A GENERALIZATION OF THE MALIGRANDA - ORLICZ LEMMA

RENÉ ERLÍN CASTILLO AND EDUARD TROUSSELOT
Departamento de Matemáticas
Universidad de Oriente
6101 Cumaná, Edo. Sucre, Venezuela
EMail: rcastill@math.ohiou.edu eddycharles2007@hotmail.com

Received: 23 August, 2007

Accepted: 3 December, 2007

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 46J10.

Key words: Banach algebra, Maligranda-Orlicz.

Abstract: In their 1987 paper, L. Maligranda and W. Orlicz gave a lemma which supplies a test to check that some function spaces are Banach algebras. In this paper we give a more general version of the Maligranda - Orlicz lemma.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

Generalization of the Maligranda - Orlicz Lemma
René Erlín Castillo and Eduard Trousselot
vol. 8, iss. 4, art. 115, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

The following lemma is due to L. Maligranda and W. Orlicz (see [1]).

Lemma 1.1. *Let* $(X, \|\cdot\|)$ *be a Banach space whose elements are bounded functions, which is closed under pointwise multiplication of functions. Let us assume that* $f \cdot g \in X$ *and*

$$\|fg\| \leq \|f\|_\infty \cdot \|g\| + \|f\| \cdot \|g\|_\infty \tag{1.1}$$

for any $f, g \in X$. *Then the space* X *equipped with the norm*

$$\|f\|_1 = \|f\|_\infty + \|f\|$$

is a normed Banach algebra. Also, if $X \hookrightarrow B[a, b]$, *then the norms* $\|\cdot\|_1$ *and* $\|\cdot\|$ *are equivalent. Moreover, if* $\|f\|_\infty \leq M\|f\|$ *for* $f \in X$, *then* $(X, \|\cdot\|_2)$ *is a normed Banach algebra with* $\|f\|_2 = 2M\|f\|$, $f \in X$ *and the norms* $\|\cdot\|_2$ *and* $\|\cdot\|$ *are equivalent.*

At least one easy example might be enlightening here. Recall that the Lipschitz function space (denoted by $\mathrm{Lip}[a, b]$) equipped with the norm

$$\|\cdot\|_{\mathrm{Lip}[a,b]} = |f(a)| + \mathrm{Lip}(f) \quad f \in \mathrm{Lip}[a, b],$$

where $\mathrm{Lip}(f) = \sup_{x \neq y} \left| \frac{f(x)-f(y)}{x-y} \right|$, is a Banach space, which is closed under the usual pointwise multiplication.

Next, we claim that $\mathrm{Lip}[a, b]$ is a Banach algebra. To see this , we just need to

journal of inequalities in pure and applied mathematics
issn: 1443-5756

check (1.1) from Lemma 1.1. Indeed,

$$\left|\frac{fg(x)-fg(y)}{x-y}\right| \le |f(x)|\left|\frac{g(x)-g(y)}{x-y}\right| + |g(y)|\left|\frac{f(x)-f(y)}{x-y}\right|, \quad x \neq y$$
$$\le \|f\|_\infty \operatorname{Lip}(g) + \|g\|_\infty \operatorname{Lip}(f), \tag{1.2}$$

since $\|fg\|_{\operatorname{Lip}_{[a,b]}} = |fg(a)| + \operatorname{Lip}(fg)$.

By (1.2) we have

$$\|fg\|_{\operatorname{Lip}_{[a,b]}} \le 2|f(a)||g(a)| + \|f\|_\infty \operatorname{Lip}(g) + \|g\|_\infty \operatorname{Lip}(f)$$
$$\le \|f\|_\infty|g(a)| + |f(a)| + |f(a)|\|g\|_\infty + \|f\|_\infty \operatorname{Lip}(g) + \|g\|_\infty \operatorname{Lip}(f). \tag{1.3}$$

Thus

$$\|fg\|_{\operatorname{Lip}[a,b]} \le \|f\|_\infty \|g\|_{\operatorname{Lip}_{[a,b]}} + \|g\|_\infty \|g\|_{\operatorname{Lip}[a,b]}.$$

On the other hand, since $BV[a,b] \hookrightarrow B[a,b]$ it is not hard to see that

$$\|f\|_\infty \le \max\{1, b-a\} \|f\|_{\operatorname{Lip}_{[a,b]}}. \tag{1.4}$$

Then by (1.3) and (1.4) we can invoke Lemma 1.1 to conclude that $\operatorname{Lip}[a,b]$ is a Banach algebra either with the norm

$$\|\cdot\|_1 = \|\cdot\|_\infty + \|\cdot\|_{\operatorname{Lip}_{[a,b]}}$$

or

$$\|\cdot\|_2 = 2\max\{1, b-a\} \|\cdot\|_{\operatorname{Lip}_{[a,b]}}$$

which are equivalent to the norm $\|\cdot\|_{\operatorname{Lip}_{[a,b]}}$.

Generalization of the Maligranda - Orlicz Lemma
René Erlín Castillo and Eduard Trousselot
vol. 8, iss. 4, art. 115, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Main Result

Theorem 2.1. *Let* $(X, \|\cdot\|)$ *be a Banach space whose elements are bounded functions, which is closed under pointwise multiplication of functions. Let us assume that* $f \cdot g \in X$ *such that*

$$\|fg\| \le \|f\|_\infty \|g\| + \|f\|\|g\|_\infty + K\|f\|\|g\|, \qquad K > 0.$$

Then $(X, \|\cdot\|_1)$ *equipped with the norm*

$$\|f\|_1 = \|f\|_\infty + K\|f\|, \qquad f \in X,$$

is a Banach algebra. If $X \hookrightarrow B[a,b]$, *then* $\|\cdot\|_1$ *and* $\|\cdot\|$ *are equivalent.*

Proof. First of all, we need to show that $\|fg\|_1 \le \|f\|_1\|g\|_1$ for all $f, g \in X$. In fact,

$$\begin{aligned} \|fg\|_1 &= \|fg\|_\infty + K\|fg\| \\ &\le \|f\|_\infty\|g\|_\infty + K\|f\|_\infty\|g\| \\ &\qquad + K\|f\|\|g\|_\infty + K^2\|f\,\|\cdot\|\,g\| \\ &= (\|f\|_\infty + K\|f\|)(\|g\|_\infty + K\|g\|) \\ &= \|f\|_1\|g\|_1. \end{aligned}$$

This tells us that $(X, \|\cdot\|)$ is a Banach algebra. It only remains to show that $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms.

Indeed, since $X \hookrightarrow B[a,b]$, there exists a constant $L > 0$ such that

$$\|\cdot\|_\infty \le L\,\|\cdot\|\,.$$

Thus

$$\begin{aligned} K\,\|\cdot\| &\le \|\cdot\|_\infty + K\,\|\cdot\| = \|\cdot\|_1 \\ &\le L\,\|\cdot\| + K\,\|\cdot\| = (L+K)\,\|\cdot\|\,. \end{aligned}$$

vol. 8, iss. 4, art. 115, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Hence

$$K\,\|\cdot\| \le \|\cdot\|_1 \le (L+K)\,\|\cdot\|\,.$$

This completes the proof of Theorem 2.1. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] L. MALIGRANDA AND W. ORLICZ, On some properties of functions of generalized variation, *Monastsh Math.*, **104** (1987), 53–65.

journal of inequalities in pure and applied mathematics
issn: 1443-5756