

Journal of Inequalities in Pure and Applied Mathematics

SOME NEW DISCRETE NONLINEAR DELAY INEQUALITIES AND APPLICATION TO DISCRETE DELAY EQUATIONS

WING-SUM CHEUNG AND SHIOJENN TSENG

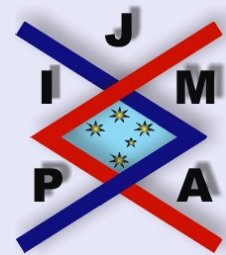
Department of Mathematics
University of Hong Kong
Hong Kong

EMail: wscheung@hku.hk

Department of Mathematics
Tamkang University
Tamsui, Taiwan 25137

EMail: tseng@math.tku.edu.tw

©2000 Victoria University
ISSN (electronic): 1443-5756
267-05



volume 7, issue 4, article 122,
2006.

*Received 07 September, 2005;
accepted 27 January, 2006.*

Communicated by: S.S. Dragomir

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper, some new discrete Gronwall-Bellman-Ou-lang-type inequalities are established. These on the one hand generalize some existing results and on the other hand provide a handy tool for the study of qualitative as well as quantitative properties of solutions of difference equations.

2000 Mathematics Subject Classification: 26D10, 26D15, 39A10, 39A70.

Key words: Gronwall-Bellman-Ou-lang-type Inequalities, Discrete inequalities, Difference equations.

Contents

| | | |
|---|--|----|
| 1 | Introduction | 3 |
| 2 | Discrete Inequalities with Delay | 5 |
| 3 | Immediate Consequences | 23 |
| 4 | Application | 32 |
| | References | |



Some New Discrete Nonlinear Delay Inequalities and Application to Discrete Delay Equations

Wing-Sum Cheung and Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 2 of 36

1. Introduction

It is widely recognized that integral inequalities in general provide an effective tool for the study of qualitative as well as quantitative properties of solutions of integral and differential equations. While most integral inequalities only give the ‘global behavior’ of the unknown functions (in the sense that bounds are only obtained for integrals of certain functions of the unknown functions), the Gronwall-Bellman type (see, e.g. [3] – [8], [10] – [12], [15] – [18]) is particularly useful as they provide explicit pointwise bounds of the unknown functions. A specific branch of this type of inequalities is originated by Ou-Iang. In his paper [13], in order to study the boundedness behavior of the solutions of some 2nd order differential equations, Ou-Iang established the following beautiful inequality.

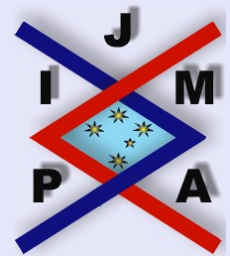
Theorem 1.1 (Ou-Iang [13]). *If u and f are non-negative functions on $[0, \infty)$ satisfying*

$$u^2(x) \leq c^2 + 2 \int_0^x f(s)u(s)ds, \quad x \in [0, \infty),$$

for some constant $c \geq 0$, then

$$u(x) \leq c + \int_0^x f(s)ds, \quad x \in [0, \infty).$$

While Ou-Iang’s inequality is interesting in its own right, it also has numerous important applications in the study of differential equations (see, e.g., [2, 3, 9, 11, 12]). Over the years, various extensions of Ou-Iang’s inequality have been established. These include, among others, works of Agarwal [1],



Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

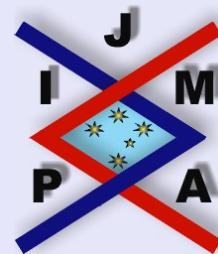
Close

Quit

Page 3 of 36

Ma-Yang [10], Pachpatte [14] – [18], Tsamatos-Ntouyas [19], and Yang [20]. Among such extensions, the discretization is of particular interest because analogous to the continuous case, discrete versions of integral inequalities should, in our opinion, play an important role in the study of qualitative as well as quantitative properties of solutions of difference equations.

It is the purpose of this paper to establish some new discrete Gronwall-Bellman-Ou-Iang-type inequalities giving explicit bounds to unknown discrete functions. These on the one hand generalize some existing results in the literature and on the other hand give a handy tool to the study of difference equations. An application to a discrete delay equation is given at the end of the paper.



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 4 of 36

2. Discrete Inequalities with Delay

Throughout this paper, $\mathbb{R}_+ = (0, \infty) \subset \mathbb{R}$, $\mathbb{Z}_+ = \mathbb{R}_+ \cap \mathbb{Z}$, and for any $a, b \in \mathbb{R}$, $\mathbb{R}_a = [a, \infty)$, $\mathbb{Z}_a = \mathbb{R}_a \cap \mathbb{Z}$, $\mathbb{Z}_{[a,b]} = \mathbb{Z} \cap [a, b]$. If X and Y are sets, the collection of functions of X into Y , the collection of continuous functions of X into Y , and that of continuously differentiable functions of X into Y are denoted by $\mathcal{F}(X, Y)$, $C(X, Y)$, and $C^1(X, Y)$, respectively. As usual, if u is a real-valued function on $\mathbb{Z}_{[a,b]}$, the difference operator Δ on u is defined as

$$\Delta u(n) = u(n+1) - u(n), \quad n \in \mathbb{Z}_{[a,b-1]}.$$

In the sequel, summations over empty sets are, as usual, defined to be zero.

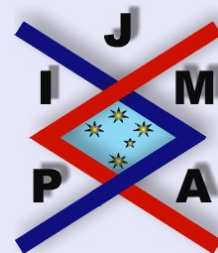
The basic assumptions and initial conditions used in this paper are the following:

Assumptions

- (A1) $f, g, h, k, p \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_0)$ with p non-decreasing;
- (A2) $w \in C(\mathbb{R}_0, \mathbb{R}_0)$ is non-decreasing with $w(r) > 0$ for $r > 0$;
- (A3) $\sigma \in \mathcal{F}(\mathbb{Z}_0, \mathbb{Z})$ with $\sigma(s) \leq s$ for all $s \in \mathbb{Z}_0$ and $-\infty < a := \inf\{\sigma(s) : s \in \mathbb{Z}_0\} \leq 0$;
- (A4) $\psi \in \mathcal{F}(\mathbb{Z}_{[a,0]}, \mathbb{R}_0)$; and
- (A5) $\phi \in C^1(\mathbb{R}_0, \mathbb{R}_0)$ with ϕ' non-decreasing and $\phi'(r) > 0$ for $r > 0$.

Initial Conditions

- (I1) $x(s) = \psi(s)$ for all $s \in \mathbb{Z}_{[a,0]}$;
- (I2) $\psi(\sigma(s)) \leq \phi^{-1}(p(s))$ for all $s \in \mathbb{Z}_0$ with $\sigma(s) \leq 0$.



Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 5 of 36

Theorem 2.1. Under Assumptions (A1) – (A5), if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ is a function satisfying the nonlinear delay inequality

$$(2.1) \quad \phi(x(n)) \leq p(n) + \sum_{s=0}^{n-1} \phi'(x(\sigma(s))) \{f(s) + g(s)x(\sigma(s)) + h(s)w(x(\sigma(s)))\}$$

for all $n \in \mathbb{Z}_0$ with initial conditions (I1) – (I2), then

$$(2.2) \quad x(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1}(p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{t=0}^{n-1} h(t) \right\}$$

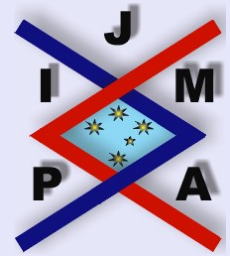
for all $n \in \mathbb{Z}_{[0, \alpha]}$, where $\Phi \in C(\mathbb{R}_0, \mathbb{R})$ is defined by

$$\Phi(r) := \int_1^r \frac{ds}{w(s)}, \quad r > 0,$$

and $\alpha \geq 0$ is chosen such that the RHS of (2.2) is well-defined, that is,

$$\Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1}(p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] + \exp \left(\sum_{s=0}^{n-1} g(s) \right) \sum_{t=0}^{n-1} h(t) \in \mathcal{I}_m \Phi$$

for all $n \in \mathbb{Z}_{[0, \alpha]}$.



Some New Discrete Nonlinear Delay Inequalities and Application to Discrete Delay Equations

Wing-Sum Cheung and Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 6 of 36

Proof. Fix $\varepsilon > 0$ and $N \in \mathbb{Z}_{[0,\alpha]}$. Define $u : \mathbb{Z}_{[0,N]} \rightarrow \mathbb{R}_0$ by

$$(2.3) \quad u(n) := \phi^{-1} \left\{ \varepsilon + p(N) + \sum_{t=0}^{n-1} \phi'(x(\sigma(t))) [f(t) + g(t)x(\sigma(t)) + h(t)w(x(\sigma(t)))] \right\}.$$

By (A5), u is non-decreasing on $\mathbb{Z}_{[0,N]}$. For any $n \in \mathbb{Z}_{[0,N]}$, by (A5) again,

$$(2.4) \quad u(n) \geq \phi^{-1}(\varepsilon + p(N)) > 0.$$

As $\phi(u(n)) > \phi(x(n))$, we have

$$(2.5) \quad u(n) > x(n).$$

Next, observe that if $\sigma(n) \geq 0$, then by (A3), $\sigma(n) \in \mathbb{Z}_{[0,N]}$ and so

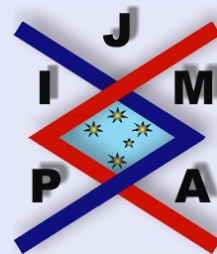
$$x(\sigma(n)) < u(\sigma(n)) \leq u(n).$$

On the other hand, if $\sigma(n) \leq 0$, then by (A3) again, $\sigma(n) \in \mathbb{Z}_{[a,0]}$ and so by (I1), (I2), (A1), (A5) and (2.4),

$$x(\sigma(n)) = \psi(\sigma(n)) \leq \phi^{-1}(p(n)) \leq \phi^{-1}(p(N)) \leq \phi^{-1}(p(N) + \varepsilon) \leq u(n).$$

Hence we always have

$$(2.6) \quad x(\sigma(n)) \leq u(n) \quad \text{for all } n \in \mathbb{Z}_{[0,N]}.$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 7 of 36

Therefore, for any $s \in \mathbb{Z}_{[0, N-1]}$, by (2.3) and (2.6),

$$\begin{aligned} \Delta(\phi \circ u)(s) &= \phi(u(s+1)) - \phi(u(s)) \\ &= \phi'(x(\sigma(s))) \{f(s) + g(s)x(\sigma(s)) + h(s)w(x(\sigma(s)))\} \\ &\leq \phi'(u(s)) \{f(s) + g(s)u(s) + h(s)w(u(s))\} . \end{aligned}$$

On the other hand, by the Mean Value Theorem, we obtain

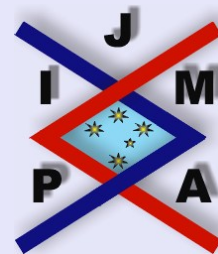
$$\begin{aligned} \Delta(\phi \circ u)(s) &= \phi(u(s+1)) - \phi(u(s)) \\ &= \phi'(\xi) \Delta u(s) \end{aligned}$$

for some $\xi \in [u(s), u(s+1)]$. Observe that by (2.4) and (A5), $\phi'(\xi) > 0$. Thus by the monotonicity of ϕ' , for any $s \in \mathbb{Z}_{[0, N-1]}$,

$$\begin{aligned} \Delta u(s) &\leq \frac{\phi'(u(s))}{\phi'(\xi)} \{f(s) + g(s)u(s) + h(s)w(u(s))\} \\ &\leq f(s) + g(s)u(s) + h(s)w(u(s)) . \end{aligned}$$

Summing up, we have

$$\begin{aligned} u(n) - u(0) &= \sum_{s=0}^{n-1} \Delta u(s) \\ &\leq \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s)w(u(s)) + \sum_{s=0}^{n-1} g(s)u(s) , \end{aligned}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 8 of 36

or

$$u(n) \leq \left[\phi^{-1}(\varepsilon + p(N)) + \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s)w(u(s)) \right] + \sum_{s=0}^{n-1} g(s)u(s)$$

for all $n \in \mathbb{Z}_{[0,N]}$. Hence by the discrete version of the Gronwall-Bellman inequality (see, e.g., [16, Corollary 1.2.5]),

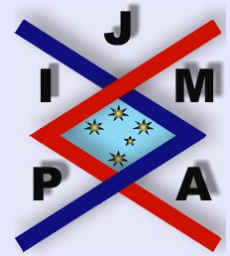
$$(2.7) \quad u(n) \leq \left[\phi^{-1}(\varepsilon + p(N)) + \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s)w(u(s)) \right] \exp \sum_{s=0}^{n-1} g(s) \\ \leq \left[\phi^{-1}(\varepsilon + p(N)) + \sum_{s=0}^{N-1} f(s) + \sum_{s=0}^{n-1} h(s)w(u(s)) \right] \exp \sum_{s=0}^{N-1} g(s)$$

for all $n \in \mathbb{Z}_{[0,N]}$. Denote by $v(n)$ the RHS of (2.7). Then v is non-decreasing and for all $n \in \mathbb{Z}_{[0,N]}$,

$$(2.8) \quad u(n) \leq v(n).$$

Therefore, for any $t \in \mathbb{Z}_{[0,N-1]}$,

$$\begin{aligned} \Delta v(t) &= v(t+1) - v(t) \\ &= h(t)w(u(t)) \exp \sum_{s=0}^{N-1} g(s) \\ &\leq h(t)w(v(t)) \exp \sum_{s=0}^{N-1} g(s). \end{aligned}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 9 of 36

