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SOME NEW DISCRETE NONLINEAR DELAY INEQUALITIES AND APPLICATION TO DISCRETE DELAY EQUATIONS

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Abstract

In this paper, some new discrete Gronwall-Bellman-Ou-Iang-type inequalities are established. These on the one hand generalize some existing results and on the other hand provide a handy tool for the study of qualitative as well as quantitative properties of solutions of difference equations.

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Contents

1. Introduction

It is widely recognized that integral inequalities in general provide an effective tool for the study of qualitative as well as quantitative properties of solutions of integral and differential equations. While most integral inequalities only give the 'global behavior' of the unknown functions (in the sense that bounds are only obtained for integrals of certain functions of the unknown functions), the Gronwall-Bellman type (see, e.g. [3] – [8], [10] – [12], [15] – [18]) is particularly useful as they provide explicit pointwise bounds of the unknown functions. A specific branch of this type of inequalities is originated by Ou-Iang. In his paper [13], in order to study the boundedness behavior of the solutions of some 2nd order differential equations, Ou-Iang established the following beautiful inequality.

Theorem 1.1 (Ou-Iang [13]). *If u and f are non-negative functions on $[0,\infty)$ satisfying*

$$u^2(x) \le c^2 + 2\int_0^x f(s)u(s)ds, \quad x \in [0,\infty),$$

for some constant $c \ge 0$, then

$$u(x) \le c + \int_0^x f(s)ds, \quad x \in [0,\infty).$$

While Ou-Iang's inequality is interesting in its own right, it also has numerous important applications in the study of differential equations (see, e.g., [2, 3, 9, 11, 12]). Over the years, various extensions of Ou-Iang's inequality have been established. These include, among others, works of Agarwal [1], Ma-Yang [10], Pachpatte [14] – [18], Tsamatos-Ntouyas [19], and Yang [20]. Among such extensions, the discretization is of particular interest because analogous to the continuous case, discrete versions of integral inequalities should, in our opinion, play an important role in the study of qualitative as well as quantitative properties of solutions of difference equations.

It is the purpose of this paper to establish some new discrete Gronwall-Bellman-Ou-Iang-type inequalities giving explicit bounds to unknown discrete functions. These on the one hand generalize some existing results in the literature and on the other hand give a handy tool to the study of difference equations. An application to a discrete delay equation is given at the end of the paper.

2. Discrete Inequalities with Delay

Throughout this paper, $\mathbb{R}_+ = (0, \infty) \subset \mathbb{R}$, $\mathbb{Z}_+ = \mathbb{R}_+ \cap \mathbb{Z}$, and for any $a, b \in \mathbb{R}$, $\mathbb{R}_a = [a, \infty)$, $\mathbb{Z}_a = \mathbb{R}_a \cap \mathbb{Z}$, $\mathbb{Z}_{[a,b]} = \mathbb{Z} \cap [a, b]$. If X and Y are sets, the collection of functions of X into Y, the collection of continuous functions of X into Y, and that of continuously differentiable functions of X into Y are denoted by $\mathcal{F}(X, Y)$, $C(X, Y)$, and $C^1(X, Y)$, respectively. As usual, if u is a real-valued function on $\mathbb{Z}_{[a,b]}$, the difference operator Δ on u is defined as

$$\Delta u(n) = u(n + 1) - u(n) \ , \quad n \in \mathbb{Z}_{[a,b-1]}.$$

In the sequel, summations over empty sets are, as usual, defined to be zero.

The basic assumptions and initial conditions used in this paper are the following:

Assumptions

(A1) $f, g, h, k, p \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_0)$ with p non-decreasing;

(A2) $w \in C(\mathbb{R}_0, \mathbb{R}_0)$ is non-decreasing with $w(r) > 0$ for $r > 0$;

(A3) $\sigma \in \mathcal{F}(\mathbb{Z}_0, \mathbb{Z})$ with $\sigma(s) \le s$ for all $s \in \mathbb{Z}_0$ and $-\infty < a := \inf\{\sigma(s) : s \in \mathbb{Z}_0\} \le 0$;

(A4) $\psi \in \mathcal{F}(\mathbb{Z}_{[a,0]}, \mathbb{R}_0)$; and

(A5) $\phi \in C^1(\mathbb{R}_0, \mathbb{R}_0)$ with ϕ' non-decreasing and $\phi'(r) > 0$ for $r > 0$.

Initial Conditions

(I1) $x(s) = \psi(s)$ for all $s \in \mathbb{Z}_{[a,0]}$;

(I2) $\psi(\sigma(s)) \le \phi^{-1}(p(s))$ for all $s \in \mathbb{Z}_0$ with $\sigma(s) \le 0$.

Theorem 2.1. *Under Assumptions (A1) – (A5), if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ is a function satisfying the nonlinear delay inequality*

$$\begin{aligned} (2.1) \quad & \phi\left(x(n)\right) \\ & \leq p(n) + \sum_{s=0}^{n-1} \phi'\left(x\left(\sigma(s)\right)\right)\left\{f(s) + g(s)x\left(\sigma(s)\right) + h(s)w\left(x\left(\sigma(s)\right)\right)\right\} \end{aligned}$$

for all $n \in \mathbb{Z}_0$ with initial conditions (I1) – (I2), then

$$\begin{aligned} (2.2) \quad x(n) \leq \Phi^{-1}\Bigg\{\Phi\Bigg[\left(\exp\sum_{s=0}^{n-1} g(s)\right)\left(\phi^{-1}\left(p(n)\right) + \sum_{s=0}^{n-1} f(s)\right)\Bigg] \\ + \left(\exp\sum_{s=0}^{n-1} g(s)\right)\sum_{t=0}^{n-1} h(t)\Bigg\} \end{aligned}$$

for all $n \in \mathbb{Z}_{[0,\alpha]}$, where $\Phi \in C(\mathbb{R}_0, \mathbb{R})$ is defined by

$$\Phi(r) := \int_1^r \frac{ds}{w(s)}, \quad r > 0,$$

and $\alpha \geq 0$ is chosen such that the RHS of (2.2) is well-defined, that is,

$$\begin{aligned} \Phi\Bigg[\left(\exp\sum_{s=0}^{n-1} g(s)\right)\left(\phi^{-1}\left(p(n)\right) + \sum_{s=0}^{n-1} f(s)\right)\Bigg] \\ + \exp\left(\sum_{s=0}^{n-1} g(s)\right)\sum_{t=0}^{n-1} h(t) \in \mathcal{I}_m\Phi \end{aligned}$$

for all $n \in \mathbb{Z}_{[0,\alpha]}$.

