A MULTIPLE HARDY-HILBERT'S INTEGRAL INEQUALITY WITH THE BEST CONSTANT FACTOR

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November 14, 2006

Abstract

In this paper, by introducing the norm $\|x\|_\alpha (x \in R^n)$, we give a multiple Hardy-Hilbert's integral inequality with a best constant factor and two parameters α, λ.

Key words: multiple Hardy-Hilbert's integral inequality, the Γ-function, the best constant factor.

Mathematics subject classification (2000): 26D15

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \geq 0$, $g \geq 0$, $0 < \int_0^\infty f^p(x)dx < +\infty$, $0 < \int_0^\infty g^q(x)dx < +\infty$, then the well known Hardy-Hilbert's integral inequality is given by (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx \right)^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible. it's equivalent form is:

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \int_0^\infty f^p(x)dx, \tag{2}$$

where the constant factor $\left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p$ is also the best possible.

Hardy-Hilbert't inequality is valuable in harmonic analysis, real analysis and operator theory. In recent years, many valuable results (see [2-5]) have been obtained in generalization and improvement of Hardy-Hilbert's inequality. In 1999,Kuang [6] gave a generalization with a parameter λ of (1) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < h_\lambda(p)\left(\int_0^\infty x^{1-\lambda}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{1-\lambda}g^q(x)dx\right)^{\frac{1}{q}}, \quad (3)$$

where $\max\{\frac{1}{p},\frac{1}{q}\} < \lambda \le 1$, $h_\lambda(p) = \pi[\lambda \sin^{\frac{1}{p}}(\frac{\pi}{p\lambda})\sin^{\frac{1}{q}}(\frac{\pi}{q\lambda})]^{-1}$. Because of the constant factor $h_\lambda(p)$ being not the best possible, Yang [7] gave a new generalization of (1) in 2002 as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy < \frac{\pi}{\lambda\sin(\frac{\pi}{p})}$$
$$\times\left(\int_0^\infty x^{(1-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{(1-\lambda)(q-1)}g^q(x)dx\right)^{\frac{1}{q}}, \quad (4)$$

it's equivalent form is:

$$\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{f(x)}{x^\lambda+y^\lambda}dx\right)^p dy < \left[\frac{\pi}{\lambda\sin(\frac{\pi}{p})}\right]^p\int_0^\infty x^{(1-\lambda)(p-1)}f^p(x)dx, \quad (5)$$

where the constant factors $\frac{\pi}{\lambda\sin(\frac{\pi}{p})}$ in (4) and $[\frac{\pi}{\lambda\sin(\frac{\pi}{p})}]^p$ in (5) are all the best possible.

At present, because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hardy-Hilbert's integral inequality is researched. Hong [8] obtained: If $a>0$, $\sum_{i=1}^n\frac{1}{p_i}=1$, $p_i>1$, $f_i\ge 0$, $r_i=\frac{1}{p_i}\prod_{j=1}^n p_j$, $\lambda > \frac{1}{a}(n-1-\frac{1}{r_i})$, $i=1,2,\cdots,n$, then

$$\int_\alpha^\infty\cdots\int_\alpha^\infty \frac{1}{[\sum_{i=1}^n(x_i-\alpha_i)^a]^\lambda}\prod_{i=1}^n f_i(x)dx_1\cdots dx_n \le \frac{\Gamma^{n-2}(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\lambda)}$$
$$\times\prod_{i=1}^n\left[\Gamma\left(\frac{1}{a}(1-\frac{1}{r_i})\right)\Gamma\left(\lambda-\frac{1}{a}(n-1-\frac{1}{r_i})\right)\int_\alpha^\infty (t-\alpha)^{n-1-\alpha\lambda}f_i^{p_i}dt\right]^{\frac{1}{p_i}}. \quad (6)$$

Afterwards, Bicheng Yang and Kuang Jichang etc. obtained some multiple Hardy-Hilbert's integral inequalities (see [9-10]).

In this paper, by introducing the Γ-function, we generalize (3) and (4) into multiple Hardy-Hilbert's integral inequalities with the best constant factors.

