A SIMULTANEOUS SYSTEM OF FUNCTIONAL INEQUALITIES AND MAPPINGS WHICH ARE WEAKLY OF A CONSTANT SIGN

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Abstract: It is shown that, under some algebraic conditions on fixed reals $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ and vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1} \in \mathbb{R}^n$, every continuous at a point function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the simultaneous system of inequalities

$$f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, i = 1, 2, \dots, n+1,$$

has to be of the form $f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0})$, $\mathbf{x} \in \mathbb{R}^n$, with uniquely determined $\mathbf{p} \in \mathbb{R}^n$. For mappings with values in a Banach space which are weakly of a constant sign, a counterpart of this result is given.

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1. Introduction

In this paper we consider the simultaneous system of functional inequalities

$$f(\mathbf{x}+\mathbf{a}_i) \le \alpha_i + f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \dots, n+1, \tag{1.1}$$

where $n \in \mathbb{N}$, $\alpha_1, \alpha_2, \dots, \alpha_{n+1} \in \mathbb{R}$, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1} \in \mathbb{R}^n$ are fixed and $f : \mathbb{R}^n \to \mathbb{R}$ is an unknown function.

In Section 3, assuming Kronecker's type conditions on the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1}$ and a inequality involving some determinants depending on these vectors and scalars $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$, we show (Theorem 3.1) that for every continuous at least at one point function f satisfying (1.1) there exists a unique vector $\mathbf{p} \in \mathbb{R}^n$ such that

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^n.$$

This result seems to be a little surprising in the context of an obvious fact that, in general, the accompanying simultaneous system of functional equations has a lot of very regular but nonlinear solutions, even "depending on an arbitrary function" (cf. M. Kuczma [4], [2] or [3], where the full construction of the solution in a special one dimensional case is given). Theorem 3.1 generalizes a suitable result of [2], where the one dimensional case

$$f(x+a) \le \alpha + f(x), \qquad f(x+b) \le \beta + f(x), \qquad x \in \mathbb{R},$$

is considered. Let us mention that in the case when $\alpha = \beta = 0$, Montel [5] considered the accompanying simultaneous system of equations.

In Section 4 we define a mapping to be weakly of a constant sign. Using this notion and a total system of linear functionals, we present a counterpart of Theorem 3.1 for functions with values in a Banach space.

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2. Kronecker's Theorem and a Lemma

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are reserved, respectively, for the set of natural, integer, rational and real numbers.

We begin this section by recalling the following:

Theorem 2.1 (Kronecker (cf. [1, p. 382])). *If the reals* $\upsilon_1, \upsilon_2, \dots, \upsilon_n, 1$ *are linearly independent over the field* $\mathbb{Q}$*, the numbers* $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ *are arbitrary, and* N *and* ϵ *are positive, then there are the integers* $p_1, p_2, \dots, p_n$*, and* $m > N$*, such that* $|m\upsilon_i - p_i - \alpha_i| < \epsilon$ *for each* $i = 1, 2, \dots, n$.

As an immediate consequence of this theorem we obtain the following

Corollary 2.2. *Let* $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ *be the standard base of the real linear space* $\mathbb{R}^n$*. If reals* $\upsilon_1, \upsilon_2, \dots, \upsilon_n, 1$ *are linearly independent over the field* $\mathbb{Q}$*, and* $\mathbf{v} := (\upsilon_1, \upsilon_2, \dots, \upsilon_n)$*, then the set*

$$\{m\mathbf{v} + p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + \cdots + p_n\mathbf{e}_n : m \in \mathbb{N},\ p_1, p_2, \dots, p_n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}^n$.

In sequel we need a more special result which guarantees that the set of all linear combinations of the elements of the set $B = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1}\}$ with natural coefficients is dense in $\mathbb{R}^n$.