On the other hand, by the Mean Value Theorem, we have

$$\begin{aligned}\Delta(\Phi \circ v)(t) &= \Phi(v(t+1)) - \Phi(v(t)) \\ &= \Phi'(\eta)\Delta v(t) \\ &= \frac{1}{w(\eta)}\Delta v(t)\end{aligned}$$

for some $\eta \in [v(t), v(t+1)]$. Observe that by (2.4), (2.8), and (A2), $w(\eta) > 0$. Therefore, as w is non-decreasing,

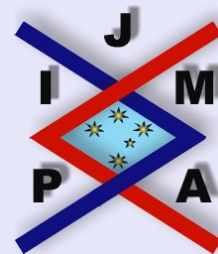
$$\begin{aligned}\Delta(\Phi \circ v)(t) &\leq \frac{1}{w(\eta)}h(t)w(v(t)) \exp \sum_{s=0}^{N-1} g(s) \\ &\leq h(t) \exp \sum_{s=0}^{N-1} g(s)\end{aligned}$$

for all $t \in \mathbb{Z}_{[0, N-1]}$. Summing up, we have

$$\sum_{t=0}^{n-1} \Delta(\Phi \circ v)(t) \leq \sum_{t=0}^{n-1} h(t) \exp \sum_{s=0}^{N-1} g(s).$$

On the other hand,

$$\sum_{t=0}^{n-1} \Delta(\Phi \circ v)(t) = \Phi(v(n)) - \Phi(v(0))$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 10 of 36

$$= \Phi(v(n)) - \Phi \left[\left(\exp \sum_{s=0}^{N-1} g(s) \right) \left(\phi^{-1}(\varepsilon + p(N)) + \sum_{s=0}^{N-1} f(s) \right) \right],$$

therefore,

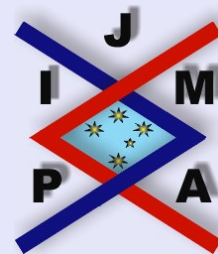
$$\begin{aligned} \Phi(v(n)) \leq \Phi \left[\left(\exp \sum_{s=0}^{N-1} g(s) \right) \left(\phi^{-1}(\varepsilon + p(N)) + \sum_{s=0}^{N-1} f(s) \right) \right] \\ + \sum_{t=0}^{n-1} h(t) \exp \sum_{s=0}^{N-1} g(s) \end{aligned}$$

for all $n \in \mathbb{Z}_{[0,N]}$. In particular, taking $n = N$ we have

$$\begin{aligned} \Phi(v(N)) \leq \Phi \left[\left(\exp \sum_{s=0}^{N-1} g(s) \right) \left(\phi^{-1}(\varepsilon + p(N)) + \sum_{s=0}^{N-1} f(s) \right) \right] \\ + \left(\exp \sum_{s=0}^{N-1} g(s) \right) \sum_{t=0}^{N-1} h(t). \end{aligned}$$

Since $N \in \mathbb{Z}_{[0,\alpha]}$ is arbitrary,

$$\begin{aligned} \Phi(v(n)) \leq \Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1}(\varepsilon + p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] \\ + \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{t=0}^{n-1} h(t) \end{aligned}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 11 of 36

for all $n \in \mathbb{Z}_{[0,\alpha]}$. Hence

$$v(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1} (\varepsilon + p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{t=0}^{n-1} h(t) \right\}$$

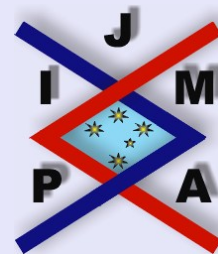
and so by (2.5) and (2.8),

$$x(n) < u(n) \leq v(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1} (\varepsilon + p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{t=0}^{n-1} h(t) \right\}$$

for all $n \in \mathbb{Z}_{[0,\alpha]}$. Finally, letting $\varepsilon \rightarrow 0^+$, we conclude that

$$x(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1} (p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{t=0}^{n-1} h(t) \right\}$$

for all $n \in \mathbb{Z}_{[0,\alpha]}$. □



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 12 of 36

Remark 1. In many cases the non-decreasing function w satisfies $\int_1^\infty \frac{ds}{w(s)} = \infty$. For example, $w = \text{constant} > 0$, $w(s) = \sqrt{s}$, etc., are such functions. In such cases $\Phi(\infty) = \infty$ and so we may take $\alpha \rightarrow \infty$, that is, (2.2) is valid for all $n \in \mathbb{Z}_0$.

Theorem 2.2. Under Assumptions (A1) – (A5), if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ is a function satisfying the nonlinear delay inequality

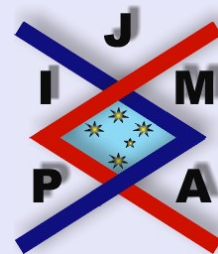
$$\phi(x(n)) \leq p(n) + \sum_{s=0}^{n-1} \phi'(x(\sigma(s))) \left\{ f(s) + g(s)x(\sigma(s)) + h(s) \sum_{t=0}^{s-1} k(t)w(x(\sigma(t))) \right\}$$

for all $n \in \mathbb{Z}_0$ with initial conditions (I1) – (I2), then

$$(2.9) \quad x(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1}(p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{s=0}^{n-1} \sum_{t=0}^{s-1} h(s)k(t) \right\}$$

for all $n \in \mathbb{Z}_{[0, \beta]}$, where $\Phi \in C(\mathbb{R}_0, \mathbb{R})$ is as defined in Theorem 2.1, and $\beta \geq 0$ is chosen such that the RHS of (2.9) is well-defined, that is,

$$\Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1}(p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right]$$



Some New Discrete Nonlinear Delay Inequalities and Application to Discrete Delay Equations

Wing-Sum Cheung and Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 13 of 36

$$+ \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{s=0}^{n-1} \sum_{t=0}^{s-1} h(s)k(t) \in \mathcal{I}_m \Phi$$

for all $n \in \mathbb{Z}_{[0,\beta]}$.