2 Some lemmas

First of all, we introduce the signs as:

$$R_+^n = \{x=(x_1,\cdots,x_n): x_1,\ldots,x_n>0\},$$
$$\|x\|_\alpha = (x_1^\alpha+\cdots+x_n^\alpha)^{\frac{1}{\alpha}}, \alpha>0.$$

Lemma 2.1 (see [11]) *If* $p_i > 0$, $a_i > 0$, $\alpha_i > 0$, $i = 1, 2, \cdots, n$, $\Psi(u)$ *is a measurable function, then*

$$\int \cdots \int_{x_1,\cdots,x_n>0;(\frac{x_1}{a_1})^{\alpha_1}+\cdots+(\frac{x_n}{a_n})^{\alpha_n}\le 1} \Psi\left((\frac{x_1}{a_1})^{\alpha_1} + \cdots + (\frac{x_n}{a_n})^{\alpha_n}\right) \times x_1^{p_1-1} \cdots x_n^{p_n-1} dx_1 \cdots dx_n$$
$$= \frac{a_1^{p_1} \cdots a_n^{p_n} \Gamma(\frac{p_1}{\alpha_1}) \cdots \Gamma(\frac{p_n}{\alpha_n})}{\alpha_1 \cdots \alpha_n \Gamma(\frac{p_1}{\alpha_1} + \cdots + \frac{p_n}{\alpha_n})} \int_0^1 \Psi(u) u^{\frac{p_1}{\alpha_1}+\cdots+\frac{p_n}{\alpha_n}-1} du. \quad (7)$$

where $\Gamma(\cdot)$ *is the* Γ*-function.*

Lemma 2.2 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \in Z_+$, $\alpha > 0$, $\lambda > 0$, *setting the weight function* $\omega_{\alpha,\lambda}(x, p, q)$ *as:*

$$\omega_{\alpha,\lambda}(x, p, q) = \int_{R_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{(n-\lambda)p} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{q}} dy,$$

then

$$\omega_{\alpha,\lambda}(x, p, q) = \|x\|_\alpha^{(n-\lambda)(p-1)} \frac{\pi \Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p}) \lambda \alpha^{n-1} \Gamma(\frac{n}{\alpha})}. \quad (8)$$

Proof By lemma 2.1, we have

$$\omega_{\alpha,\lambda}(x, p, q) = \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \int_{R_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \|y\|_\alpha^{-(n-\lambda+\frac{\lambda}{q})} dy$$
$$= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \lim_{R\to+\infty} \int \cdots \int_{y_1,\cdots,y_n>0;y_1^\alpha+\cdots+y_n^\alpha<r^\alpha}$$
$$\times \frac{\left[r((\frac{y_1}{r})^\alpha + \cdots + (\frac{y_n}{r})^\alpha)^{\frac{1}{\alpha}}\right]^{-(n-\lambda+\frac{\lambda}{q})}}{\|x\|_\alpha^\lambda + \left[r((\frac{y_1}{r})^\alpha + \cdots + (\frac{y_n}{r})^\alpha)^{\frac{1}{\alpha}}\right]^\lambda} y_1^{1-1} \cdots y_n^{1-1} dy_1 \cdots dy_n$$
$$= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \lim_{R\to+\infty} \frac{r^n \Gamma^n(\frac{1}{\alpha})}{\alpha^n \Gamma(\frac{n}{\alpha})} \int_0^1 \frac{(ru^{\frac{1}{\alpha}})^{-(n-\lambda+\frac{\lambda}{q})}}{\|x\|_\alpha^\lambda + (ru^{\frac{1}{\alpha}})^\lambda} u^{\frac{n}{\alpha}-1} du$$
$$= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \lim_{R\to+\infty} \int_0^r \frac{1}{\|x\|_\alpha^\lambda + u^\lambda} u^{\lambda-\frac{\lambda}{q}-1} du$$
$$= \|x\|_\alpha^{(n-\lambda)(p-1)+\frac{\lambda}{q}} \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \int_0^\infty \frac{1}{\|x\|_\alpha^\lambda + u^\lambda} u^{\lambda-\frac{\lambda}{q}-1} du$$
$$= \|x\|_\alpha^{(n-\lambda)(p-1)} \frac{\Gamma^n(\frac{1}{\alpha})}{\lambda \alpha^{n-1}\Gamma(\frac{n}{\alpha})} \int_0^\infty \frac{1}{1+u} u^{\frac{1}{p}-1} du$$

$$
\begin{aligned}
&= \|x\|_\alpha^{(n-\lambda)(p-1)} \frac{\Gamma^n(\frac{1}{\alpha})}{\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\Gamma(\frac{1}{p})\Gamma(1-\frac{1}{p}) \\
&= \|x\|_\alpha^{(n-\lambda)(p-1)} \frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})},
\end{aligned}
$$

hence (8) is valid.

