

# A GENERALIZED CLASS OF $k$ -STARLIKE FUNCTIONS WITH VARYING ARGUMENTS OF COEFFICIENTS

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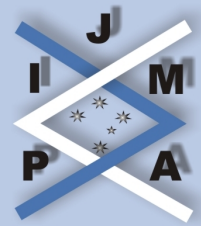
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*Key words:* Univalent functions, starlike functions, varying arguments, coefficient estimates.

*Abstract:* In terms of Wright generalized hypergeometric function we define a class of analytic functions. The class generalize well known classes of  $k$ -starlike functions and  $k$ -uniformly convex functions. Necessary and sufficient coefficient bounds are given for functions in this class. Further distortion bounds, extreme points and results on partial sums are investigated.



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# 1. Introduction

Let  $A$  denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc  $U = \{z : |z| < 1\}$ . We denote by  $S$  the subclass of  $A$  consisting of functions  $f$  which are univalent in  $U$ .

Also we denote by  $V$ , the class of analytic functions with varying arguments (introduced by Silverman [16]) consisting of functions  $f$  of the form (1.1) for which there exists a real number  $\eta$  such that

$$(1.2) \quad \theta_n + (n - 1)\eta = \pi \pmod{2\pi}, \quad \text{where} \quad \arg(a_n) = \theta_n \quad \text{for all} \quad n \geq 2.$$

Let  $k, \gamma$  be real parameters with  $k \geq 0$ ,  $-1 \leq \gamma < 1$ .

**Definition 1.1.** A function  $f \in A$  is said to be in the class  $UCV(k, \gamma)$  of  $k$ -uniformly convex functions of order  $\gamma$  if it satisfies the condition

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} - \gamma \right\} > k \left| \frac{z f''(z)}{f'(z)} \right|, \quad z \in U.$$

In particular, the classes  $UCV := UCV(1, 0)$ ,  $k - UCV := UCV(k, 0)$  were introduced by Goodman [6] (see also [10, 13]), and Kanas and Wisniowska [8] (see also [7]), respectively, where their geometric definition and connections with the conic domains were considered.

Related to the class  $UCV(k, \gamma)$  by means of the well-known Alexander equivalence between the usual classes of convex and starlike functions, we define the class  $SP(k, \gamma)$  of  $k$ -starlike functions of order  $\gamma$ .

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**Definition 1.2.** A function  $f \in A$  is said to be in the class  $SP(k, \gamma)$  of  $k$ -starlike functions of order  $\gamma$  if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in U.$$

The classes  $S_p := SP(1, 0)$ ,  $k-ST := SP(k, 0)$  were investigated by Rønning [13, 14], Kanas and Wisniowska [9], Kanas and Srivastava [7].

Note that the classes

$$ST := SP(0, 0), \quad CV := UCV(0, 0)$$

are the well known classes of starlike and convex functions, respectively.

For functions  $f \in A$  given by (1.1) and  $g \in A$  given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in U,$$

we define the Hadamard product (or convolution) of  $f$  and  $g$  by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

For positive real parameters  $\alpha_1, A_1, \dots, \alpha_p, A_p$  and  $\beta_1, B_1, \dots, \beta_q, B_q$  ( $p, q \in N = 1, 2, 3, \dots$ ) such that

$$(1.3) \quad 1 + \sum_{n=1}^q B_n - \sum_{n=1}^p A_n \geq 0,$$

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the Wright generalized hypergeometric function [24]

$${}_p\Psi_q[(\alpha_1, A_1), \dots, (\alpha_p, A_p); (\beta_1, B_1), \dots, (\beta_q, B_q); z] \\ = {}_p\Psi_q[(\alpha_n, A_n)_{1,p}; (\beta_n, B_n)_{1,q}; z]$$

is defined by

$${}_p\Psi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] \\ = \sum_{n=0}^{\infty} \left\{ \prod_{t=0}^p \Gamma(\alpha_t + nA_t) \right\} \left\{ \prod_{t=0}^q \Gamma(\beta_t + nB_t) \right\}^{-1} \frac{z^n}{n!}, \quad z \in U.$$