Proof. Fix $\varepsilon > 0$ and $N \in \mathbb{Z}_{[0,\alpha]}$. Define $u : \mathbb{Z}_{[0,N]} \to \mathbb{R}_0$ by

$$u(n) := \phi^{-1}\left\{\varepsilon + p(N) + \sum_{t=0}^{n-1} \phi'\left(x\left(\sigma(t)\right)\right)\left[f(t) + g(t)x\left(\sigma(t)\right) + h(t)w\left(x\left(\sigma(t)\right)\right)\right]\right\}. \tag{2.3}$$

By (A5), u is non-decreasing on $\mathbb{Z}_{[0,N]}$. For any $n \in \mathbb{Z}_{[0,N]}$, by (A5) again,

$$u(n) \geq \phi^{-1}\left(\varepsilon + p(N)\right) > 0\,. \tag{2.4}$$

As $\phi\left(u(n)\right) > \phi\left(x(n)\right)$, we have

$$u(n) > x(n)\,. \tag{2.5}$$

Next, observe that if $\sigma(n) \geq 0$, then by (A3), $\sigma(n) \in \mathbb{Z}_{[0,N]}$ and so

$$x\left(\sigma(n)\right) < u\left(\sigma(n)\right) \leq u(n)\,.$$

On the other hand, if $\sigma(n) \leq 0$, then by (A3) again, $\sigma(n) \in \mathbb{Z}_{[a,0]}$ and so by (I1), (I2), (A1), (A5) and (2.4),

$$x\left(\sigma(n)\right) = \psi\left(\sigma(n)\right) \leq \phi^{-1}\left(p(n)\right) \leq \phi^{-1}\left(p(N)\right) \leq \phi^{-1}\left(p(N) + \varepsilon\right) \leq u(n)\,.$$

Hence we always have

$$x\left(\sigma(n)\right) \leq u(n) \quad \text{for all } n \in \mathbb{Z}_{[0,N]}\,. \tag{2.6}$$

Therefore, for any $s \in \mathbb{Z}_{[0,N-1]}$, by (2.3) and (2.6),

$$\begin{aligned}\Delta(\phi \circ u)(s) &= \phi\left(u(s+1)\right) - \phi\left(u(s)\right) \\ &= \phi'\left(x\left(\sigma(s)\right)\right)\left\{f(s) + g(s)x\left(\sigma(s)\right) + h(s)w\left(x\left(\sigma(s)\right)\right)\right\} \\ &\leq \phi'\left(u(s)\right)\left\{f(s) + g(s)u(s) + h(s)w\left(u(s)\right)\right\} .\end{aligned}$$

On the other hand, by the Mean Value Theorem, we obtain

$$\begin{aligned}\Delta(\phi \circ u)(s) &= \phi\left(u(s+1)\right) - \phi\left(u(s)\right) \\ &= \phi'(\xi)\Delta u(s)\end{aligned}$$

for some $\xi \in [u(s), u(s+1)]$. Observe that by (2.4) and (A5), $\phi'(\xi) > 0$. Thus by the monotonicity of ϕ', for any $s \in \mathbb{Z}_{[0,N-1]}$,

$$\begin{aligned}\Delta u(s) &\leq \frac{\phi'\left(u(s)\right)}{\phi'(\xi)}\left\{f(s) + g(s)u(s) + h(s)w\left(u(s)\right)\right\} \\ &\leq f(s) + g(s)u(s) + h(s)w\left(u(s)\right) .\end{aligned}$$

Summing up, we have

$$\begin{aligned}u(n) - u(0) &= \sum_{s=0}^{n-1} \Delta u(s) \\ &\leq \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s)w\left(u(s)\right) + \sum_{s=0}^{n-1} g(s)u(s) ,\end{aligned}$$

or

$$u(n) \le \left[\phi^{-1}\left(\varepsilon + p(N)\right) + \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s) w\left(u(s)\right) \right] + \sum_{s=0}^{n-1} g(s) u(s)$$

for all $n \in \mathbb{Z}_{[0,N]}$. Hence by the discrete version of the Gronwall-Bellman inequality (see, e.g., [16, Corollary 1.2.5]),

$$\begin{aligned} u(n) &\le \left[\phi^{-1}\left(\varepsilon + p(N)\right) + \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s) w\left(u(s)\right) \right] \exp \sum_{s=0}^{n-1} g(s) \\ &\le \left[\phi^{-1}\left(\varepsilon + p(N)\right) + \sum_{s=0}^{N-1} f(s) + \sum_{s=0}^{n-1} h(s) w\left(u(s)\right) \right] \exp \sum_{s=0}^{N-1} g(s) \end{aligned} \tag{2.7}$$

for all $n \in \mathbb{Z}_{[0,N]}$. Denote by $v(n)$ the RHS of (2.7). Then v is non-decreasing and for all $n \in \mathbb{Z}_{[0,N]}$,

$$u(n) \le v(n)\,. \tag{2.8}$$

Therefore, for any $t \in \mathbb{Z}_{[0,N-1]}$,

$$\begin{aligned} \Delta v(t) &= v(t+1) - v(t) \\ &= h(t) w\left(u(t)\right) \exp \sum_{s=0}^{N-1} g(s) \\ &\le h(t) w\left(v(t)\right) \exp \sum_{s=0}^{N-1} g(s)\,. \end{aligned}$$

On the other hand, by the Mean Value Theorem, we have

$$\begin{aligned}\Delta(\Phi\circ v)(t) &= \Phi\left(v(t+1)\right)-\Phi\left(v(t)\right)\\ &= \Phi'(\eta)\Delta v(t)\\ &= \frac{1}{w(\eta)}\Delta v(t)\end{aligned}$$

for some $\eta\in[v(t),v(t+1)]$. Observe that by (2.4), (2.8), and (A2), $w(\eta)>0$. Therefore, as w is non-decreasing,

$$\begin{aligned}\Delta(\Phi\circ v)(t) &\le \frac{1}{w(\eta)}h(t)w\left(v(t)\right)\exp\sum_{s=0}^{N-1}g(s)\\ &\le h(t)\exp\sum_{s=0}^{N-1}g(s)\end{aligned}$$

for all $t\in\mathbb{Z}_{[0,N-1]}$. Summing up, we have

$$\sum_{t=0}^{n-1}\Delta(\Phi\circ v)(t)\le\sum_{t=0}^{n-1}h(t)\exp\sum_{s=0}^{N-1}g(s)\,.$$