Lemma 2.3 *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $n\in Z_+$, $\alpha>0$, $\lambda>0$, $0<\varepsilon<\lambda(q-1)$, setting $\tilde{\omega}_{\alpha,\lambda}(x,q,\varepsilon)$ as:*

$$
\tilde{\omega}_{\alpha,\lambda}(x,q,\varepsilon) = \int_{R_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda}\|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}} dy,
$$

then we have

$$
\tilde{\omega}_{\alpha,\lambda}(x,q,\varepsilon) = \|x\|_\alpha^{-\frac{\lambda}{q}-\frac{\varepsilon}{q}} \frac{\Gamma^n(\frac{1}{\alpha})}{\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\Gamma(\frac{1}{p}-\frac{\varepsilon}{\lambda q})\Gamma(\frac{1}{q}+\frac{\varepsilon}{\lambda q}). \tag{9}
$$

Proof By the same way of lemma 2.2, lemma 2.3 can be proved.

3 Main Results

Theorem 3.1 *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $n\in Z_+$, $\alpha>0$, $\lambda>0$, $f\geq 0$, $g\geq 0$, and*

$$
0<\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx<\infty, \qquad 0<\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(q-1)}g^q(x)dx<\infty, \tag{10}
$$

then

$$
\begin{aligned}
\int_{R_+^n}\int_{R_+^n}\frac{f(x)g(y)}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}dxdy &< \frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \\
&\times\left(\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(q-1)}g^q(x)dx\right)^{\frac{1}{q}};
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\int_{R_+^n}\|y\|_\alpha^{\lambda-n}\left(\int_{R_+^n}\frac{f(x)}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}dx\right)^p dy &< \left[\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right]^p \\
&\times\int_{R_+^n}\|x\|_\alpha^{(n-\lambda)(p-1)}f^p(x)dx,
\end{aligned} \tag{12}
$$

where the constant factors $\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}$ and $[\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}]^p$ are all the best possible.

Proof By Hölder's inequality, we have

$$A := \int_{R_+^n}\int_{R_+^n} \frac{f(x)g(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dxdy$$

$$= \int_{R_+^n}\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda)^{\frac{1}{p}}} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{n-\lambda} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{pq}}$$

$$\times \frac{g(y)}{(\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda)^{\frac{1}{q}}} \left(\frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}}\right)^{n-\lambda} \left(\frac{\|y\|_\alpha}{\|x\|_\alpha}\right)^{\frac{\lambda}{pq}} dxdy$$

$$\leq \left[\int_{R_+^n}\int_{R_+^n} \frac{f^p(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{(n-\lambda)p} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{q}} dxdy\right]^{\frac{1}{p}}$$

$$\times \left[\int_{R_+^n}\int_{R_+^n} \frac{g^q(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}}\right)^{(n-\lambda)q} \left(\frac{\|y\|_\alpha}{\|x\|_\alpha}\right)^{\frac{\lambda}{p}} dxdy\right]^{\frac{1}{q}}$$

$$= \left[\int_{R_+^n} f^p(x)\left(\int_{R_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{(n-\lambda)p} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{q}} dy\right) dx\right]^{\frac{1}{p}}$$

$$\times \left[\int_{R_+^n} g^q(y)\left(\int_{R_+^n} \frac{1}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}}\right)^{(n-\lambda)q} \left(\frac{\|y\|_\alpha}{\|x\|_\alpha}\right)^{\frac{\lambda}{p}} dx\right) dy\right]^{\frac{1}{q}}$$

$$= \left(\int_{R_+^n} f^p(x)\omega_{\alpha,\lambda}(x,p,q)dx\right)^{\frac{1}{p}} \left(\int_{R_+^n} g^q(y)\omega_{\alpha,\lambda}(y,q,p)dy\right)^{\frac{1}{q}},$$

according to the condition of taking equality in Hölder's inequality, if this inequality takes the form of an equality, then there exist constants C_1 and C_2, such that they are not all zero, and

$$\frac{C_1 f^p(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|x\|_\alpha^{\frac{1}{q}}}{\|y\|_\alpha^{\frac{1}{p}}}\right)^{(n-\lambda)p} \left(\frac{\|x\|_\alpha}{\|y\|_\alpha}\right)^{\frac{\lambda}{q}}$$

$$= \frac{C_2 g^q(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} \left(\frac{\|y\|_\alpha^{\frac{1}{p}}}{\|x\|_\alpha^{\frac{1}{q}}}\right)^{(n-\lambda)q} \left(\frac{\|y\|_\alpha}{\|x\|_\alpha}\right)^{\frac{\lambda}{p}}, \quad a.e.\ in\ R_+^n \times R_+^n,$$

it follows that

$$C_1\|x\|_\alpha^{(n-\lambda)(p-1)+n} f^p(x) = C_2\|y\|_\alpha^{(n-\lambda)(q-1)+n} g^q(y) = C(constant), \quad a.e.\ in\ R_+^n \times R_+^n,$$

which contradicts (10), hence we have

$$A < \left(\int_{R_+^n} f^p(x)\omega_{\alpha,\lambda}(x,p,q)dx \right)^{\frac{1}{p}} \left(\int_{R_+^n} g^q(y)\omega_{\alpha,\lambda}(y,q,p)dy \right)^{\frac{1}{q}}.$$