If  $p \leq q + 1$ ,  $A_n = 1$  ( $n = 1, \dots, p$ ) and  $B_n = 1$  ( $n = 1, \dots, q$ ), we have the relationship:

$$(1.4) \quad \Omega {}_p\Psi_q[(\alpha_n, 1)_{1,p}; (\beta_n, 1)_{1,q}; z] = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z), \quad z \in U,$$

where  ${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$  is the generalized hypergeometric function and

$$(1.5) \quad \Omega = \left( \prod_{t=0}^p \Gamma(\alpha_t) \right)^{-1} \left( \prod_{t=0}^q \Gamma(\beta_t) \right).$$

In [3] Dziok and Raina defined the linear operator by using Wright generalized hypergeometric function. Let

$${}_p\phi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] = \Omega z {}_p\Psi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z], \quad z \in U,$$

and

$$\mathcal{W} = \mathcal{W}[(\alpha_n, A_n)_{1,p}; (\beta_n, B_n)_{1,q}] : A \rightarrow A$$



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be a linear operator defined by

$$\mathcal{W}f(z) := z {}_p\phi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] * f(z), \quad z \in U.$$

We observe that, for  $f$  of the form (1.1), we have

$$(1.6) \quad \mathcal{W}f(z) = z + \sum_{n=2}^{\infty} \sigma_n a_n z^n, \quad z \in U,$$

where

$$\sigma_n = \frac{\Omega \Gamma(\alpha_1 + A_1(n-1)) \cdots \Gamma(\alpha_p + A_p(n-1))}{(n-1)! \Gamma(\beta_1 + B_1(n-1)) \cdots \Gamma(\beta_q + B_q(n-1))},$$

and  $\Omega$  is given by (1.5).

In view of the relationship (1.4), the linear operator (1.6) includes the Dziok-Srivastava operator (see [5]) and other operators. For more details on these operators, see [1], [2], [4], [11], [12], [15] and [19].

Motivated by the earlier works of Kanas and Srivastava [7], Srivastava and Mishra [20] and Vijaya and Murugusundaramoorthy [23], we define a new class of functions based on generalized hypergeometric functions.

Corresponding to the family  $SP(\gamma, k)$ , we define the class  $W_q^p(k, \gamma)$  for a function  $f$  of the form (1.1) such that

$$(1.7) \quad \operatorname{Re} \left\{ \frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)} - \gamma \right\} \geq k \left| \frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)} - 1 \right|, \quad z \in U.$$

We also let

$$VW_q^p(k, \gamma) = V \cap W_q^p(k, \gamma).$$

The class  $W_q^p(k, \gamma)$  generalizes the classes of  $k$ -uniformly convex functions and  $k$ -starlike functions. If  $p = 2$ ,  $q = 1$ ,  $A_1 = A_2 = B_1 = \alpha_1 = \beta_1 = 1$ , then for

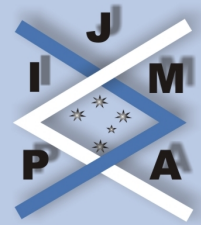
$\alpha_2 = 2$  we have

$$W_1^2(k, 0) = k - UCV,$$

and for  $\alpha_2 = 1$  we have

$$W_1^2(k, 0) = k - ST.$$

In this paper we obtain a sufficient coefficient condition for functions  $f$  given by (1.1) to be in the class  $W_q^p(k, \gamma)$  and we show that it is also a necessary condition for functions to belong to this class. Distortion results and extreme points for functions in  $VW_q^p(k, \gamma)$  are obtained. Finally, we investigate partial sums for the class  $VW_q^p(k, \gamma)$ .



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## 2. Main Results

First we obtain a sufficient condition for functions from the class  $A$  to belong to the class  $W_q^p(k, \gamma)$ .

**Theorem 2.1.** *Let  $f$  be given by (1.1). If*

$$(2.1) \quad \sum_{n=2}^{\infty} (kn + n - k - \gamma) \sigma_n |a_n| \leq 1 - \gamma,$$

then  $f \in W_q^p(k, \gamma)$ .