On the other hand,

$$\sum_{t=0}^{n-1}\Delta(\Phi\circ v)(t)=\Phi\left(v(n)\right)-\Phi\left(v(0)\right)$$

$$= \Phi\left(v(n)\right) - \Phi\left[\left(\exp\sum_{s=0}^{N-1} g(s)\right)\left(\phi^{-1}\left(\varepsilon + p(N)\right) + \sum_{s=0}^{N-1} f(s)\right)\right],$$

therefore,

$$\Phi\left(v(n)\right) \leq \Phi\left[\left(\exp\sum_{s=0}^{N-1} g(s)\right)\left(\phi^{-1}\left(\varepsilon + p(N)\right) + \sum_{s=0}^{N-1} f(s)\right)\right] + \sum_{t=0}^{n-1} h(t)\exp\sum_{s=0}^{N-1} g(s)$$

for all $n \in \mathbb{Z}_{[0,N]}$. In particular, taking $n = N$ we have

$$\Phi\left(v(N)\right) \leq \Phi\left[\left(\exp\sum_{s=0}^{N-1} g(s)\right)\left(\phi^{-1}\left(\varepsilon + p(N)\right) + \sum_{s=0}^{N-1} f(s)\right)\right] + \left(\exp\sum_{s=0}^{N-1} g(s)\right)\sum_{t=0}^{N-1} h(t).$$

Since $N \in \mathbb{Z}_{[0,\alpha]}$ is arbitrary,

$$\Phi\left(v(n)\right) \leq \Phi\left[\left(\exp\sum_{s=0}^{n-1} g(s)\right)\left(\phi^{-1}\left(\varepsilon + p(n)\right) + \sum_{s=0}^{n-1} f(s)\right)\right] + \left(\exp\sum_{s=0}^{n-1} g(s)\right)\sum_{t=0}^{n-1} h(t)$$

for all $n \in \mathbb{Z}_{[0,\alpha]}$. Hence

$$v(n) \leq \Phi^{-1}\left\{\Phi\left[\left(\exp\sum_{s=0}^{n-1} g(s)\right)\left(\phi^{-1}\left(\varepsilon + p(n)\right) + \sum_{s=0}^{n-1} f(s)\right)\right] + \left(\exp\sum_{s=0}^{n-1} g(s)\right)\sum_{t=0}^{n-1} h(t)\right\}$$

and so by (2.5) and (2.8),

$$\begin{aligned} x(n) < u(n) \leq v(n) \\ \leq \Phi^{-1}\left\{\Phi\left[\left(\exp\sum_{s=0}^{n-1} g(s)\right)\left(\phi^{-1}\left(\varepsilon + p(n)\right) + \sum_{s=0}^{n-1} f(s)\right)\right] + \left(\exp\sum_{s=0}^{n-1} g(s)\right)\sum_{t=0}^{n-1} h(t)\right\} \end{aligned}$$

for all $n \in \mathbb{Z}_{[0,\alpha]}$. Finally, letting $\varepsilon \to 0^+$, we conclude that

$$x(n) \leq \Phi^{-1}\left\{\Phi\left[\left(\exp\sum_{s=0}^{n-1} g(s)\right)\left(\phi^{-1}\left(p(n)\right) + \sum_{s=0}^{n-1} f(s)\right)\right] + \left(\exp\sum_{s=0}^{n-1} g(s)\right)\sum_{t=0}^{n-1} h(t)\right\}$$

for all $n \in \mathbb{Z}_{[0,\alpha]}$. □

Remark 1. *In many cases the non-decreasing function* w *satisfies* $\int_1^\infty \frac{ds}{w(s)} = \infty$. *For example,* w = *constant* > 0, $w(s) = \sqrt{s}$, *etc., are such functions. In such cases* $\Phi(\infty) = \infty$ *and so we may take* $\alpha \to \infty$, *that is, (2.2) is valid for all* $n \in \mathbb{Z}_0$.

Theorem 2.2. *Under Assumptions (A1) – (A5), if* $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ *is a function satisfying the nonlinear delay inequality*

$$\phi(x(n)) \le p(n) + \sum_{s=0}^{n-1} \phi'(x(\sigma(s))) \Big\{ f(s) + g(s) x(\sigma(s)) + h(s) \sum_{t=0}^{s-1} k(t) w(x(\sigma(t))) \Big\}$$

for all $n \in \mathbb{Z}_0$ *with initial conditions (I1) – (I2), then*

$$x(n) \le \Phi^{-1} \Bigg\{ \Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1}(p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \sum_{s=0}^{n-1} g(s) \right) \sum_{s=0}^{n-1} \sum_{t=0}^{s-1} h(s) k(t) \Bigg\} \tag{2.9}$$

for all $n \in \mathbb{Z}_{[0,\beta]}$, *where* $\Phi \in C(\mathbb{R}_0, \mathbb{R})$ *is as defined in Theorem 2.1, and* $\beta \ge 0$ *is chosen such that the RHS of (2.9) is well-defined, that is,*

$$\Phi \left[\left(\exp \sum_{s=0}^{n-1} g(s) \right) \left(\phi^{-1}(p(n)) + \sum_{s=0}^{n-1} f(s) \right) \right]$$

$$+\left(\exp\sum_{s=0}^{n-1} g(s)\right)\sum_{s=0}^{n-1}\sum_{t=0}^{s-1} h(s)k(t) \in \mathcal{I}_m\Phi$$

for all $n \in \mathbb{Z}_{[0,\beta]}$.