By lemma 2.2 and $\sin(\frac{\pi}{p}) = \sin(\frac{\pi}{q})$, we have

$$\begin{aligned} A &< \left[\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \right]^{\frac{1}{p}} \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx \right)^{\frac{1}{p}} \\ &\quad \times \left[\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{q})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \right]^{\frac{1}{q}} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g^q(y)dy \right)^{\frac{1}{q}} \\ &= \frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(q-1)} g^q(x)dx \right)^{\frac{1}{q}}. \end{aligned}$$

Hence (11) is valid.

For $0 < a < b < \infty$, setting

$$g_{a,b}(y) = \begin{cases} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx \right)^{p-1}, & a < \|y\|_\alpha < b \\ 0, & 0 < \|y\|_\alpha \le a \quad or \quad \|y\|_\alpha \ge b \end{cases}$$

$$\tilde{g}(y) = \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx \right)^{p-1}, \ \ y \in R_+^n,$$

by (10), for sufficiently small $a > 0$ and sufficiently large $b > 0$, we have

$$0 < \int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{(n-\lambda)(q-1)} g_{a,b}^q(y)dy < \infty.$$

Hence by (11), we have

$$\begin{aligned} &\int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy = \int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx \right)^p dy \\ &= \int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx \right)^{p-1} \left(\int_{R_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx \right) dy \\ &= \int_{R_+^n} \int_{R_+^n} \frac{f(x)g_{a,b}(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dxdy \\ &< \frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g_{a,b}^q(y)dy \right)^{\frac{1}{q}} \\ &= \frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy \right)^{\frac{1}{q}}, \end{aligned}$$

it follows that

$$\int_{a<\|y\|_\alpha<b} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy < \left[\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right]^p \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx.$$

For $a \to 0^+$, $b \to +\infty$, by (10), we obtain

$$0 < \int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy \leq \left[\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right]^p \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx < \infty,$$

hence by (11), we have

$$\begin{aligned}
&\int_{R_+^n} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx\right)^p dy = \int_{R_+^n}\int_{R_+^n} \frac{f(x)\tilde{g}(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dxdy \\
< \ & \frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} \tilde{g}^q(y)dy\right)^{\frac{1}{q}} \\
= \ & \frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}} \\
&\times \left[\int_{R_+^n} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx\right)^p dy\right]^{\frac{1}{q}},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_{R_+^n} \|y\|_\alpha^{\lambda-n} \left(\int_{R_+^n} \frac{f(x)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dx\right)^p dy \\
< \ & \left[\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right]^p \int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx.
\end{aligned}$$

Hence (12) is valid.

Remark: By (12), we can also obtain (11), hence (12) and (11) are equivalent.

If the constant factor $C_{n,\alpha}(\lambda,p) := \frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}$ in (11) is not the best possible, then there exists a positive constant $K < C_{n,\alpha}(\lambda,p)$, such that

$$\begin{aligned}
&\int_{R_+^n}\int_{R_+^n} \frac{f(x)g(y)}{\|x\|_\alpha^\lambda + \|y\|_\alpha^\lambda} dxdy \\
< \ & K\left(\int_{R_+^n} \|x\|_\alpha^{(n-\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\lambda)(q-1)} g^q(y)dy\right)^{\frac{1}{q}}. \qquad (13)
\end{aligned}$$

In particular, setting

$$f(x)=\|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}},\quad g(y)=\|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}},\quad 0<\varepsilon<\lambda(q-1),$$

(13) is still true. By the properties of limit, there exists a sufficiently small $a>0$, such that

$$\begin{aligned}
&\int_{\|x\|_\alpha>a}\int_{R_+^n}\frac{1}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}\|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}}\|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}}dxdy\\
<\ & K\left(\int_{\|x\|_\alpha>a}\|x\|_\alpha^{(n-\lambda)(p-1)}\|x\|_\alpha^{-(n-\lambda)(p-1)-n-\varepsilon}dx\right)^{\frac{1}{p}}\\
&\times\left(\int_{\|y\|_\alpha>a}\|y\|_\alpha^{(n-\lambda)(q-1)}\|y\|_\alpha^{-(n-\lambda)(q-1)-n-\varepsilon}dy\right)^{\frac{1}{q}}\\
=\ & K\int_{\|x\|_\alpha>a}\|x\|_\alpha^{-n-\varepsilon}dx.
\end{aligned}$$