*Proof.* By definition of the class  $W_q^p([\alpha_1], \gamma)$ , it suffices to show that

$$k \left| \frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)} - 1 \right\} \leq 1 - \gamma, \quad z \in U.$$

Simple calculations give

$$\begin{aligned} & k \left| \frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)} - \gamma \right\} \\ & \leq (k + 1) \left| \frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)} - 1 \right| \\ & \leq (k + 1) \frac{\sum_{n=2}^{\infty} (n - 1) \sigma_n |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \sigma_n |a_n| |z|^{n-1}}. \end{aligned}$$

Now the last expression is bounded above by  $(1 - \gamma)$  if (2.1) holds. □

In the next theorem, we show that the condition (2.1) is also necessary for functions from the class  $VW_q^p(k, \gamma)$ .

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**Theorem 2.2.** Let  $f$  be given by (1.1) and satisfy (1.2). Then the function  $f$  belongs to the class  $VW_q^p(k, \gamma)$  if and only if (2.1) holds.

*Proof.* In view of Theorem 2.1 we need only to show that  $f \in VW_q^p(k, \gamma)$  satisfies the coefficient inequality (2.1). If  $f \in VW_q^p(k, \gamma)$  then by definition, we have

$$k \left| \frac{z + \sum_{n=2}^{\infty} n\sigma_n a_n z^n}{z + \sum_{n=2}^{\infty} \sigma_n a_n z^n} - 1 \right| \leq \operatorname{Re} \left\{ \frac{z + \sum_{n=2}^{\infty} n\sigma_n a_n z^n}{z + \sum_{n=2}^{\infty} \sigma_n a_n z^n} - \gamma \right\},$$

or

$$k \left| \frac{\sum_{n=2}^{\infty} (n-1)\sigma_n a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \sigma_n a_n z^{n-1}} \right| \leq \operatorname{Re} \left\{ \frac{(1-\gamma) + \sum_{n=2}^{\infty} (n-\gamma)\sigma_n a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \sigma_n a_n z^{n-1}} \right\}.$$

In view of (1.2), we set  $z = r^{in}$  in the above inequality to obtain

$$\frac{\sum_{n=2}^{\infty} k(n-1)\sigma_n |a_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} \sigma_n |a_n| r^{n-1}} \leq \frac{(1-\gamma) - \sum_{n=2}^{\infty} (n-\gamma)\sigma_n |a_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} \sigma_n |a_n| r^{n-1}}.$$

Thus

$$(2.2) \quad \sum_{n=2}^{\infty} (kn + n - k - \gamma)\sigma_n |a_n| r^{n-1} \leq 1 - \gamma,$$

and letting  $r \rightarrow 1^-$  in (2.2), we obtain the desired inequality (2.1).  $\square$

**Corollary 2.3.** If a function  $f$  of the form (1.1) belongs to the class  $VW_q^p(k, \gamma)$ , then

$$|a_n| \leq \frac{1-\gamma}{(kn + n - k - \gamma)\sigma_n}, \quad n = 2, 3, \dots$$

The equality holds for the functions

$$(2.3) \quad h_{n,\eta}(z) = z - \frac{(1-\gamma)e^{i(1-n)\eta}}{(kn + n - k - \gamma)\sigma_n} z^n, \quad z \in U; 0 \leq \eta < 2\pi, n = 2, 3, \dots$$



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Next we obtain the distortion bounds for functions belonging to the class  $VW_q^p(k, \gamma)$ .

**Theorem 2.4.** Let  $f$  be in the class  $VW_q^p(k, \gamma)$ ,  $|z| = r < 1$ . If the sequence

$$\{(kn + n - k - \gamma)\sigma_n\}_{n=2}^{\infty}$$

is nondecreasing, then

$$(2.4) \quad r - \frac{1 - \gamma}{(k - \gamma + 2)\sigma_2} r^2 \leq |f(z)| \leq r + \frac{1 - \gamma}{(k - \gamma + 2)\sigma_2} r^2.$$

If the sequence  $\left\{\frac{kn+n-k-\gamma}{n}\sigma_n\right\}_{n=2}^{\infty}$  is nondecreasing, then

$$(2.5) \quad 1 - \frac{2(1 - \gamma)}{(k - \gamma + 2)\sigma_2} r \leq |f'(z)| \leq 1 + \frac{2(1 - \gamma)}{(k - \gamma + 2)\sigma_2} r.$$

The result is sharp. The extremal functions are the functions  $h_{2,n}$  of the form (2.3).