Proof. Fix $\varepsilon > 0$ and $M \in \mathbb{Z}_{[0,\beta]}$. Define $u : \mathbb{Z}_{[0,M]} \to \mathbb{R}_0$ by

$$u(n) := \phi^{-1}\left\{\varepsilon + p(M) + \sum_{\delta=0}^{n-1} \phi'\left(x\left(\sigma(\delta)\right)\right) \cdot \left[f(\delta) + g(\delta)x\left(\sigma(\delta)\right) + h(\delta)\sum_{t=0}^{\delta-1} k(t)w\left(x\left(\sigma(t)\right)\right)\right]\right\}. \tag{2.10}$$

By (A5), u is non-decreasing on $\mathbb{Z}_{[0,M]}$. For any $n \in \mathbb{Z}_{[0,M]}$, by (A5) again,

$$u(n) \geq \phi^{-1}\left(\varepsilon + p(M)\right) > 0\,. \tag{2.11}$$

As $\phi\left(u(n)\right) > \phi\left(x(n)\right)$, we have

$$u(n) > x(n)\,. \tag{2.12}$$

Using the same arguments as in the derivation of (2.6) in the proof of Theorem 2.1, we have

$$x\left(\sigma(n)\right) \leq u(n) \quad \text{for all } n \in \mathbb{Z}_{[0,M]}\,. \tag{2.13}$$

Hence for any $s \in \mathbb{Z}_{[0,M-1]}$, by (2.10) and (2.13),

$$\begin{aligned}\Delta(\phi \circ u)(s) &= \phi(u(s+1)) - \phi(u(s)) \\ &= \phi'(x(\sigma(s)))\left\{f(s) + g(s)x(\sigma(s)) + h(s)\sum_{t=0}^{s-1} k(t)w(x(\sigma(t)))\right\} \\ &\le \phi'(u(s))\left\{f(s) + g(s)u(s) + h(s)\sum_{t=0}^{s-1} k(t)w(u(t))\right\}.\end{aligned}$$

On the other hand, by the Mean Value Theorem,

$$\begin{aligned}\Delta(\phi \circ u)(s) &= \phi(u(s+1)) - \phi(u(s)) \\ &= \phi'(\xi)\Delta u(s)\end{aligned}$$

for some $\xi \in [u(s), u(s+1)]$. Observe that by (2.12) and (A5), $\phi'(\xi) > 0$. Thus by the monotonicity of ϕ', for any $s \in \mathbb{Z}_{[0,M-1]}$,

$$\begin{aligned}\Delta u(s) &\le \frac{\phi'(u(s))}{\phi'(\xi)}\left\{f(s) + g(s)u(s) + h(s)\sum_{t=0}^{s-1} k(t)w(u(t))\right\} \\ &\le f(s) + g(s)u(s) + h(s)\sum_{t=0}^{s-1} k(t)w(u(t)).\end{aligned}$$

Summing up, we have

$$u(n) - u(0) = \sum_{s=0}^{n-1} \Delta u(s)$$

$$\leq \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) w\left(u(t)\right) + \sum_{s=0}^{n-1} g(s) u(s)\ ,$$

or

$$u(n) \leq \left[\phi^{-1}\left(\varepsilon + p(M)\right) + \sum_{s=0}^{n-1} f(s) + \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) w\left(u(t)\right)\right] + \sum_{s=0}^{n-1} g(s) u(s)$$

for all $n \in \mathbb{Z}_{[0,M]}$. Hence by the discrete version of the Gronwall-Bellman inequality (see, e.g., [16, Corollary 1.2.5]),

$$\begin{aligned} u(n) &\leq \left[\phi^{-1}\left(\varepsilon + p(M)\right) + \sum_{s=0}^{n-1} f(s) \right. \\ &\qquad \left. + \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) w\left(u(t)\right)\right] \exp \sum_{s=0}^{n-1} g(s) \\ &\leq \left[\phi^{-1}\left(\varepsilon + p(M)\right) + \sum_{s=0}^{M-1} f(s) \right. \\ &\qquad \left. + \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) w\left(u(t)\right)\right] \exp \sum_{s=0}^{M-1} g(s) \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{Z}_{[0,M]}$. Denote by $v(n)$ the RHS of (2.14). Then v is non-decreasing and for all $n \in \mathbb{Z}_{[0,M]}$,

$$u(n) \leq v(n)\ . \tag{2.15}$$

Therefore, for any $\delta \in \mathbb{Z}_{[0,M-1]}$,

$$\begin{aligned}\Delta v(\delta) &= v(\delta+1) - v(\delta)\\ &= h(\delta)\left(\sum_{t=0}^{\delta-1} k(t) w\left(u(t)\right)\right) \exp \sum_{s=0}^{M-1} g(s)\\ &\leq h(\delta)\left(\sum_{t=0}^{\delta-1} k(t) w\left(v(t)\right)\right) \exp \sum_{s=0}^{M-1} g(s)\\ &\leq h(\delta) w\left(v(\delta)\right)\left(\sum_{t=0}^{\delta-1} k(t)\right) \exp \sum_{s=0}^{M-1} g(s) .\end{aligned}$$

On the other hand, by the Mean Value Theorem,

$$\begin{aligned}\Delta(\Phi \circ v)(\delta) &= \Phi\left(v(\delta+1)\right) - \Phi\left(v(\delta)\right)\\ &= \Phi'(\eta)\Delta v(\delta) = \frac{1}{w(\eta)}\Delta v(\delta)\end{aligned}$$

for some $\eta \in [v(\delta), v(\delta+1)]$. Observe that by (2.11), (2.14), and (A2), $w(\eta) > 0$. Therefore, as w is non-decreasing,

$$\begin{aligned}\Delta(\Phi \circ v)(\delta) &\leq \frac{1}{w(\eta)} h(\delta) w\left(v(\delta)\right)\left(\sum_{t=0}^{\delta-1} k(t)\right) \exp \sum_{s=0}^{M-1} g(s)\\ &\leq h(\delta)\left(\sum_{t=0}^{\delta-1} k(t)\right) \exp \sum_{s=0}^{M-1} g(s)\end{aligned}$$

for all $\delta \in \mathbb{Z}_{[0,M-1]}$. Summing up, we have

$$\sum_{\delta=0}^{n-1} \Delta(\Phi \circ v)(\delta) \leq \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s) ,$$

or

$$\begin{aligned}
\Phi\left(v(n)\right) &\leq \Phi\left(v(0)\right) + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s) \\
&= \Phi\left[\left(\phi^{-1}\left(\varepsilon + p(M)\right) + \sum_{s=0}^{M-1} f(s) \right) \exp \sum_{s=0}^{M-1} g(s) \right] \\
&\quad + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s)
\end{aligned}$$

for all $n \in \mathbb{Z}_{[0,M]}$. In particular, taking $n = M$ this yields

$$\begin{aligned}
\Phi\left(v(M)\right) \leq \Phi&\left[\left(\phi^{-1}\left(\varepsilon + p(M)\right) + \sum_{s=0}^{M-1} f(s) \right) \exp \sum_{s=0}^{M-1} g(s) \right] \\
&\quad + \sum_{\delta=0}^{M-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t) \right) \exp \sum_{s=0}^{M-1} g(s) .
\end{aligned}$$

