sigma_e(T_1) = \sigma_e(T_2)$.*

Proof. Obvious from Theorem 3.5 and Lemma 3.6. □

REFERENCES

[1] L. CHEN, R. YINGBIN AND Y. ZIKUN, w-Hyponormal operators are subscalar, *Integr. Equat. Oper. Th.*, **50** (2004), 165–168.

[2] L. CHEN AND Y. ZIKUN, Bishop's property (β) and essential spectra of quasisimilar operators, *Proc. Amer. Math. Soc.*, **128** (2000), 485–493.

[3] I. COLOJOARĀ AND C. FOIAS, *Theory of Generalized Spectral Operators*, New York, Gordon and Breach, 1968.

[4] T. FURUTA, *Invitation to Linear Operators – From Matrices to Bounded Linear Operators on a Hilbert Space*, London: Taylor & Francis, 2001.

[5] T. HURUYA, A note on $p-$hyponormal operators, *Proc. Amer. Math. Soc.*, **125** (1997), 3617–3624.

[6] M. ITO AND T. YAMAZAKI, Relations between two inequalities $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^r$ and $\left(A^{\frac{p}{2}}B^rA^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \leq A^p$ and its applications, *Integr. Equat. Oper. Th.*, **44** (2002), 442–450.

[7] C.R. PUTNAM, On normal operators in Hilbert space, *Amer. J. Math.*, **73** (1951), 357–362.

[8] C.R. PUTNAM, Hyponormal contractions and strong power convergence, *Pacific J. Math.*, **57** (1975), 531–538.

[9] C.R. PUTNAM, Ranges of normal and subnormal operators, *Michigan Math. J.*, **18** (1971) 33–36.

[10] C. YANG AND J. YUAN, Spectrum of class $wF(p,r,q)$ operators for $p+r \leq 1$ and $q \geq 1$, *Acta. Sci. Math.* (Szeged), **71** (2005), 767–779.

[11] C. YANG AND J. YUAN, On class $wF(p,r,q)$ operators, preprint.