*Proof.* Since  $f \in VW_q^p(k, \gamma)$ , we apply Theorem 2.2 to obtain

$$(k - \gamma + 2)\sigma_2 \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} (kn + n - k - \gamma)\sigma_n |a_n| \leq 1 - \gamma.$$

Thus

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq r + \frac{1 - \gamma}{(k - \gamma + 2)\sigma_2} r^2.$$

Also we have

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \geq r - \frac{1 - \gamma}{(k - \gamma + 2)\sigma_2} r^2$$



and (2.4) follows. In similar manner for  $f'$ , the inequalities

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \leq 1 + |z| \sum_{n=2}^{\infty} na_n$$

and

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{2(1-\gamma)}{(k-\gamma+2)\sigma_2}$$

lead to (2.5). This completes the proof. □

**Corollary 2.5.** *Let  $f$  be in the class  $VW_q^p(k, \gamma)$ ,  $|z| = r < 1$ . If*

$$(2.6) \quad p > q, \alpha_{q+1} \geq 1, \alpha_j \geq \beta_j \quad \text{and} \quad A_j \geq B_j \quad (j = 2, \dots, q),$$

*then the assertions (2.4), (2.5) hold true.*

*Proof.* From (2.6) we have that the sequences  $\{(kn + n - k - \gamma)\sigma_n\}_{n=2}^{\infty}$  and  $\{\frac{kn+n-k-\gamma}{n}\sigma_n\}_{n=2}^{\infty}$  are nondecreasing. Thus, by Theorem 2.4, we have Corollary 2.5. □

**Theorem 2.6.** *Let  $f$  be given by (1.1) and satisfy (1.2). Then the function  $f$  belongs to the class  $VW_q^p(k, \gamma)$  if and only if  $f$  can be expressed in the form*

$$(2.7) \quad f(z) = \sum_{n=1}^{\infty} \mu_n h_{n,\eta}(z), \quad \mu_n \geq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \mu_n = 1,$$

where  $h_1(z) = z$  and  $h_{n,\eta}$  are defined by (2.3).

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*Proof.* If a function  $f$  is of the form (2.7), then by (1.2) we have

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\gamma)e^{i\theta_n}}{(kn+n-k-\gamma)\sigma_n} \mu_n z^n, \quad z \in U.$$

Since

$$\begin{aligned} \sum_{n=2}^{\infty} (kn+n-k-\gamma)\sigma_n \frac{1-\gamma}{(kn+n-k-\gamma)\sigma_n} \mu_n \\ = \sum_{n=2}^{\infty} \mu_n (1-\gamma) = (1-\mu_1)(1-\gamma) \leq 1-\gamma, \end{aligned}$$

by Theorem 2.2 we have  $f \in VW_q^p(k, \gamma)$ .

Conversely, if  $f$  is in the class  $VW_q^p(k, \gamma)$ , then we may set  $\mu_n = \frac{(kn+n-k-\gamma)\sigma_n}{1-\gamma}$ ,  $n \geq 2$  and  $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$ . Then the function  $f$  is of the form (2.7) and this completes the proof.  $\square$

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### 3. Partial Sums

For a function  $f \in A$  given by (1.1), Silverman [17] and Silvia [18] investigated the partial sums  $f_1$  and  $f_m$  defined by

$$(3.1) \quad f_1(z) = z; \quad \text{and} \quad f_m(z) = z + \sum_{n=2}^m a_n z^n, \quad (m = 2, 3, \dots).$$

We consider in this section partial sums of functions in the class  $VW_q^p(k, \gamma)$  and obtain sharp lower bounds for the ratios of the real part of  $f$  to  $f_m(z)$  and  $f'$  to  $f'_m$ .