Lemma 2.3. *Let* $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ *be an arbitrary base of linear space* $\mathbb{R}^n$ *over* $\mathbb{R}$*. Suppose that* $\upsilon_1, \upsilon_2, \dots, \upsilon_n \in \mathbb{R}$ *are negative and the system of numbers* $\upsilon_1, \upsilon_2, \dots, \upsilon_n, 1$ *is linearly independent over the field* $\mathbb{Q}$*. If*

$$\mathbf{a}_{n+1} = \upsilon_1\mathbf{a}_1 + \upsilon_2\mathbf{a}_2 + \cdots + \upsilon_n\mathbf{a}_n,$$

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then the set

$$A := \left\{ \sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i : m^{(i)} \in \mathbb{N},\, i = 1, 2, \dots, n+1 \right\}$$

is dense in $\mathbb{R}^n$.

Proof. Let $N > 0$ be fixed. Take $\mathbf{x} \in \mathbb{R}^n$. Then there exists a unique system of numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ such that

$$\mathbf{x} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n.$$

By Kronecker's theorem, for every $k \in \mathbb{N}$ there exist $m_k > N$ and $p_{ik} \in \mathbb{Z}$ such that

$$|m_k v_i + p_{ik} - x_i| = |m_k v_i - (-p_{ik}) - x_i| < \frac{1}{k}, \qquad i = 1, 2, \dots, n,$$

whence

$$\lim_{k \to \infty} (m_k v_i + p_{ik}) = x_i, \qquad i = 1, 2, \dots, n.$$

Since $m_k \in \mathbb{N}$, $v_i < 0$ for $i = 1, 2, \dots, n$, it follows that $p_{ik} \in \mathbb{N}$ for k large enough (in the opposite case we would have $\lim_{k\to\infty} (m_k v_i + p_{ik}) = -\infty$). It follows that, for k large enough, setting $m_k^{(n+1)} := m_k$ and $m_k^{(i)} := p_{ik}$, we get

$$m_k^{(i)} \in \mathbb{N}, \qquad i = 1, 2, \dots, n+1.$$

Put

$$\mathbf{t}_k := \sum_{i=1}^{n} \left(m_k^{(n+1)} v_i + m_k^{(i)} \right) \mathbf{a}_i, \quad k \in \mathbb{N}.$$

By the definition of $\mathbf{a}_{n+1}$ we hence get

$$\mathbf{t}_k = \sum_{i=1}^{n+1} m_k^{(i)} \mathbf{a}_i, \quad k \in \mathbb{N},$$

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whence $\mathbf{t}_k \in A$ for k sufficiently large. Moreover

$$\begin{aligned}\lim_{k\to\infty} \mathbf{t}_k &= \lim_{k\to\infty} \sum_{i=1}^{n} \left(m_k^{(n+1)} v_i + m_k^{(i)} \right) \mathbf{a}_i \\ &= \sum_{i=1}^{n} \lim_{k\to\infty} \left(m_k^{(n+1)} v_i + m_k^{(i)} \right) \mathbf{a}_i \\ &= \sum_{i=1}^{n} x_i \mathbf{a}_i = \mathbf{x}.\end{aligned}$$

This completes the proof of the density of A in $\mathbb{R}^n$. □

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3. Main Result

Theorem 3.1. *Let* $n \in \mathbb{N}$, $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,n}) \in \mathbb{R}^n$, $i = 1, 2, \dots, n$, *negative reals* $v_1, v_2, \dots, v_n$ *and* $\alpha_1, \alpha_2, \dots, \alpha_{n+1} \in \mathbb{R}$ *be fixed and such that:*

(i) $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ *form a base of the linear space* $\mathbb{R}^n$ *over* $\mathbb{R}$*;*

(ii) $v_1, v_2, \dots, v_n, 1$ *are linearly independent over* $\mathbb{Q}$*,*

(iii) $\mathbf{a}_{n+1} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \dots + v_n\mathbf{a}_n$*,*

(iv)

$$(-1)^n \left(\operatorname{sgn} \mathbf{W}^{(n)}\right) \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix} \le 0,$$

where

$$\mathbf{W}^{(n)} := \begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{vmatrix}.$$