Proof. Fix $\varepsilon > 0$ and $M \in \mathbb{Z}_{[0,\beta]}$. Define $u : \mathbb{Z}_{[0,M]} \rightarrow \mathbb{R}_0$ by

$$(2.10) \quad u(n) := \phi^{-1} \left\{ \varepsilon + p(M) + \sum_{\delta=0}^{n-1} \phi' (x(\sigma(\delta))) \cdot \left[f(\delta) + g(\delta)x(\sigma(\delta)) + h(\delta) \sum_{t=0}^{\delta-1} k(t)w(x(\sigma(t))) \right] \right\}.$$

By (A5), u is non-decreasing on $\mathbb{Z}_{[0,M]}$. For any $n \in \mathbb{Z}_{[0,M]}$, by (A5) again,

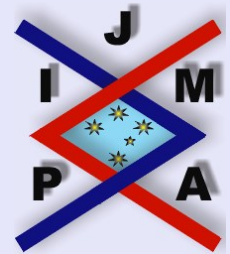
$$(2.11) \quad u(n) \geq \phi^{-1} (\varepsilon + p(M)) > 0.$$

As $\phi(u(n)) > \phi(x(n))$, we have

$$(2.12) \quad u(n) > x(n).$$

Using the same arguments as in the derivation of (2.6) in the proof of Theorem 2.1, we have

$$(2.13) \quad x(\sigma(n)) \leq u(n) \quad \text{for all } n \in \mathbb{Z}_{[0,M]}.$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 14 of 36

Hence for any $s \in \mathbb{Z}_{[0, M-1]}$, by (2.10) and (2.13),

$$\begin{aligned} \Delta(\phi \circ u)(s) &= \phi(u(s+1)) - \phi(u(s)) \\ &= \phi'(x(\sigma(s))) \left\{ f(s) + g(s)x(\sigma(s)) + h(s) \sum_{t=0}^{s-1} k(t)w(x(\sigma(t))) \right\} \\ &\leq \phi'(u(s)) \left\{ f(s) + g(s)u(s) + h(s) \sum_{t=0}^{s-1} k(t)w(u(t)) \right\}. \end{aligned}$$

On the other hand, by the Mean Value Theorem,

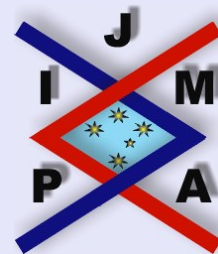
$$\begin{aligned} \Delta(\phi \circ u)(s) &= \phi(u(s+1)) - \phi(u(s)) \\ &= \phi'(\xi)\Delta u(s) \end{aligned}$$

for some $\xi \in [u(s), u(s+1)]$. Observe that by (2.12) and (A5), $\phi'(\xi) > 0$. Thus by the monotonicity of ϕ' , for any $s \in \mathbb{Z}_{[0, M-1]}$,

$$\begin{aligned} \Delta u(s) &\leq \frac{\phi'(u(s))}{\phi'(\xi)} \left\{ f(s) + g(s)u(s) + h(s) \sum_{t=0}^{s-1} k(t)w(u(t)) \right\} \\ &\leq f(s) + g(s)u(s) + h(s) \sum_{t=0}^{s-1} k(t)w(u(t)). \end{aligned}$$

Summing up, we have

$$u(n) - u(0) = \sum_{s=0}^{n-1} \Delta u(s)$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 15 of 36

$$\leq \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t)w(u(t)) + \sum_{s=0}^{n-1} g(s)u(s),$$

or

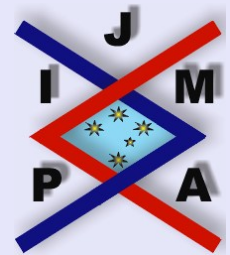
$$u(n) \leq \left[\phi^{-1}(\varepsilon + p(M)) + \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t)w(u(t)) \right] + \sum_{s=0}^{n-1} g(s)u(s)$$

for all $n \in \mathbb{Z}_{[0,M]}$. Hence by the discrete version of the Gronwall-Bellman inequality (see, e.g., [16, Corollary 1.2.5]),

$$\begin{aligned} u(n) &\leq \left[\phi^{-1}(\varepsilon + p(M)) + \sum_{s=0}^{n-1} f(s) \right. \\ &\quad \left. + \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t)w(u(t)) \right] \exp \sum_{s=0}^{n-1} g(s) \\ &\leq \left[\phi^{-1}(\varepsilon + p(M)) + \sum_{s=0}^{M-1} f(s) \right. \\ &\quad \left. + \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t)w(u(t)) \right] \exp \sum_{s=0}^{M-1} g(s) \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{Z}_{[0,M]}$. Denote by $v(n)$ the RHS of (2.14). Then v is non-decreasing and for all $n \in \mathbb{Z}_{[0,M]}$,

$$u(n) \leq v(n). \tag{2.15}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 16 of 36

Therefore, for any $\delta \in \mathbb{Z}_{[0, M-1]}$,

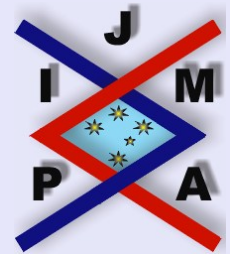
$$\begin{aligned} \Delta v(\delta) &= v(\delta + 1) - v(\delta) \\ &= h(\delta) \left(\sum_{t=0}^{\delta-1} k(t)w(u(t)) \right) \exp \sum_{s=0}^{M-1} g(s) \\ &\leq h(\delta) \left(\sum_{t=0}^{\delta-1} k(t)w(v(t)) \right) \exp \sum_{s=0}^{M-1} g(s) \\ &\leq h(\delta)w(v(\delta)) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s). \end{aligned}$$

On the other hand, by the Mean Value Theorem,

$$\begin{aligned} \Delta(\Phi \circ v)(\delta) &= \Phi(v(\delta + 1)) - \Phi(v(\delta)) \\ &= \Phi'(\eta)\Delta v(\delta) = \frac{1}{w(\eta)}\Delta v(\delta) \end{aligned}$$

for some $\eta \in [v(\delta), v(\delta + 1)]$. Observe that by (2.11), (2.14), and (A2), $w(\eta) > 0$. Therefore, as w is non-decreasing,

$$\begin{aligned} \Delta(\Phi \circ v)(\delta) &\leq \frac{1}{w(\eta)}h(\delta)w(v(\delta)) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s) \\ &\leq h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s) \end{aligned}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 17 of 36

for all $\delta \in \mathbb{Z}_{[0, M-1]}$. Summing up, we have

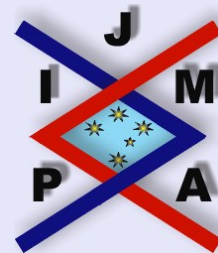
$$\sum_{\delta=0}^{n-1} \Delta(\Phi \circ v)(\delta) \leq \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s),$$

or

$$\begin{aligned} \Phi(v(n)) &\leq \Phi(v(0)) + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s) \\ &= \Phi \left[\left(\phi^{-1}(\varepsilon + p(M)) + \sum_{s=0}^{M-1} f(s) \right) \exp \sum_{s=0}^{M-1} g(s) \right] \\ &\quad + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s) \end{aligned}$$

for all $n \in \mathbb{Z}_{[0, M]}$. In particular, taking $n = M$ this yields

$$\begin{aligned} \Phi(v(M)) &\leq \Phi \left[\left(\phi^{-1}(\varepsilon + p(M)) + \sum_{s=0}^{M-1} f(s) \right) \exp \sum_{s=0}^{M-1} g(s) \right] \\ &\quad + \sum_{\delta=0}^{M-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s). \end{aligned}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 18 of 36

