

A GENERALIZATION OF THE MALIGRANDA - ORLICZ LEMMA

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Abstract: In their 1987 paper, L. Maligranda and W. Orlicz gave a lemma which supplies a test to check that some function spaces are Banach algebras. In this paper we give a more general version of the Maligranda - Orlicz lemma.



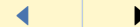
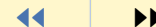
Generalization of the Maligranda -
Orlicz Lemma

René Erlín Castillo and
Eduard Trousselot

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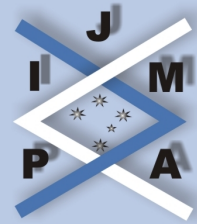
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1. Introduction

The following lemma is due to L. Maligranda and W. Orlicz (see [1]).

Lemma 1.1. *Let $(X, \|\cdot\|)$ be a Banach space whose elements are bounded functions, which is closed under pointwise multiplication of functions. Let us assume that $f \cdot g \in X$ and*

$$(1.1) \quad \|fg\| \leq \|f\|_\infty \cdot \|g\| + \|f\| \cdot \|g\|_\infty$$

for any $f, g \in X$. Then the space X equipped with the norm

$$\|f\|_1 = \|f\|_\infty + \|f\|$$

is a normed Banach algebra. Also, if $X \hookrightarrow B[a, b]$, then the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent. Moreover, if $\|f\|_\infty \leq M\|f\|$ for $f \in X$, then $(X, \|\cdot\|_2)$ is a normed Banach algebra with $\|f\|_2 = 2M\|f\|$, $f \in X$ and the norms $\|\cdot\|_2$ and $\|\cdot\|$ are equivalent.

At least one easy example might be enlightening here. Recall that the Lipschitz function space (denoted by $\text{Lip}[a, b]$) equipped with the norm

$$\|\cdot\|_{\text{Lip}[a,b]} = |f(a)| + \text{Lip}(f) \quad f \in \text{Lip}[a, b],$$

where $\text{Lip}(f) = \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|$, is a Banach space, which is closed under the usual pointwise multiplication.

Next, we claim that $\text{Lip}[a, b]$ is a Banach algebra. To see this, we just need to

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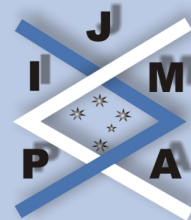


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check (1.1) from Lemma 1.1. Indeed,

$$(1.2) \quad \left| \frac{fg(x) - fg(y)}{x - y} \right| \leq |f(x)| \left| \frac{g(x) - g(y)}{x - y} \right| \\ + |g(y)| \left| \frac{f(x) - f(y)}{x - y} \right|, \quad x \neq y \\ \leq \|f\|_\infty \text{Lip}(g) + \|g\|_\infty \text{Lip}(f),$$

since $\|fg\|_{\text{Lip}[a,b]} = |fg(a)| + \text{Lip}(fg)$.

By (1.2) we have

$$(1.3) \quad \|fg\|_{\text{Lip}[a,b]} \leq 2|f(a)||g(a)| + \|f\|_\infty \text{Lip}(g) \\ + \|g\|_\infty \text{Lip}(f) \\ \leq \|f\|_\infty |g(a)| + |f(a)| + |f(a)| \|g\|_\infty \\ + \|f\|_\infty \text{Lip}(g) + \|g\|_\infty \text{Lip}(f).$$

Thus

$$\|fg\|_{\text{Lip}[a,b]} \leq \|f\|_\infty \|g\|_{\text{Lip}[a,b]} + \|g\|_\infty \|f\|_{\text{Lip}[a,b]}.$$

On the other hand, since $BV[a, b] \hookrightarrow B[a, b]$ it is not hard to see that

$$(1.4) \quad \|f\|_\infty \leq \max\{1, b - a\} \|f\|_{\text{Lip}[a,b]}.$$

Then by (1.3) and (1.4) we can invoke Lemma 1.1 to conclude that $\text{Lip}[a, b]$ is a Banach algebra either with the norm

$$\|\cdot\|_1 = \|\cdot\|_\infty + \|\cdot\|_{\text{Lip}[a,b]}$$

or

$$\|\cdot\|_2 = 2 \max\{1, b - a\} \|\cdot\|_{\text{Lip}[a,b]}$$

which are equivalent to the norm $\|\cdot\|_{\text{Lip}[a,b]}$.



2. Main Result

Theorem 2.1. Let $(X, \|\cdot\|)$ be a Banach space whose elements are bounded functions, which is closed under pointwise multiplication of functions. Let us assume that $f \cdot g \in X$ such that

$$\|fg\| \leq \|f\|_\infty \|g\| + \|f\| \|g\|_\infty + K\|f\| \|g\|, \quad K > 0.$$

Then $(X, \|\cdot\|_1)$ equipped with the norm

$$\|f\|_1 = \|f\|_\infty + K\|f\|, \quad f \in X,$$

is a Banach algebra. If $X \hookrightarrow B[a, b]$, then $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

Proof. First of all, we need to show that $\|fg\|_1 \leq \|f\|_1 \|g\|_1$ for all $f, g \in X$. In fact,

$$\begin{aligned} \|fg\|_1 &= \|fg\|_\infty + K\|fg\| \\ &\leq \|f\|_\infty \|g\|_\infty + K\|f\|_\infty \|g\| \\ &\quad + K\|f\| \|g\|_\infty + K^2\|f\| \|g\| \\ &= (\|f\|_\infty + K\|f\|)(\|g\|_\infty + K\|g\|) \\ &= \|f\|_1 \|g\|_1. \end{aligned}$$

This tells us that $(X, \|\cdot\|)$ is a Banach algebra. It only remains to show that $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms.

Indeed, since $X \hookrightarrow B[a, b]$, there exists a constant $L > 0$ such that

$$\|\cdot\|_\infty \leq L \|\cdot\|.$$

Thus

$$\begin{aligned} K \|\cdot\| &\leq \|\cdot\|_\infty + K \|\cdot\| = \|\cdot\|_1 \\ &\leq L \|\cdot\| + K \|\cdot\| = (L + K) \|\cdot\|. \end{aligned}$$

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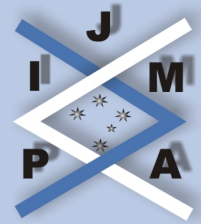
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Hence

$$K \|\cdot\| \leq \|\cdot\|_1 \leq (L + K) \|\cdot\|.$$

This completes the proof of Theorem 2.1.

□



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