If a continuous at least at one point function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *satisfies the simultaneous system of functional inequalities*

$$f(\mathbf{x} + \mathbf{a}_i) \le \alpha_i + f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n,\ i = 1, 2, \dots, n+1, \tag{3.1}$$

then there exists a unique $\mathbf{p} \in \mathbb{R}^n$ *such that*

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^n.$$

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Moreover,

$$\mathbf{p} = \frac{1}{\mathbf{W}^{(n)}} \left[p_1^{(n)}, p_2^{(n)}, \dots, p_n^{(n)} \right],$$

where

$$p_i^{(n)} := \begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,i-1} & a_{2,i-1} & \cdots & a_{n,i-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{1,i+1} & a_{2,i+1} & \cdots & a_{n,i+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{vmatrix}, \qquad i = 1, 2, \dots, n.$$

Proof. Suppose that the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies system (3.1). Using an easy induction argument, one can show that f satisfies the inequality

$$f\left(\sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i + \mathbf{x} \right) \leq \sum_{i=1}^{n+1} m^{(i)} \alpha_i + f(\mathbf{x}), \tag{3.2}$$

for all $m^{(i)} \in \mathbb{N}$, $i = 1, 2, \dots, n+1$ and all $\mathbf{x} \in \mathbb{R}^n$. By Lemma 2.3, the set

$$A := \left\{ \sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i : m^{(i)} \in \mathbb{N}, \quad i = 1, 2, \dots, n+1 \right\}$$

is dense in $\mathbb{R}^n$. Thus there exist the sequences of positive integers $(m_k^{(i)})_{k \in \mathbb{N}}$ for

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$i = 1, 2, \ldots, n+1$ such that

$$\lim_{k\to\infty} \sum_{i=1}^{n+1} m_k^{(i)} \mathbf{a}_i = \lim_{k\to\infty} \left(\sum_{i=1}^{n+1} m_k^{(i)} a_{i,1}, \sum_{i=1}^{n+1} m_k^{(i)} a_{i,2}, \ldots, \sum_{i=1}^{n+1} m_k^{(i)} a_{i,n} \right) = (0, 0, \ldots, 0),$$

and, obviously,

$$\lim_{k\to\infty} m_k^{(i)} = \infty, \qquad i = 1, 2, \ldots, n+1.$$

It follows that, for each $i = 1, 2, \ldots, n$, the limit $\lim_{k\to\infty} \frac{m_k^{(i)}}{m_k^{(n+1)}}$ exists, and

$$\sum_{i=1}^{n+1} \left(\lim_{k\to\infty} \frac{m_k^{(i)}}{m_k^{(n+1)}} a_{i,j} \right) = \lim_{k\to\infty} \frac{\sum_{i=1}^{n+1} m_k^{(i)} a_{i,j}}{m_k^{(n+1)}} = 0, \qquad j = 1, 2, \ldots, n.$$

Consequently,

$$\lim_{k\to\infty} \frac{m_k^{(i)}}{m_k^{(n+1)}} = \frac{\mathbf{W}_i^{(n)}}{\mathbf{W}^{(n)}}, \qquad i = 1, 2, \ldots, n, \tag{3.3}$$

where

$$\mathbf{W}_i^{(n)} := \begin{vmatrix} a_{1,1} & \cdots & a_{i-1,1} & -a_{n+1,1} & a_{i+1,1} & \cdots & a_{n,1} \\ a_{1,2} & \cdots & a_{i-1,2} & -a_{n+1,2} & a_{i+1,2} & \cdots & a_{n,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & \cdots & a_{i-1,n} & -a_{n+1,n} & a_{i+1,n} & \cdots & a_{n,n} \end{vmatrix}, \qquad i = 1, 2, \ldots, n.$$

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Let $\mathbf{x}_0$ be a point of the continuity of f. From inequality (3.2), we get

$$\frac{f\left(\sum_{i=1}^{n+1} m_k^{(i)}\mathbf{a}_i + \mathbf{x}_0\right)}{m_k^{(n+1)}} \leq \sum_{i=1}^{n+1} \frac{m_k^{(i)}}{m_k^{(n+1)}}\alpha_i + \frac{f(\mathbf{x}_0)}{m_k^{(n+1)}}.$$