Since $M \in \mathbb{Z}_{[0,\beta]}$ is arbitrary,

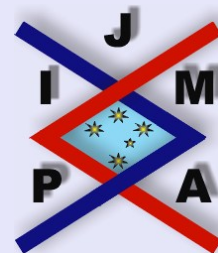
$$\Phi(v(n)) \leq \Phi \left[\left(\phi^{-1}(\varepsilon + p(n)) + \sum_{s=0}^{n-1} f(s) \right) \exp \sum_{s=0}^{n-1} g(s) \right] + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{n-1} g(s)$$

for all $n \in \mathbb{Z}_{[0,\beta]}$. Hence

$$v(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\phi^{-1}(\varepsilon + p(n)) + \sum_{s=0}^{n-1} f(s) \right) \exp \sum_{s=0}^{n-1} g(s) \right] + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{n-1} g(s) \right\}$$

and so by (2.12) and (2.15),

$$x(n) < u(n) \leq v(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\phi^{-1}(\varepsilon + p(n)) + \sum_{s=0}^{n-1} f(s) \right) \exp \sum_{s=0}^{n-1} g(s) \right] + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{n-1} g(s) \right\}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 19 of 36

for all $n \in \mathbb{Z}_{[0,\beta]}$. Finally, letting $\varepsilon \rightarrow 0^+$, we conclude that

$$x(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1}(p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{\delta=0}^{n-1} \sum_{t=0}^{\delta-1} h(\delta)k(t) \right\}$$

for all $n \in \mathbb{Z}_{[0,\beta]}$. □

Remark 2. *Similar to the previous remark, in case $\Phi(\infty) = \infty$, (2.9) holds for all $n \in \mathbb{Z}_0$.*

Theorem 2.3. *Under Assumptions (A1), (A3) and (A4), if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ is a function satisfying the nonlinear delay inequality*

$$x^r(n) \leq c^r + \sum_{s=0}^{n-1} x^r(\sigma(s)) \{f(s) + g(s)x^r(\sigma(s))\}, \quad n \in \mathbb{Z}_0,$$

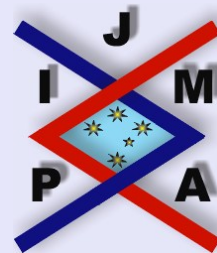
with initial conditions (II) and

$$(I3) \quad \psi(\sigma(s)) \leq c \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0,$$

where $r, c > 0$ are constants, then

$$(2.16) \quad x(n) \leq \left[c^{-r} \prod_{s=0}^{n-1} (1 - f(s)) - \sum_{s=1}^n g(s) \prod_{t=s}^{n-1} (1 - f(t)) \right]^{-\frac{1}{r}}$$

for all $n \in \mathbb{Z}_{[0,\gamma]}$, where $\gamma \geq 0$ is chosen such that the RHS of (2.16) is well-defined.



Some New Discrete Nonlinear Delay Inequalities and Application to Discrete Delay Equations

Wing-Sum Cheung and Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 20 of 36

Proof. Define $u \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_0)$ by

$$(2.17) \quad u^r(n) := c^r + \sum_{s=0}^{n-1} x^r(\sigma(s)) \{f(s) + g(s)x^r(\sigma(s))\}, \quad n \in \mathbb{Z}_0.$$

Clearly, $u \geq 0$ is non-decreasing and

$$(2.18) \quad x(n) \leq u(n) \quad \text{for all } n \in \mathbb{Z}_0.$$

Similar to the derivation of (2.6) in the proof of Theorem 2.1, we easily establish

$$x(\sigma(n)) \leq u(n) \quad \text{for all } n \in \mathbb{Z}_0.$$

By (2.17), for any $n \in \mathbb{Z}_0$,

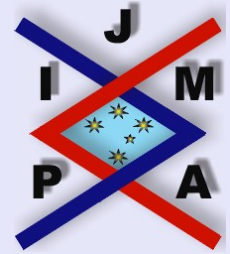
$$\begin{aligned} \Delta u^r(n) &= u^r(n+1) - u^r(n) \\ &= x^r(\sigma(n)) \{f(n) + g(n)x^r(\sigma(n))\} \\ &\leq u^r(n) \{f(n) + g(n)u^r(n)\} \\ &\leq u^r(n+1) \{f(n) + g(n)u^r(n)\}. \end{aligned}$$

As $u(0) = c$, by elementary analysis, we infer from (2.17) that

$$(2.19) \quad u(n) \leq y(n) \quad \text{for all } n \in \mathbb{Z}_{[0,\rho]}$$

where $\mathbb{Z}_{[0,\rho]}$ is the maximal lattice on which the unique solution $y(n)$ to the discrete Bernoulli equation

$$(2.20) \quad \begin{cases} \Delta y^r(n) = y^r(n+1) \{f(n) + g(n)y^r(n)\}, & n \in \mathbb{Z}_0 \\ y(0) = c \end{cases}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

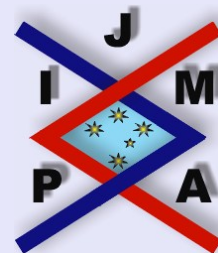
Quit

Page 21 of 36

is defined. Now the unique solution for (2.20) is (see, e.g., [1])

$$(2.21) \quad y(n) = \left[c^{-r} \prod_{s=0}^{n-1} (1 - f(s)) - \sum_{s=1}^n g(s) \prod_{t=s}^{n-1} (1 - f(t)) \right]^{-\frac{1}{r}}$$

for all $n \in \mathbb{Z}_{[0,\gamma]}$. The assertion now follows from (2.18), (2.19) and (2.21). \square



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 22 of 36

3. Immediate Consequences

Direct application of the results in Section 2 yields the following consequences immediately.

Corollary 3.1. *Under Assumptions (A1) – (A4), if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ is a function satisfying the nonlinear delay inequality*

$$(3.1) \quad x^\alpha(n) \leq p(n) + \sum_{s=0}^{n-1} x^{\alpha-1}(\sigma(s)) \{f(s) + g(s)x(\sigma(s)) + h(s)w(x(\sigma(s)))\}$$

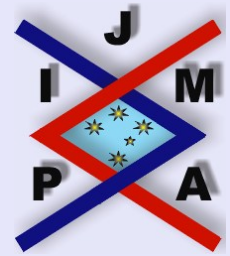
for all $n \in \mathbb{Z}_0$ with initial conditions (I1) and

$$(I4) \quad \psi(\sigma(s)) \leq p^{\frac{1}{\alpha}}(s) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0,$$

where $\alpha \geq 1$ is a constant, then

$$(3.2) \quad x(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(\alpha) \right) \left(p^{\frac{1}{\alpha}}(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(\alpha) \right) \frac{1}{\alpha} \sum_{t=0}^{n-1} h(t) \right\}$$

for all $n \in \mathbb{Z}_{[0, \mu]}$, where $\mu \geq 0$ is chosen such that the RHS of (3.2) is well-defined for all $n \in \mathbb{Z}_{[0, \mu]}$, and Φ is defined as in Theorem 2.1.



Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 23 of 36

Proof. Let $\phi : \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be defined by $\phi(r) = r^\alpha$, $r \in \mathbb{R}_0$. Then ϕ satisfies Assumption (A5). By (3.1) we have

$$\phi(x(n)) \leq p(n) + \sum_{s=0}^{n-1} \phi'(x(\sigma(s))) \left\{ \frac{f(s)}{\alpha} + \frac{g(s)}{\alpha} x(\sigma(s)) + \frac{h(s)}{\alpha} w(x(\sigma(s))) \right\}.$$

Furthermore, it is easy to see that

$$\phi(x(s)) \leq p^{\frac{1}{\alpha}}(s) = \phi^{-1}(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0.$$

Thus Theorem 2.1 applies and the assertion follows. \square

Remark 3.

(i) In Corollary 3.1, if we set $\alpha = 2$, $p(n) \equiv c^2$, $g(n) \equiv 0$, we have

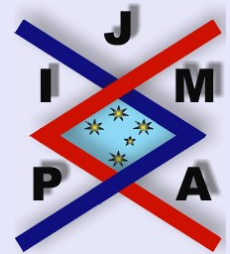
$$x^2(n) \leq c^2 + \sum_{s=0}^{n-1} x(\sigma(s)) \{f(s) + h(s)w(x(\sigma(s)))\}, \quad n \in \mathbb{Z}_0$$

implies

$$x(n) \leq \Phi^{-1} \left\{ \Phi \left[c + \frac{1}{2} \sum_{s=0}^{n-1} f(s) \right] + \frac{1}{2} \sum_{s=0}^{n-1} h(s) \right\}, \quad n \in \mathbb{Z}_{[0, \mu]}.$$

This is the discrete analogue of a result of Pachpatte in [14]. Furthermore, if $\sigma = id$, this reduces to a result of Pachpatte in [18].

(ii) In case $\Phi(\infty) = \infty$, (3.2) holds for all $n \in \mathbb{Z}_0$.



Some New Discrete Nonlinear Delay Inequalities and Application to Discrete Delay Equations

Wing-Sum Cheung and Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 24 of 36

Corollary 3.2. Under Assumptions (A1) – (A4) with $p \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+)$, if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_1)$ satisfies the nonlinear delay inequality

$$(3.3) \quad x^\alpha(n) \leq p(n) + \sum_{s=0}^{n-1} x^\alpha(\sigma(s)) \{f(s) + g(s) \ln x(\sigma(s)) + h(s)w(\ln x(\sigma(s)))\}$$

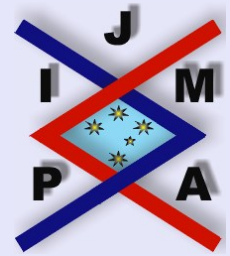
for all $n \in \mathbb{Z}_0$ with initial conditions (I1) and

$$(I5) \quad \psi(\sigma(s)) \leq \frac{1}{\alpha} \ln(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0,$$

where $\alpha > 0$ is a constant, then

$$(3.4) \quad x(n) \leq \exp \left\{ \Phi^{-1} \left[\Phi \left(\left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \times \left(\frac{1}{\alpha} \ln p(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \right) \right) + \left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \frac{1}{\alpha} \sum_{t=0}^{n-1} h(t) \right] \right\}$$

for all $n \in \mathbb{Z}_{[0, \nu]}$, where $\nu \geq 0$ is chosen such that the RHS of (3.4) is well-defined for all $n \in \mathbb{Z}_{[0, \nu]}$, and Φ is defined as in Theorem 2.1.



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 25 of 36

Proof. Letting $y(n) = \ln x(n)$, (3.3) becomes

$$(3.5) \quad \exp(\alpha y(n)) \leq p(n) + \sum_{s=0}^{n-1} \exp(\alpha y(\sigma(s))) \{f(s) + g(s)y(\sigma(s)) + h(s)w(y(\sigma(s)))\}.$$

Let $\phi : \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be defined by $\phi(r) = \exp(\alpha r)$, $r \in \mathbb{R}_0$. Then ϕ satisfies Assumption (A5). Hence from (3.5), we have

$$\phi(y(n)) \leq p(n) + \sum_{s=0}^{n-1} \phi'(y(\sigma(s))) \left\{ \frac{f(s)}{\alpha} + \frac{g(s)}{\alpha} y(\sigma(s)) + \frac{h(s)}{\alpha} w(y(\sigma(s))) \right\}.$$

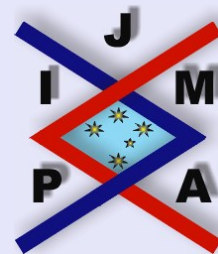
Furthermore, it is easy to see that

$$\psi(\sigma(s)) \leq \frac{1}{\alpha} \ln(p(s)) = \phi^{-1}(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0.$$

Thus Theorem 2.1 applies and we have

$$y(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \left(\frac{1}{\alpha} \ln p(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \frac{1}{\alpha} \sum_{t=0}^{n-1} h(t) \right\}$$

for all $n \in \mathbb{Z}_{[0, \nu]}$, and from this the assertion follows. \square



Some New Discrete Nonlinear Delay Inequalities and Application to Discrete Delay Equations

Wing-Sum Cheung and Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 26 of 36

Remark 4. In case $\Phi(\infty) = \infty$, (3.4) holds for all $n \in \mathbb{Z}_0$.

Corollary 3.3. Under Assumptions (A1) – (A4), if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ satisfies the nonlinear delay inequality

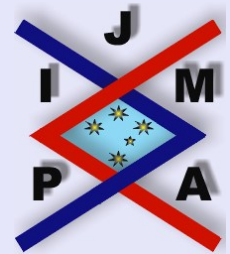
$$(3.6) \quad x^\alpha(n) \leq p(n) + \sum_{s=0}^{n-1} x^{\alpha-1}(\sigma(s)) \left\{ f(s) + g(s)x(\sigma(s)) + h(s) \sum_{t=0}^{s-1} k(t)w(x(\sigma(t))) \right\}$$

for all $n \in \mathbb{Z}_0$ with initial conditions (II) and (I4), where $\alpha \geq 1$ is a constant, then

$$(3.7) \quad x(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \left(p^{\frac{1}{\alpha}}(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \left(\frac{1}{\alpha} \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) \right) \right\}$$

for all $n \in \mathbb{Z}_{[0,\eta]}$, where $\eta \geq 0$ is chosen such that the RHS of (3.7) is well-defined for all $n \in \mathbb{Z}_{[0,\eta]}$, and Φ is defined as in Theorem 2.1.