**Theorem 3.1.** *Let a function  $f$  of the form (1.1) belong to the class  $VW_q^p(k, \gamma)$  and assume (2.6). Then*

$$(3.2) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \geq 1 - \frac{1}{d_{m+1}}, \quad z \in U, m \in N$$

and

$$(3.3) \quad \operatorname{Re} \left\{ \frac{f'_m(z)}{f'(z)} \right\} \geq \frac{d_{m+1}}{1 + d_{m+1}}, \quad z \in U, m \in N,$$

where

$$(3.4) \quad d_n := \frac{kn + n - k - \gamma}{1 - \gamma} \sigma_n.$$

*Proof.* By (2.6) it is not difficult to verify that

$$(3.5) \quad d_{n+1} > d_n > 1, \quad n = 2, 3, \dots$$

Thus by Theorem 2.1 we have

$$(3.6) \quad \sum_{n=2}^m |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} d_n |a_n| \leq 1.$$

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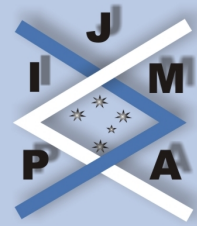
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Setting

$$(3.7) \quad g(z) = d_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left( 1 - \frac{1}{d_{m+1}} \right) \right\} = 1 + \frac{d_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^m a_n z^{n-1}},$$

it suffices to show that

$$\operatorname{Re} g(z) \geq 0, \quad z \in U.$$

Applying (3.6), we find that

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{d_{m+1} \sum_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^m |a_n| - d_{m+1} \sum_{n=m+1}^{\infty} |a_n|} \leq 1, \quad z \in U,$$

which readily yields the assertion (3.2) of Theorem 3.1. In order to see that

$$(3.8) \quad f(z) = z + \frac{z^{m+1}}{d_{m+1}}, \quad z \in U,$$

gives sharp the result, we observe that for  $z = re^{i\pi/m}$  we have

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{d_{m+1}} \xrightarrow{z \rightarrow 1^-} 1 - \frac{1}{d_{m+1}}.$$

Similarly, if we take

$$\begin{aligned} h(z) &= (1 + d_{m+1}) \left\{ \frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1 + d_{m+1}} \right\} \\ &= 1 - \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}}, \quad z \in U, \end{aligned}$$

and making use of (3.6), we can deduce that

$$\left| \frac{h(z) - 1}{h(z) + 1} \right| \leq \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^m |a_n| - (1 + d_{m+1}) \sum_{n=m+1}^{\infty} |a_n|} \leq 1, \quad z \in U,$$

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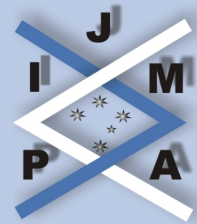
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which leads us immediately to the assertion (3.3) of Theorem 3.1. The bound in (3.3) is sharp for each  $m \in N$  with the extremal function  $f$  given by (3.8), and the proof is complete.  $\square$

**Theorem 3.2.** Let a function  $f$  of the form (1.1) belong to the class  $VW_q^p(k, \gamma)$  and assume (2.6). Then

$$(3.9) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_m(z)} \right\} \geq 1 - \frac{m+1}{d_{m+1}}$$

and

$$(3.10) \quad \operatorname{Re} \left\{ \frac{f'_m(z)}{f'(z)} \right\} \geq \frac{d_{m+1}}{m+1+d_{m+1}},$$

where  $d_m$  is defined by (3.4)

*Proof.* By setting

$$g(z) = d_{m+1} \left\{ \frac{f'(z)}{f'_m(z)} - \left( 1 - \frac{m+1}{d_{m+1}} \right) \right\}, \quad z \in U,$$

and

$$h(z) = [(m+1) + d_{m+1}] \left\{ \frac{f'_m(z)}{f'(z)} - \frac{d_{m+1}}{m+1+d_{m+1}} \right\}, \quad z \in U,$$

the proof is analogous to that of Theorem 3.1, and we omit the details.  $\square$

**Concluding Remarks:** Observe that, if we specialize the parameters of the class  $VW_q^p(k, \gamma)$ , we obtain various classes introduced and studied by Goodman [6], Kanas and Srivastava [7], Ma and Minda [10], Rønning [13, 14], Murugusundaramoorthy *et al.* [22, 23], and others.