Since $M \in \mathbb{Z}_{[0,\beta]}$ is arbitrary,

$$\Phi\left(v(n)\right) \leq \Phi\left[\left(\phi^{-1}\left(\varepsilon + p(n)\right) + \sum_{s=0}^{n-1} f(s)\right) \exp \sum_{s=0}^{n-1} g(s)\right] + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t)\right) \exp \sum_{s=0}^{n-1} g(s)$$

for all $n \in \mathbb{Z}_{[0,\beta]}$. Hence

$$v(n) \leq \Phi^{-1}\left\{\Phi\left[\left(\phi^{-1}\left(\varepsilon + p(n)\right) + \sum_{s=0}^{n-1} f(s)\right) \exp \sum_{s=0}^{n-1} g(s)\right] + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t)\right) \exp \sum_{s=0}^{n-1} g(s)\right\}$$

and so by (2.12) and (2.15),

$$\begin{aligned} x(n) < u(n) \leq v(n) \\ \leq \Phi^{-1}\left\{\Phi\left[\left(\phi^{-1}\left(\varepsilon + p(n)\right) + \sum_{s=0}^{n-1} f(s)\right) \exp \sum_{s=0}^{n-1} g(s)\right] + \sum_{\delta=0}^{n-1} h(\delta) \left(\sum_{t=0}^{\delta-1} k(t)\right) \exp \sum_{s=0}^{n-1} g(s)\right\} \end{aligned}$$

for all $n \in \mathbb{Z}_{[0,\beta]}$. Finally, letting $\varepsilon \to 0^+$, we conclude that

$$x(n) \leq \Phi^{-1}\left\{\Phi\left[\left(\exp\sum_{s=0}^{n-1} g(s)\right)\left(\phi^{-1}(p(n)) + \sum_{s=0}^{n-1} f(s)\right)\right]\right. \\ \left. + \left(\exp\sum_{s=0}^{n-1} g(s)\right)\sum_{\delta=0}^{n-1}\sum_{t=0}^{\delta-1} h(\delta)k(t)\right\}$$

for all $n \in \mathbb{Z}_{[0,\beta]}$. □

Remark 2. *Similar to the previous remark, in case $\Phi(\infty) = \infty$, (2.9) holds for all $n \in \mathbb{Z}_0$.*

Theorem 2.3. *Under Assumptions (A1), (A3) and (A4), if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ is a function satisfying the nonlinear delay inequality*

$$x^r(n) \leq c^r + \sum_{s=0}^{n-1} x^r(\sigma(s))\{f(s) + g(s)x^r(\sigma(s))\}, \quad n \in \mathbb{Z}_0,$$

with initial conditions (I1) and

(I3) $$\psi(\sigma(s)) \leq c \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0,$$

where $r, c > 0$ are constants, then

(2.16) $$x(n) \leq \left[c^{-r}\prod_{s=0}^{n-1}(1 - f(s)) - \sum_{s=1}^{n} g(s)\prod_{t=s}^{n-1}(1 - f(t))\right]^{-\frac{1}{r}}$$

for all $n \in \mathbb{Z}_{[0,\gamma]}$, where $\gamma \geq 0$ is chosen such that the RHS of (2.16) is well-defined.

Proof. Define $u \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_0)$ by

$$u^r(n) := c^r + \sum_{s=0}^{n-1} x^r(\sigma(s))\{f(s) + g(s)x^r(\sigma(s))\}, \quad n \in \mathbb{Z}_0 . \tag{2.17}$$

Clearly, $u \geq 0$ is non-decreasing and

$$x(n) \leq u(n) \quad \text{for all } n \in \mathbb{Z}_0 . \tag{2.18}$$

Similar to the derivation of (2.6) in the proof of Theorem 2.1, we easily establish

$$x(\sigma(n)) \leq u(n) \quad \text{for all } n \in \mathbb{Z}_0 .$$

By (2.17), for any $n \in \mathbb{Z}_0$,

$$\begin{aligned} \Delta u^r(n) &= u^r(n+1) - u^r(n) \\ &= x^r(\sigma(n))\{f(n) + g(n)x^r(\sigma(n))\} \\ &\leq u^r(n)\{f(n) + g(n)u^r(n)\} \\ &\leq u^r(n+1)\{f(n) + g(n)u^r(n)\} . \end{aligned}$$

As $u(0) = c$, by elementary analysis, we infer from (2.17) that

$$u(n) \leq y(n) \quad \text{for all } n \in \mathbb{Z}_{[0,\rho]} \tag{2.19}$$

where $\mathbb{Z}_{[0,\rho]}$ is the maximal lattice on which the unique solution $y(n)$ to the discrete Bernoulli equation

$$\begin{cases} \Delta y^r(n) = y^r(n+1)\{f(n) + g(n)y^r(n)\}, & n \in \mathbb{Z}_0 \\ y(0) = c \end{cases} \tag{2.20}$$

is defined. Now the unique solution for (2.20) is (see, e.g., [1])

$$y(n) = \left[c^{-r} \prod_{s=0}^{n-1} (1 - f(s)) - \sum_{s=1}^{n} g(s) \prod_{t=s}^{n-1} (1 - f(t)) \right]^{-\frac{1}{r}} \tag{2.21}$$

for all $n \in \mathbb{Z}_{[0,\gamma]}$. The assertion now follows from (2.18), (2.19) and (2.21). □

3. Immediate Consequences

Direct application of the results in Section 2 yields the following consequences immediately.