On the other hand, by lemma 2.3, we have

$$\begin{aligned}
&\int_{\|x\|_\alpha>a}\int_{R_+^n}\frac{1}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}\|x\|_\alpha^{-\frac{(n-\lambda)(p-1)+n+\varepsilon}{p}}\|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}}dxdy\\
=\ & \int_{\|x\|_\alpha>a}\|x\|_\alpha^{-n+\frac{\lambda}{q}-\frac{\varepsilon}{p}}\int_{R_+^n}\frac{1}{\|x\|_\alpha^\lambda+\|y\|_\alpha^\lambda}\|y\|_\alpha^{-\frac{(n-\lambda)(q-1)+n+\varepsilon}{q}}dydx\\
=\ & \int_{\|x\|_\alpha>a}\|x\|_\alpha^{-n+\frac{\lambda}{q}-\frac{\varepsilon}{p}}\tilde{\omega}_{\alpha,\lambda}(x,q,\varepsilon)dx\\
=\ & \frac{\Gamma^n(\frac{1}{\alpha})}{\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\Gamma\left(\frac{1}{p}-\frac{\varepsilon}{\lambda q}\right)\Gamma\left(\frac{1}{q}+\frac{\varepsilon}{\lambda q}\right)\int_{\|x\|_\alpha>a}\|x\|_\alpha^{-n-\varepsilon}dx,
\end{aligned}$$

hence we obtain

$$\frac{\Gamma^n(\frac{1}{\alpha})}{\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\Gamma\left(\frac{1}{p}-\frac{\varepsilon}{\lambda q}\right)\Gamma\left(\frac{1}{q}+\frac{\varepsilon}{\lambda q}\right)<K.$$

for $\varepsilon\to0^+$, we have

$$C_{n,\alpha}(\lambda,p)=\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})\lambda\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\le K,$$

this contradicts the fact that $K<C_{n,\alpha}(\lambda,p)$.Hence the constant factor in (11) is the best possible.

Since (12) and (11) are equivalent, the constant factor in (12) is also the best possible.

Corollary 3.1 *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $n\in Z_+$, $\alpha>0$, $f\ge0$, $g\ge0$, and*

$$0<\int_{R_+^n}f^p(x)dx<\infty,\qquad 0<\int_{R_+^n}g^q(x)dx<\infty,\tag{14}$$

then

$$\int_{R_+^n}\int_{R_+^n}\frac{f(x)g(y)}{\|x\|_\alpha^n+\|y\|_\alpha^n}dxdy<\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})n\alpha^{n-1}\Gamma(\frac{n}{\alpha})}$$
$$\times\left(\int_{R_+^n}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{R_+^n}g^q(x)dx\right)^{\frac{1}{q}}; \qquad (15)$$

$$\int_{R_+^n}\left(\int_{R_+^n}\frac{f(x)}{\|x\|_\alpha^n+\|y\|_\alpha^n}dx\right)^p dy<\left[\frac{\pi\Gamma^n(\frac{1}{\alpha})}{\sin(\frac{\pi}{p})n\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right]^p\int_{R_+^n}f^p(x)dx, \qquad (16)$$

where the constant factors in (15) and (16) are all the best possible.

Proof By taking $\lambda=n$ in theorem 3.1, (15) and (16) can be obtained.

Corollary 3.2 *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $n\in Z_+$, $f\geq 0$, $g\geq 0$, *and*

$$0<\int_{R_+^n}f^p(x)dx<\infty, \qquad 0<\int_{R_+^n}g^q(x)dx<\infty, \qquad (17)$$

then

$$\int_{R_+^n}\int_{R_+^n}\frac{f(x)g(y)}{\left(\sum_{i=1}^n x_i\right)^n+\left(\sum_{i=1}^n y_i\right)^n}dxdy<\frac{\pi}{n!\sin(\frac{\pi}{p})}$$
$$\times\left(\int_{R_+^n}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{R_+^n}g^q(x)dx\right)^{\frac{1}{q}}; \qquad (18)$$

$$\int_{R_+^n}\left(\int_{R_+^n}\frac{f(x)}{\left(\sum_{i=1}^n x_i\right)^n+\left(\sum_{i=1}^n y_i\right)^n}dx\right)^p dy<\left[\frac{\pi}{n!\sin(\frac{\pi}{p})}\right]^p\int_{R_+^n}f^p(x)dx, \qquad (19)$$

where the constant factors in (18) and (19) are all the best possible.

Proof By taking $\alpha=1$ in corollary 3.1, (18) and (19) can be obtained.

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