Letting here $k \to \infty$ and applying (3.3), we obtain

$$0 \leq \sum_{i=1}^{n} \frac{\mathbf{W}_i^{(n)}}{\mathbf{W}^{(n)}}\alpha_i + \alpha_{n+1}. \tag{3.4}$$

Setting

$$\overline{\mathbf{W}_i^{(n)}} := -\begin{vmatrix} a_{1,1} & \cdots & a_{i-1,1} & a_{i+1,1} & \cdots & a_{n,1} & a_{n+1,1} \\ a_{1,2} & \cdots & a_{i-1,2} & a_{i+1,2} & \cdots & a_{n,2} & a_{n+1,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & \cdots & a_{i-1,n} & a_{i+1,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix}, \qquad i = 1, 2, \dots, n,$$

we can write inequality (3.4) in the form

$$0 \leq \operatorname{sgn} \mathbf{W}^{(n)} \left(\sum_{i=1}^{n} (-1)^{n-i+1} \overline{\mathbf{W}_i^{(n)}} \alpha_i + \mathbf{W}^{(n)} \alpha_{n+1} \right),$$

whence, by the Laplace expansion of a determinant,

$$0 \leq (-1)^n \operatorname{sgn} \mathbf{W}^{(n)} \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix}.$$

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Hence, taking into account condition (iv), we infer that

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix} = 0.$$

Now, by Laplace's expansion theorem,

$$\alpha_{n+1} = (-1)^{n+1} \frac{\sum_{i=1}^{n} (-1)^{1-i} \overline{\mathbf{W}_i^{(n)}} \alpha_i}{\mathbf{W}^{(n)}},$$

whence by (3.2),

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i + \mathbf{x}\right) &\le \sum_{i=1}^{n} m^{(i)} \alpha_i + m^{(n+1)} (-1)^{n+1} \frac{\sum_{i=1}^{n} (-1)^{1-i} \overline{\mathbf{W}_i^{(n)}} \alpha_i}{\mathbf{W}^{(n)}} + f(\mathbf{x}) \\ &= \frac{1}{\mathbf{W}^{(n)}} \left[\sum_{i=1}^{n} m^{(i)} \mathbf{W}^{(n)} \alpha_i + m^{(n+1)} \sum_{i=1}^{n} (-1)^{n+2-i} \overline{\mathbf{W}_i^{(n)}} \alpha_i\right] + f(\mathbf{x}), \end{aligned}$$

for all $m^{(i)} \in \mathbb{N}$, and $x \in \mathbb{R}^n$.

Applying Laplace's theorem for the ith column of $\mathbf{W}^{(n)}$ and for the last columne

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of $\overline{\mathbf{W}_i^{(n)}}$, we have

$$\sum_{i=1}^{n} m^{(i)}\mathbf{W}^{(n)}\alpha_i + m^{(n+1)}\sum_{i=1}^{n}(-1)^{n+2-i}\overline{\mathbf{W}_i^{(n)}}\alpha_i$$
$$= \sum_{i=1}^{n} m^{(i)}\alpha_i \sum_{j=1}^{n}(-1)^{i+j}a_{i,j}\mathbf{W}_{ij}^{(n)}$$
$$+ m^{(n+1)}\left(\sum_{i=1}^{n}(-1)^{n-i}a_{i,j}\alpha_i\left(\sum_{j=1}^{n}(-1)^{n+j}a_{n+1,j}\overline{\mathbf{W}_{ij}^{(n)}}\right)\right),$$

where $\mathbf{W}_{ij}^{(n)}$ is obtained from $\mathbf{W}_i^{(n)}$ by deleting the jth row and ith column, and $\overline{\mathbf{W}_{ij}^{(n)}}$ is obtained from $\overline{\mathbf{W}_i^{(n)}}$ by deleting the jth row and the last column. Since

$$\mathbf{W}_{ij}^{(n)} = \overline{\mathbf{W}_{ij}^{(n)}}, \qquad i,j = 1,2,\dots,n,$$

applying Fubini's theorem for sums, we have

$$\sum_{i=1}^{n} m^{(i)}\mathbf{W}^{(n)}\alpha_i + m^{(n+1)}\sum_{i=1}^{n}(-1)^{n+2-i}\overline{\mathbf{W}_i^{(n)}}\alpha_i$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{n} m^{(k)}a_{k,j}\left((-1)^{k+j}\mathbf{W}_{kj}^{(n)}\alpha_k\right) + m^{(n+1)}a_{n+1,j}\sum_{i=1}^{n}(-1)^{j-i}\mathbf{W}_{ij}^{(n)}\alpha_i.$$