Corollary 3.1. *Under Assumptions (A1) – (A4), if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ is a function satisfying the nonlinear delay inequality*

$$(3.1)\quad x^{\alpha}(n) \le p(n) + \sum_{s=0}^{n-1} x^{\alpha-1}(\sigma(s))\left\{f(s) + g(s)x(\sigma(s)) + h(s)w(x(\sigma(s)))\right\}$$

for all $n \in \mathbb{Z}_0$ with initial conditions (I1) and

$$(\text{I4})\qquad \psi(\sigma(s)) \le p^{\frac{1}{\alpha}}(s) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \le 0\,,$$

where $\alpha \ge 1$ is a constant, then

$$(3.2)\quad x(n) \le \Phi^{-1}\left\{\Phi\left[\left(\exp\frac{1}{\alpha}\sum_{s=0}^{n-1} g(\alpha)\right)\left(p^{\frac{1}{\alpha}}(n) + \frac{1}{\alpha}\sum_{s=0}^{n-1} f(s)\right)\right] + \left(\exp\frac{1}{\alpha}\sum_{s=0}^{n-1} g(\alpha)\right)\frac{1}{\alpha}\sum_{t=0}^{n-1} h(t)\right\}$$

for all $n \in \mathbb{Z}_{[0,\mu]}$, where $\mu \ge 0$ is chosen such that the RHS of (3.2) is well-defined for all $n \in \mathbb{Z}_{[0,\mu]}$, and Φ is defined as in Theorem 2.1.

Proof. Let $\phi : \mathbb{R}_0 \to \mathbb{R}_0$ be defined by $\phi(r) = r^{\alpha}$, $r \in \mathbb{R}_0$. Then ϕ satisfies Assumption (A5). By (3.1) we have

$$\phi\left(x(n)\right) \leq p(n) + \sum_{s=0}^{n-1} \phi'\left(x\left(\sigma(s)\right)\right) \left\{ \frac{f(s)}{\alpha} + \frac{g(s)}{\alpha} x\left(\sigma(s)\right) + \frac{h(s)}{\alpha} w\left(x\left(\sigma(s)\right)\right) \right\} .$$

Furthermore, it is easy to see that

$$\phi\left(x(s)\right) \leq p^{\frac{1}{\alpha}}(s) = \phi^{-1}\left(p(s)\right) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0 .$$

Thus Theorem 2.1 applies and the assertion follows. □

Remark 3.

(i) *In Corollary 3.1, if we set* $\alpha = 2$, $p(n) \equiv c^2$, $g(n) \equiv 0$, *we have*

$$x^2(n) \leq c^2 + \sum_{s=0}^{n-1} x\left(\sigma(s)\right) \left\{ f(s) + h(s) w\left(x\left(\sigma(s)\right)\right) \right\}, \quad n \in \mathbb{Z}_0$$

implies

$$x(n) \leq \Phi^{-1} \left\{ \Phi\left[c + \frac{1}{2} \sum_{s=0}^{n-1} f(s) \right] + \frac{1}{2} \sum_{s=0}^{n-1} h(s) \right\}, \quad n \in \mathbb{Z}_{[0,\mu]} .$$

This is the discrete analogue of a result of Pachpatte in [14]. Furthermore, if $\sigma = id$, *this reduces to a result of Pachpatte in [18].*

(ii) *In case* $\Phi(\infty) = \infty$, *(3.2) holds for all* $n \in \mathbb{Z}_0$.

Corollary 3.2. *Under Assumptions (A1) – (A4) with* $p \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+)$*, if* $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_1)$ *satisfies the nonlinear delay inequality*

$$x^{\alpha}(n) \leq p(n) + \sum_{s=0}^{n-1} x^{\alpha}(\sigma(s)) \{f(s) + g(s) \ln x(\sigma(s)) + h(s) w(\ln x(\sigma(s)))\} \tag{3.3}$$

for all $n \in \mathbb{Z}_0$ *with initial conditions (I1) and*

$$\psi(\sigma(s)) \leq \frac{1}{\alpha} \ln(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0, \tag{I5}$$

where $\alpha > 0$ *is a constant, then*

$$x(n) \leq \exp\left\{\Phi^{-1}\left[\Phi\left(\left(\exp \frac{1}{\alpha}\sum_{s=0}^{n-1} g(s)\right) \times \left(\frac{1}{\alpha}\ln p(n) + \frac{1}{\alpha}\sum_{s=0}^{n-1} f(s)\right)\right) + \left(\exp \frac{1}{\alpha}\sum_{s=0}^{n-1} g(s)\right)\frac{1}{\alpha}\sum_{t=0}^{n-1} h(t)\right]\right\} \tag{3.4}$$

for all $n \in \mathbb{Z}_{[0,\nu]}$*, where* $\nu \geq 0$ *is chosen such that the RHS of (3.4) is well-defined for all* $n \in \mathbb{Z}_{[0,\nu]}$*, and* Φ *is defined as in Theorem 2.1.*

Proof. Letting $y(n) = \ln x(n)$, (3.3) becomes

$$\text{(3.5)}\quad \exp\left(\alpha y(n)\right) \le p(n) + \sum_{s=0}^{n-1} \exp\left(\alpha y\left(\sigma(s)\right)\right)\left\{f(s) + g(s)y\left(\sigma(s)\right) + h(s)w\left(y\left(\sigma(s)\right)\right)\right\}.$$

Let $\phi : \mathbb{R}_0 \to \mathbb{R}_0$ be defined by $\phi(r) = \exp(\alpha r)$, $r \in \mathbb{R}_0$. Then ϕ satisfies Assumption (A5). Hence from (3.5), we have

$$\phi\left(y(n)\right) \le p(n) + \sum_{s=0}^{n-1} \phi'\left(y\left(\sigma(s)\right)\right)\left\{\frac{f(s)}{\alpha} + \frac{g(s)}{\alpha} y\left(\sigma(s)\right) + \frac{h(s)}{\alpha} w\left(y\left(\sigma(s)\right)\right)\right\}.$$

Furthermore, it is easy to see that

$$\psi\left(\sigma(s)\right) \le \frac{1}{\alpha}\ln\left(p(s)\right) = \phi^{-1}\left(p(s)\right) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \le 0\,.$$

Thus Theorem 2.1 applies and we have

$$y(n) \le \Phi^{-1}\left\{\Phi\left[\left(\exp\frac{1}{\alpha}\sum_{s=0}^{n-1} g(s)\right)\left(\frac{1}{\alpha}\ln p(n) + \frac{1}{\alpha}\sum_{s=0}^{n-1} f(s)\right)\right] + \left(\exp\frac{1}{\alpha}\sum_{s=0}^{n-1} g(s)\right)\frac{1}{\alpha}\sum_{t=0}^{n-1} h(t)\right\}$$

for all $n \in \mathbb{Z}_{[0,\nu]}$, and from this the assertion follows. □

Remark 4. *In case* $\Phi(\infty) = \infty$, *(3.4) holds for all* $n \in \mathbb{Z}_0$.