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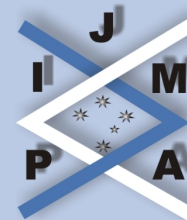
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## References

- [1] B.C. CARLSON AND S.B. SHAFFER, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (2002), 737–745.
- [2] B.C. CARLSON, *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.
- [3] J. DZIOK AND R.K. RAINA, Families of analytic functions associated with the Wright generalized hypergeometric function, *Demonstratio Math.*, **37**(3) (2004), 533–542.
- [4] J. DZIOK AND H.M. SRIVASTAVA, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transform Spec. Funct.*, **14** (2003), 7–18.
- [5] J. DZIOK AND H.M. SRIVASTAVA, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, **103** (1999), 1–13.
- [6] A.W. GOODMAN, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87–92.
- [7] S. KANAS AND H.M. SRIVASTAVA, Linear operators associated with  $k$ -uniformly convex functions, *Integral Transform and Spec. Funct.*, **9** (2000), 121–132.
- [8] S. KANAS AND A. WISNIOWSKA, Conic regions and  $k$ -uniform convexity, *J. Comput. Appl. Math.*, **105** (1999), 327–336.
- [9] S. KANAS AND A. WISNIOWSKA, Conic regions and  $k$ -starlike functions, *Rev. Roumaine Math. Pures Appl.* **45** (2000), 647–657.



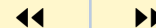
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- [10] W. MA AND D. MINDA, Uniformly convex functions, *Ann. Polon. Math.*, **57** (1992), 165–175.
- [11] S. PONNUSAMY AND S. SABAPATHY, Geometric properties of generalized hypergeometric functions, *Ramanujan Journal*, **1** (1997), 187–210.
- [12] E.D. RAINVILLE, *Special Functions*, Chelsea Publishing Co., New York, 1960.
- [13] F. RØNNING, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 189–196.
- [14] F. RØNNING, Integral representations for bounded starlike functions, *Annal. Polon. Math.*, **60** (1995), 289–297.
- [15] S. RUSCHEWEYH, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109–115.
- [16] H. SILVERMAN, Univalent functions with varying arguments, *Houston J. Math.*, **7** (1981), 283–287.
- [17] H. SILVERMAN, Partial sums of starlike and convex functions, *J. Math. Anal. & Appl.*, **209** (1997), 221–227.
- [18] E.M. SILVIA, Partial sums of convex functions of order  $\alpha$ , *Houston. J. Math., Math. Soc.*, **11**(3) (1985), 397–404.
- [19] H.M. SRIVASTAVA AND S. OWA, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions, *Nagoya Math. J.*, **106** (1987), 1–28.

---

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- [20] H.M. SRIVASTAVA AND A.K. MISHRA, Applications of fractional calculus to parabolic starlike and uniformly convex functions, *Computers Math. Applic.*, **39** (2000), 57–69.
- [21] K.G. SUBRAMANIAN, G. MURUGUSUNDARAMOORTHY, P. BALASUBRAHMANYAM AND H. SILVERMAN, Subclasses of uniformly convex and uniformly starlike functions, *Math. Japonica*, **42**(3) (1995), 517–522.
- [22] T. ROSY, K.G. SUBRAMANIAN AND G. MURUGUSUNDARAMOORTHY, Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives, *J. Ineq. Pure and Appl. Math.*, **4**(4) (2003), Art. 64. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=305>].
- [23] K. VIJAYA AND G. MURUGUSUNDARAMOORTHY, Some uniformly starlike functions with varying arguments, *Tamkang. J. Math.*, **35**(1) (2004), 23–28.
- [24] E.M. WRIGHT, The asymptotic expansion of the generalized hypergeometric function, *Proc. London. Math. Soc.*, **46** (1946), 389–408.

---

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