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Adding and subtracting the term $\sum_{i=1}^{n}(-1)^{j-i}\mathbf{W}_{ij}^{(n)}\alpha_i$ in the sum over k gives

$$\begin{aligned}\sum_{i=1}^{n} m^{(i)}\mathbf{W}^{(n)}\alpha_i + m^{(n+1)}\sum_{i=1}^{n}(-1)^{n+2-i}\overline{\mathbf{W}_i^{(n)}}\alpha_i \\ = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} m^{(k)}a_{k,j} + m^{(n+1)}a_{n+1,j}\right)\left(\sum_{i=1}^{n}(-1)^{j-i}\mathbf{W}_{ij}^{(n)}\alpha_i\right) \\ + \sum_{j=1}^{n}\left(\sum_{k=1}^{n} m^{(k)}a_{k,j}\left(\sum_{i=1}^{n}(-1)^{j-i+1}\mathbf{W}_{ij}^{(n)}\alpha_i + (-1)^{k+j}\mathbf{W}_{kj}^{(n)}\alpha_k\right)\right).\end{aligned}$$

Let us note that

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{n} m^{(k)}a_{k,j}\left(\sum_{i=1}^{n}(-1)^{j-i+1}\mathbf{W}_{ij}^{(n)}\alpha_i + (-1)^{k+j}\mathbf{W}_{kj}^{(n)}\alpha_k\right)\right) = 0.$$

Indeed, we have

$$\begin{aligned}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} m^{(k)}a_{k,j}\left(\sum_{i=1}^{n}(-1)^{j-i+1}\mathbf{W}_{ij}^{(n)}\alpha_i + (-1)^{k+j}\mathbf{W}_{kj}^{(n)}\alpha_k\right)\right) \\ = \sum_{j=1}^{n}\sum_{k=1}^{n} m^{(k)}a_{k,j}\left(\sum_{i\in\{1,2,\dots,n\}-k}(-1)^{j-i+1}\mathbf{W}_{ij}^{(n)}\alpha_i\right) \\ = \sum_{k=1}^{n} m^{(k)}\left(\sum_{i\in\{1,2,\dots,n\}-k}\alpha_i\left(\sum_{j=1}^{n}(-1)^{j-i+1}a_{k,j}\mathbf{W}_{ij}^{(n)}\right)\right)\end{aligned}$$

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and

$$\sum_{j=1}^{n}(-1)^{j-i+1}a_{k,j}\mathbf{W}_{ij}^{(n)}$$

is a determinant with two equal columns. Thus we have shown that f satisfies the inequality

$$f\left(\sum_{i=1}^{n+1}m^{(i)}\mathbf{a}_i+\mathbf{x}\right)\le\sum_{j=1}^{n}\left(\left(\frac{1}{\mathbf{W}^n}\sum_{i=1}^{n}(-1)^{j-i}\mathbf{W}_{ij}^{(n)}\alpha_i\right)\left(\sum_{i=1}^{n+1}m^{(i)}a_{ij}\right)\right)+f(\mathbf{x}),$$

for all $m^{(i)}\in\mathbb{N}$, and $x\in\mathbb{R}^n$, which can be written in the form

$$f(\mathbf{t}+\mathbf{x})\le\mathbf{p}\cdot\mathbf{t}+f(\mathbf{x}),\qquad \mathbf{t}\in A,\mathbf{x}\in\mathbb{R}^n. \tag{3.5}$$

Now take an arbitrary $\mathbf{x}\in\mathbb{R}^n$. By the density of A there is a sequence $(\mathbf{t}_n)$ such that

$$\mathbf{t}_n\in A\ (n\in\mathbf{N}),\qquad \lim_{n\to\infty}\mathbf{t}_n=\mathbf{x}_0-\mathbf{x}.$$