Corollary 3.3. *Under Assumptions (A1) – (A4), if* $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ *satisfies the nonlinear delay inequality*

$$x^{\alpha}(n) \le p(n) + \sum_{s=0}^{n-1} x^{\alpha-1}(\sigma(s)) \Bigg\{ f(s) + g(s)x(\sigma(s)) + h(s) \sum_{t=0}^{s-1} k(t) w(x(\sigma(t))) \Bigg\} \tag{3.6}$$

for all $n \in \mathbb{Z}_0$ *with initial conditions (I1) and (I4), where* $\alpha \ge 1$ *is a constant, then*

$$x(n) \le \Phi^{-1} \Bigg\{ \Phi \Bigg[\Bigg(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \Bigg) \Bigg(p^{\frac{1}{\alpha}}(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \Bigg) \Bigg] + \Bigg(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \Bigg) \Bigg(\frac{1}{\alpha} \sum_{s=0}^{n-1} h(s) \sum_{t=0}^{s-1} k(t) \Bigg) \Bigg\} \tag{3.7}$$

for all $n \in \mathbb{Z}_{[0,\eta]}$, *where* $\eta \ge 0$ *is chosen such that the RHS of (3.7) is well-defined for all* $n \in \mathbb{Z}_{[0,\eta]}$, *and* Φ *is defined as in Theorem 2.1.*

Proof. Let $\phi : \mathbb{R}_0 \to \mathbb{R}_0$ be defined by $\phi(r) = r^{\alpha}$, $r \in \mathbb{R}_0$. Then ϕ satisfies

Assumption (A5). By (3.6),

$$\phi\left(x(n)\right) \leq p(n) + \sum_{s=0}^{n-1} \phi'\left(x\left(\sigma(s)\right)\right) \left\{ \frac{f(s)}{\alpha} + \frac{g(s)}{\alpha} x\left(\sigma(s)\right) + \frac{h(s)}{\alpha} \sum_{t=0}^{s-1} k(t) w\left(x\left(\sigma(t)\right)\right) \right\}$$

for all $n \in \mathbb{Z}_0$. Furthermore, it is easy to see that

$$\psi\left(\sigma(s)\right) \leq p^{\frac{1}{\alpha}}(s) = \phi^{-1}\left(p(s)\right) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0\,.$$

Thus Theorem 2.2 applies and we have

$$x(n) \leq \Phi^{-1} \left\{ \Phi \left[\left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \left(p^{\frac{1}{\alpha}}(n) + \frac{1}{\alpha} \sum_{s=0}^{n-1} f(s) \right) \right] + \left(\exp \frac{1}{\alpha} \sum_{s=0}^{n-1} g(s) \right) \cdot \frac{1}{\alpha} \sum_{s=0}^{n-1} \sum_{t=0}^{s-1} h(s) k(t) \right\}$$

for all $n \in \mathbb{Z}_{[0,\eta]}$. □

Remark 5.

(i) *In Corollary 3.3, if we put* $\alpha = 2$, $p(n) \equiv c^2$, $g(n) \equiv 0$, *we have*

$$x^2(n) \leq c^2 + \sum_{s=0}^{n-1} x\left(\sigma(s)\right) \left\{ f(s) + h(s) \sum_{t=0}^{s-1} k(t) w\left(x\left(\sigma(t)\right)\right) \right\}, \quad n \in \mathbb{Z}_0$$

implies

$$x(n) \leq \Phi^{-1}\left\{\Phi\left[c + \frac{1}{2}\sum_{s=0}^{n-1} f(s)\right] + \frac{1}{2}\sum_{s=0}^{n-1} h(s)\sum_{t=0}^{s-1} k(t)\right\}, \quad n \in \mathbb{Z}_{[0,\eta]} \ .$$

This is the discrete analogue of a result of Pachpatte in [14]. Furthermore, if $\sigma = id$ and $w = id$, this reduces to a result of Pachpatte in [18].

(ii) *In case $\Phi(\infty) = \infty$, (3.7) holds for all $n \in \mathbb{Z}_0$.*

Corollary 3.4. *Under Assumptions (A1) – (A4) with $p \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+)$, if $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_1)$ satisfies the nonlinear delay inequality*

$$x^{\alpha}(n) \leq p(n) + \sum_{s=0}^{n-1} x^{\alpha}(\sigma(s))\left\{f(s) + g(s)\ln x(\sigma(s)) + h(s)\sum_{t=0}^{s-1} k(t) w(\ln x(\sigma(t)))\right\} \tag{3.8}$$

for all $n \in \mathbb{Z}_0$ with initial conditions (I1) and

$$\psi(\sigma(s)) \leq \frac{1}{\alpha}\ln(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0 \ , \tag{I6}$$

where $\alpha > 0$ is any constant, then

$$x(n) \leq \exp\left\{\Phi^{-1}\left[\Phi\left(\left(\exp\frac{1}{\alpha}\sum_{s=0}^{n-1} g(s)\right)\right.\right.\right. \tag{3.9}$$

$$\times\left(\frac{1}{\alpha}\ln p(n)+\frac{1}{\alpha}\sum_{s=0}^{n-1}f(s)\right)\Bigg)$$
$$+\left(\exp\frac{1}{\alpha}\sum_{s=0}^{n-1}g(s)\right)\cdot\frac{1}{\alpha}\sum_{s=0}^{n-1}h(s)\sum_{t=0}^{s-1}k(t)\Bigg]\Bigg\}$$

for all $n\in\mathbb{Z}_{[0,\lambda]}$*, where* $\lambda\geq 0$ *is chosen such that the RHS of (3.9) is well-defined for all* $n\in\mathbb{Z}_{[0,\lambda]}$*, and* Φ *is defined as in Theorem 2.1.*