Proof. Let $\phi : \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be defined by $\phi(r) = r^\alpha$, $r \in \mathbb{R}_0$. Then ϕ satisfies



Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 27 of 36

Assumption (A5). By (3.6),

$$\phi(x(n)) \leq p(n) + \sum_{s=0}^{n-1} \phi'(x(\sigma(s))) \left\{ \frac{f(s)}{\alpha} + \frac{g(s)}{\alpha} x(\sigma(s)) + \frac{h(s)}{\alpha} \sum_{t=0}^{s-1} k(t)w(x(\sigma(t))) \right\}$$

for all $n \in \mathbb{Z}_0$. Furthermore, it is easy to see that

$$\psi(\sigma(s)) \leq p^{\frac{1}{\alpha}}(s) = \phi^{-1}(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0.$$

Thus Theorem 2.2 applies and we have

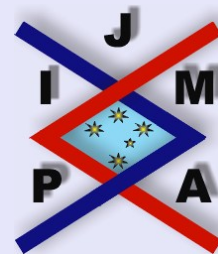
$$x(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \left(p^{\frac{1}{\alpha}}(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \cdot \frac{1}{\alpha} \sum_{s=0}^{n-1} \sum_{t=0}^{s-1} h(s)k(t) \right\}$$

for all $n \in \mathbb{Z}_{[0,n]}$. □

Remark 5.

(i) In Corollary 3.3, if we put $\alpha = 2$, $p(n) \equiv c^2$, $g(n) \equiv 0$, we have

$$x^2(n) \leq c^2 + \sum_{s=0}^{n-1} x(\sigma(s)) \left\{ f(s) + h(s) \sum_{t=0}^{s-1} k(t)w(x(\sigma(t))) \right\}, \quad n \in \mathbb{Z}_0$$



Some New Discrete Nonlinear Delay Inequalities and Application to Discrete Delay Equations

Wing-Sum Cheung and Shiojenn Tseng

Title Page

Contents

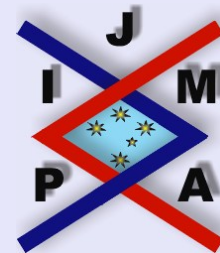


Go Back

Close

Quit

Page 28 of 36



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 29 of 36

implies

$$x(n) \leq \Phi^{-1} \left\{ \Phi \left[c + \frac{1}{2} \sum_{s=0}^{n-1} f(s) \right] + \frac{1}{2} \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) \right\}, \quad n \in \mathbb{Z}_{[0,n]}.$$

This is the discrete analogue of a result of Pachpatte in [14]. Furthermore, if $\sigma = id$ and $w = id$, this reduces to a result of Pachpatte in [18].

(ii) In case $\Phi(\infty) = \infty$, (3.7) holds for all $n \in \mathbb{Z}_0$.

Corollary 3.4. Under Assumptions (A1) – (A4) with $p \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+)$, if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_1)$ satisfies the nonlinear delay inequality

$$(3.8) \quad x^\alpha(n) \leq p(n) + \sum_{s=0}^{n-1} x^\alpha(\sigma(s)) \left\{ f(s) + g(s) \ln x(\sigma(s)) + h(s) \sum_{t=0}^{s-1} k(t) w(\ln x(\sigma(t))) \right\}$$

for all $n \in \mathbb{Z}_0$ with initial conditions (I1) and

$$(I6) \quad \psi(\sigma(s)) \leq \frac{1}{\alpha} \ln(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0,$$

where $\alpha > 0$ is any constant, then

$$(3.9) \quad x(n) \leq \exp \left\{ \Phi^{-1} \left[\Phi \left(\left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \right) \right] \right\}$$

$$\times \left(\frac{1}{\alpha} \ln p(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \right) + \left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \cdot \frac{1}{\alpha} \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) \Bigg\}$$

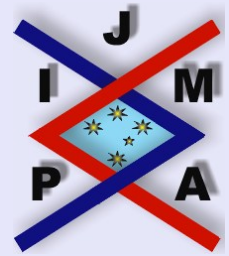
for all $n \in \mathbb{Z}_{[0,\lambda]}$, where $\lambda \geq 0$ is chosen such that the RHS of (3.9) is well-defined for all $n \in \mathbb{Z}_{[0,\lambda]}$, and Φ is defined as in Theorem 2.1.

Proof. Letting $y(n) = \ln x(n)$, (3.8) becomes

$$(3.10) \quad \exp(\alpha y(n)) \leq p(n) + \sum_{s=0}^{n-1} \exp(\alpha y(\sigma(s))) \left\{ f(s) + g(s)y(\sigma(s)) + h(s) \sum_{t=0}^{s-1} k(t)w(y(\sigma(t))) \right\}$$

for all $n \in \mathbb{Z}_0$. Let $\phi : \mathbb{R}_0 \rightarrow \mathbb{R}_0$ be defined by $\phi(r) = \exp(\alpha r)$, $r \in \mathbb{R}_0$. Then ϕ satisfies Assumption (A5). Hence from (3.10), we have

$$\phi(y(n)) \leq p(n) + \sum_{s=0}^{n-1} \phi'(y(\sigma(s))) \times \left\{ \frac{f(s)}{\alpha} + \frac{g(s)}{\alpha} y(\sigma(s)) + \frac{h(s)}{\alpha} \sum_{t=0}^{s-1} k(t)w(y(\sigma(t))) \right\}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 30 of 36

for all $n \in \mathbb{Z}_0$. Furthermore, it is easy to check that

$$\psi(\sigma(s)) \leq \frac{1}{\alpha} \ln(p(s)) = \phi^{-1}(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0.$$

Thus Theorem 2.2 applies and we have

$$y(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \left(\frac{1}{\alpha} \ln p(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \right) \right] + \exp \left(\frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \cdot \frac{1}{\alpha} \sum_{s=0}^{n-1} \sum_{t=0}^{s-1} h(s)k(t) \right\}$$

for all $n \in \mathbb{Z}_{[0,\lambda]}$, and from this the assertion follows. \square

Remark 6.

(i) In Corollary 3.4, if we set $\alpha = 2$, $p(n) \equiv c^2$, $g(n) \equiv 0$, then

$$x^2(n) \leq c^2 + \sum_{s=0}^{n-1} x^2(\sigma(s)) \left\{ f(s) + h(s) \sum_{t=0}^{s-1} k(t) w(\ln x(\sigma(t))) \right\}, \quad n \in \mathbb{Z}_0$$

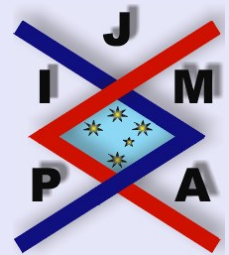
implies

$$x(n) \leq \exp \left\{ \Phi^{-1} \left[\Phi \left(\frac{1}{2} \ln p(n) + \frac{1}{2} \sum_{s=0}^{n-1} f(s) \right) + \frac{1}{2} \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) \right] \right\}$$

$n \in \mathbb{Z}_{[0,\lambda]}.$

This is the discrete version of a result of Pachpatte in [14].