From (3.5) we have

$$f(\mathbf{t}_n+\mathbf{x})\le\mathbf{p}\cdot\mathbf{t}_n+f(\mathbf{x}),\qquad n\in\mathbf{N}.$$

Letting here $n\to\infty$, and making use of the continuity of f at $\mathbf{x}_0$, we obtain

$$f(\mathbf{x}_0)\le\mathbf{p}\cdot(\mathbf{x}_0-\mathbf{x})+f(\mathbf{x}),\qquad \mathbf{x}\in\mathbb{R}^n.$$

To prove the converse inequality, note that replacing $\mathbf{x}$ by $\mathbf{x}-\mathbf{t}$ in (3.5) we get

$$f(\mathbf{x})\le\mathbf{p}\cdot\mathbf{t}+f(\mathbf{x}-\mathbf{t}),\qquad \mathbf{t}\in A,\mathbf{x}\in\mathbb{R}^n.$$

Taking a sequence $(\mathbf{t}_n)$ such that

$$\mathbf{t}_n\in A\ (n\in\mathbf{N}),\qquad \lim_{n\to\infty}\mathbf{t}_n=\mathbf{x}-\mathbf{x}_0,$$

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(which, by the density of A, exists) we hence get

$$f(\mathbf{x}) \leq \mathbf{p} \cdot \mathbf{t}_n + f(\mathbf{x} - \mathbf{t}_n), \qquad n \in \mathbf{N}.$$

Letting here $n \to \infty$, and again making use of the continuity of f at $\mathbf{x}_0$, we obtain the inequality

$$f(\mathbf{x}) \leq \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0) + f(\mathbf{x}_0), \qquad \mathbf{x} \in \mathbb{R}^n.$$

Thus

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + (f(\mathbf{x}_0) - \mathbf{p} \cdot \mathbf{x}_0), \qquad \mathbf{x} \in \mathbb{R}^n.$$

Since $f(\mathbf{0}) = f(\mathbf{x}_0) - \mathbf{p} \cdot \mathbf{x}_0$, the proof is completed. $\square$

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4. Application for Mappings Which are Weakly of a Constant Sign

We start this section with the following.

Definition 4.1. *Let X be an arbitrary nonempty set, Y - an arbitrary real linear topological space, Y^*- the conjugate space of Y, and $T \subset Y^*$.*

(i) *We say that a mapping $G : X \to Y$ is T-weakly of a constant sign if for each functional $\phi \in T$ either $\phi \circ G$ is nonpositive or $\phi \circ G$ is nonnegative.*

(ii) *The mappings $G_1 : X \to Y$ and $G_2 : X \to Y$ are said to be T-weakly of the same sign if $(\phi \circ G_1) \cdot (\phi \circ G_2) \geq 0$ for every functional $\phi \in T$.*

Now applying Theorem 3.1 we prove:

Theorem 4.1. *Let Y be a real Banach space and $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \subset \mathbb{R}^n$, $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n})$, $i = 1, 2, \ldots, n$ be a base of the real linear space $\mathbb{R}^n$. Suppose that reals $\upsilon_1, \upsilon_2, \ldots, \upsilon_n$ are negative and $\upsilon_1, \upsilon_2, \ldots, \upsilon_n, 1$ are linearly independent over the field $\mathbb{Q}$. Let $\mathbf{a}_{n+1} = \upsilon_1\mathbf{a}_1 + \upsilon_2\mathbf{a}_2 + \cdots + \upsilon_n\mathbf{a}_n$.*

If a mapping $F : \mathbb{R}^n \to Y$ is continuous at least at one point and there exist a total system $T \subset Y^*$ and the vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n+1} \in Y$ such that

(i) *the mappings*

$$\textbf{(*)} \qquad \mathbb{R}^n \ni \mathbf{x} \to F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i, \qquad i \in \{1, 2, \ldots, n+1\},$$

are T-weakly of a constant and the same sign;