Proof. Letting $y(n)=\ln x(n)$, (3.8) becomes

$$\exp\left(\alpha y(n)\right)\leq p(n)+\sum_{s=0}^{n-1}\exp\left(\alpha y\left(\sigma(s)\right)\right)\Bigg\{f(s)+g(s)y\left(\sigma(s)\right)+h(s)\sum_{t=0}^{s-1}k(t)w\left(y\left(\sigma(t)\right)\right)\Bigg\}\tag{3.10}$$

for all $n\in\mathbb{Z}_0$. Let $\phi:\mathbb{R}_0\to\mathbb{R}_0$ be defined by $\phi(r)=\exp(\alpha r)$, $r\in\mathbb{R}_0$. Then ϕ satisfies Assumption (A5). Hence from (3.10), we have

$$\phi\left(y(n)\right)\leq p(n)+\sum_{s=0}^{n-1}\phi'\left(y\left(\sigma(s)\right)\right)\times\left\{\frac{f(s)}{\alpha}+\frac{g(s)}{\alpha}y\left(\sigma(s)\right)+\frac{h(s)}{\alpha}\sum_{t=0}^{s-1}k(t)w\left(y\left(\sigma(t)\right)\right)\right\}$$

for all $n \in \mathbb{Z}_0$. Furthermore, it is easy to check that

$$\psi(\sigma(s)) \leq \frac{1}{\alpha} \ln(p(s)) = \phi^{-1}(p(s)) \quad \text{for all } s \in \mathbb{Z}_0 \text{ with } \sigma(s) \leq 0\,.$$

Thus Theorem 2.2 applies and we have

$$y(n) \leq \Phi^{-1}\left\{\Phi\left[\left(\exp\frac{1}{\alpha}\sum_{s=0}^{n-1} g(s)\right)\left(\frac{1}{\alpha}\ln p(n) + \frac{1}{\alpha}\sum_{s=0}^{n-1} f(s)\right)\right]\right.$$
$$\left. + \exp\left(\frac{1}{\alpha}\sum_{s=0}^{n-1} g(s)\right) \cdot \frac{1}{\alpha}\sum_{s=0}^{n-1}\sum_{t=0}^{s-1} h(s)k(t)\right\}$$

for all $n \in \mathbb{Z}_{[0,\lambda]}$, and from this the assertion follows. □

Remark 6.

(i) *In Corollary 3.4, if we set* $\alpha = 2$, $p(n) \equiv c^2$, $g(n) \equiv 0$, *then*

$$x^2(n) \leq c^2 + \sum_{s=0}^{n-1} x^2(\sigma(s))\left\{f(s) + h(s)\sum_{t=0}^{s-1} k(t) w(\ln x(\sigma(t)))\right\}, \quad n \in \mathbb{Z}_0$$

implies

$$x(n) \leq \exp\left\{\Phi^{-1}\left[\Phi\left(\frac{1}{2}\ln p(n) + \frac{1}{2}\sum_{s=0}^{n-1} f(s)\right) + \frac{1}{2}\sum_{s=0}^{n-1} h(s)\sum_{t=0}^{s-1} k(t)\right]\right\}$$
$$n \in \mathbb{Z}_{[0,\lambda]}\,.$$

This is the discrete version of a result of Pachpatte in [14].

(ii) *In case* $\Phi(\infty) = \infty$, *(3.9) holds for all* $n \in \mathbb{Z}_0$.

4. Application

Consider the discrete delay equation

$$x^{\alpha}(n) = F\left(n, x(\sigma(n)), \sum_{s=0}^{n-1} G(n, s, x(\sigma(s)))\right), \quad n \in \mathbb{Z}_0 \tag{4.1}$$

with initial conditions (I1) and (I4), where $\alpha \geq 1$ is a constant, σ, ψ satisfy Assumptions (A3), (A4), $x \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R})$, $F \in C(\mathbb{Z}_0 \times \mathbb{R}^2, \mathbb{R})$, and $G \in C(\mathbb{Z}_0^2 \times \mathbb{R}, \mathbb{R})$. If F, G satisfy

$$\begin{aligned} |F(n,u,v)| &\leq p(n) + K|v|, \quad n \in \mathbb{Z}_0, u, v \in \mathbb{R}, \\ |G(n,s,v)| &\leq [f(s) + g(s)|v| + h(s)w(|v|)]\,|v|^{\alpha-1}, \quad n, s \in \mathbb{Z}_0, v \in \mathbb{R}, \end{aligned}$$

for some p, f, g, h, w satisfying (A1) and (A2), and some constant $K > 0$, then every solution of (4.1) satisfies

$$\begin{aligned} |x(n)|^{\alpha} &= \left| F\left(n, x(\sigma(n)), \sum_{s=0}^{n-1} G(n, s, x(\sigma(s)))\right) \right| \\ &\leq p(n) + K\left|\sum_{s=0}^{n-1} G(n, s, x(\sigma(s)))\right| \\ &\leq p(n) + K\sum_{s=0}^{n-1} |G(n, s, x(\sigma(s)))| \\ &\leq p(n) + K\sum_{s=0}^{n-1} [f(s) + g(s)|x(\sigma(s))| + h(s)w(|x(\sigma(s))|)]\,|x(\sigma(s))|^{\alpha-1} \end{aligned}$$

for all $n \in J(x) :=$ the maximal existence lattice on which x is defined. Applying Corollary 3.1, this yields

$$|x(n)| \leq \Phi^{-1}\left\{\Phi\left[\left(\exp\frac{K}{\alpha}\sum_{s=0}^{n-1}g(\alpha)\right)\left(p^{\frac{1}{\alpha}}(n)+\frac{K}{\alpha}\sum_{s=0}^{n-1}f(s)\right)\right]\right.$$
$$\left.+\left(\exp\frac{K}{\alpha}\sum_{s=0}^{n-1}g(\alpha)\right)\frac{K}{\alpha}\sum_{t=0}^{n-1}h(t)\right\}$$

for all $n \in J(x) \cap \mathbb{Z}_{[0,\mu]}$. This gives the boundedness of solutions of (4.1).

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