(ii) In case $\Phi(\infty) = \infty$, (3.9) holds for all $n \in \mathbb{Z}_0$.



Some New Discrete Nonlinear Delay Inequalities and Application to Discrete Delay Equations

Wing-Sum Cheung and Shiojenn Tseng

| | |
|---------------|---|
| Title Page | |
| Contents | |
| ⏪ | ⏩ |
| ◀ | ▶ |
| Go Back | |
| Close | |
| Quit | |
| Page 31 of 36 | |

4. Application

Consider the discrete delay equation

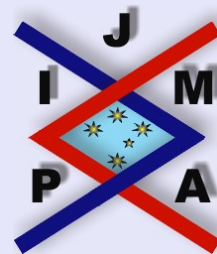
$$(4.1) \quad x^\alpha(n) = F \left(n, x(\sigma(n)), \sum_{s=0}^{n-1} G(n, s, x(\sigma(s))) \right), \quad n \in \mathbb{Z}_0$$

with initial conditions (I1) and (I4), where $\alpha \geq 1$ is a constant, σ, ψ satisfy Assumptions (A3), (A4), $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R})$, $F \in C(\mathbb{Z}_0 \times \mathbb{R}^2, \mathbb{R})$, and $G \in C(\mathbb{Z}_0^2 \times \mathbb{R}, \mathbb{R})$. If F, G satisfy

$$\begin{aligned} |F(n, u, v)| &\leq p(n) + K|v|, \quad n \in \mathbb{Z}_0, u, v \in \mathbb{R}, \\ |G(n, s, v)| &\leq [f(s) + g(s)|v| + h(s)w(|v|)]|v|^{\alpha-1}, \quad n, s \in \mathbb{Z}_0, v \in \mathbb{R}, \end{aligned}$$

for some p, f, g, h, w satisfying (A1) and (A2), and some constant $K > 0$, then every solution of (4.1) satisfies

$$\begin{aligned} |x(n)|^\alpha &= \left| F \left(n, x(\sigma(n)), \sum_{s=0}^{n-1} G(n, s, x(\sigma(s))) \right) \right| \\ &\leq p(n) + K \left| \sum_{s=0}^{n-1} G(n, s, x(\sigma(s))) \right| \\ &\leq p(n) + K \sum_{s=0}^{n-1} |G(n, s, x(\sigma(s)))| \\ &\leq p(n) + K \sum_{s=0}^{n-1} [f(s) + g(s)|x(\sigma(s))| + h(s)w(|x(\sigma(s))|)] |x(\sigma(s))|^{\alpha-1} \end{aligned}$$



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

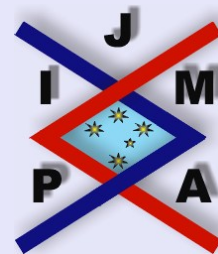
Quit

Page 32 of 36

for all $n \in J(x) :=$ the maximal existence lattice on which x is defined. Applying Corollary 3.1, this yields

$$|x(n)| \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \frac{K}{\alpha} \sum_{s=0}^{n-1} g(\alpha) \right) \left(p^{\frac{1}{\alpha}}(n) + \frac{K}{\alpha} \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \frac{K}{\alpha} \sum_{s=0}^{n-1} g(\alpha) \right) \frac{K}{\alpha} \sum_{t=0}^{n-1} h(t) \right\}$$

for all $n \in J(x) \cap \mathbb{Z}_{[0, \mu]}$. This gives the boundedness of solutions of (4.1).



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

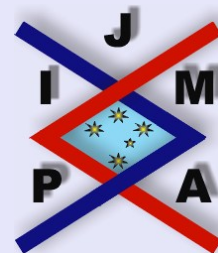
Close

Quit

Page 33 of 36

References

- [1] R.P. AGARWAL, *Difference Equations and Inequalities*, Marcel Dekker, New York, 2000.
- [2] D. BAINOV AND P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1992.
- [3] E.F. BECKENBACH AND R. BELLMAN, *Inequalities*, Springer-Verlag, New York, 1961.
- [4] R. BELLMAN, The stability of solutions of linear differential equations, *Duke Math. J.*, **10** (1943), 643–647.
- [5] I. BIHARI, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Sci. Hungar.*, **7** (1956), 71–94.
- [6] W.S. CHEUNG, On some new integrodifferential inequalities of the Gronwall and Wendroff type, *J. Math. Anal. Appl.*, **178** (1993), 438–449.
- [7] W.S. CHEUNG AND Q.H. MA, Nonlinear retarded integral inequalities for functions in two variables, to appear in *J. Concrete Appl. Math.*
- [8] T.H. GRONWALL, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Ann. Math.*, **20** (1919), 292–296.
- [9] H. HARAUX, *Nonlinear Evolution Equation: Global Behavior of Solutions*, Lecture Notes in Mathematics, v.**841**, Springer-Verlag, Berlin, 1981.



Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



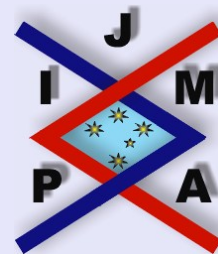
Go Back

Close

Quit

Page 34 of 36

- [10] Q.M. MA AND E.H. YANG, On some new nonlinear delay integral inequalities, *J. Math. Anal. Appl.*, **252** (2000), 864–878.
- [11] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, New York, 1970.
- [12] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.
- [13] L. OU-IANG, The boundedness of solutions of linear differential equations $y'' + A(t)y = 0$, *Shuxue Jinzhan*, **3** (1957), 409–415.
- [14] B.G. PACHPATTE, A note on certain integral inequalities with delay, *Period. Math. Hungar.*, **31** (1995), 229–234.
- [15] B.G. PACHPATTE, Explicit bounds on certain integral inequalities, *J. Math. Anal. Appl.*, **267** (2002), 48–61.
- [16] B.G. PACHPATTE, *Inequalities for Finite Difference Equations*, Marcel Dekker, New York, 2002.
- [17] B.G. PACHPATTE, On some new inequalities related to a certain inequality arising in the theory of differential equations, *J. Math. Anal. Appl.*, **251** (2000), 736–751.
- [18] B.G. PACHPATTE, On some new inequalities related to certain inequalities in the theory of differential equations, *J. Math. Anal. Appl.*, **189** (1995), 128–144.



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 35 of 36

- [19] P. Ch. TSAMATOS AND S.K. NTOUYAS, On a Bellman-Bihari type inequality with delay, *Period. Math. Hungar.*, **23** (1991), 91–94.
- [20] E.H. YANG, Generalizations of Pachpatte’s integral and discrete inequalities, *Ann. Differential Equations*, **13** (1997), 180–188.



**Some New Discrete Nonlinear
Delay Inequalities and
Application to Discrete Delay
Equations**

Wing-Sum Cheung and
Shiojenn Tseng

Title Page

Contents



Go Back

Close

Quit

Page 36 of 36