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(ii) *for every* $\phi \in T$,

$$(-1)^n \left(\operatorname{sgn} \mathbf{W}^{(n)}\right) \begin{vmatrix} \phi(\mathbf{y}_1) & \phi(\mathbf{y}_2) & \cdots & \phi(\mathbf{y}_n) & \phi(\mathbf{y}_{n+1}) \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix} \leq 0,$$

then there exists a unique linear continuous mapping $L : \mathbb{R}^n \to Y$ *such that*

$$F(\mathbf{x}) = L(\mathbf{x}) + F(\mathbf{0}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{4.1}$$

Proof. Assume that $F : \mathbb{R}^n \to Y$ satisfies assumptions (i) and (ii). By Definition 4.1, we have either, for all $\phi \in T$ and $i = 1, 2, \ldots, n+1$,

$$\phi\left(F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i\right) \leq 0, \quad \mathbf{x} \in \mathbb{R}^n,$$

or, for all $\phi \in T$ and $i = 1, 2, \ldots, n+1$,

$$\phi\left(F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i\right) \geq 0, \quad \mathbf{x} \in \mathbb{R}^n.$$

Without any loss of generality we may confine the consideration to the first case. From the linearity of ϕ we obtain the simultaneous system of inequalities

$$\phi\left(F(\mathbf{x} + \mathbf{a}_i)\right) \leq \phi\left(F(\mathbf{x})\right) + \phi\left(\mathbf{y}_i\right), \quad \mathbf{x} \in \mathbb{R}^n; \quad \mathbf{y}_i \in Y, i = 1, 2, \ldots, n+1; \quad \phi \in T.$$

Let us fix $\phi \in T$. Setting $f := \phi \circ F$ and $\alpha_i := \phi\left(\mathbf{y}_i\right)$, we hence get

$$f\left(\mathbf{x} + \mathbf{a}_i\right) \leq \alpha_i + f\left(\mathbf{x}\right), \qquad \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \ldots, n+1,$$

i.e. f satisfies system (3.1). Since all the required assumptions of Theorem 3.1 are satisfied, there exists a vector $\mathbf{p}$, such that

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^n.$$

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Hence, by the definition of f,

$$\phi\left(F(\mathbf{x})-F(\mathbf{0})\right)=\mathbf{p}\cdot\mathbf{x},\qquad \mathbf{x}\in\mathbb{R}^n.$$

It follows that for every $\phi\in T$, there exists $\mathbf{p}:=\mathbf{p}_\phi$, such that

$$\phi\left(F(\mathbf{x})-F(\mathbf{0})\right)=\mathbf{p}_\phi\cdot\mathbf{x},\qquad \mathbf{x}\in\mathbb{R}^n.$$

Setting $L:=F-F(\mathbf{0})$, we hence get

$$\begin{aligned}\phi\left(L(\mathbf{x}+\mathbf{y})\right)&=\phi\left(F(\mathbf{x}+\mathbf{y})-F(\mathbf{0})\right)\\&=\mathbf{p}_\phi\cdot(\mathbf{x}+\mathbf{y})\\&=\mathbf{p}_\phi\cdot\mathbf{x}+\mathbf{p}_\phi\cdot\mathbf{y}\\&=\phi\left(F(\mathbf{x})-F(\mathbf{0})\right)+\phi\left(F(\mathbf{y})-F(\mathbf{0})\right)\\&=\phi\left(L(\mathbf{x})\right)+\phi\left(L(\mathbf{y})\right),\end{aligned}$$

for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, whence

$$\phi\left(L(\mathbf{x}+\mathbf{y})-L(\mathbf{x})-L(\mathbf{y})\right)=0,\qquad \mathbf{x},\mathbf{y}\in\mathbb{R}^n,$$

for every functional $\phi\in T$. Since T is total set of functionals (cf. [6]),

$$L(\mathbf{x}+\mathbf{y})-L(\mathbf{x})-L(\mathbf{y})=0,\qquad \mathbf{x},\mathbf{y}\in\mathbb{R}^n,$$

that is, L is an additive mapping. Now the continuity of F at least at one point implies that $L:=F-F(\mathbf{0})$ is continuous and, consequently, a linear map. The proof is completed. □

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