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A GENERALIZED OSTROWSKI-GRÜSS TYPE INEQUALITY FOR TWICE DIFFERENTIABLE MAPPINGS AND APPLICATIONS

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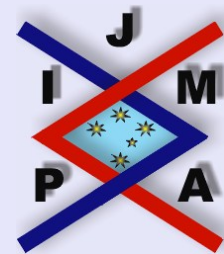
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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

A generalized Ostrowski type inequality for twice differentiable mappings in terms of the upper and lower bounds of the second derivative is established. The inequality is applied to numerical integration.

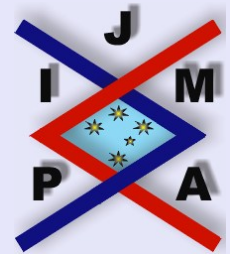
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Contents

1	Introduction	3
2	Main Results	5
3	Applications in Numerical Integration	11
	References	



A Generalized Ostrowski-Grüss Type Inequality for Twice Differentiable Mappings and Applications

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 2 of 15

1. Introduction

The integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality. The inequality is as follows:

Theorem 1.1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable functions such that $\Psi \leq f(x) \leq \varphi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$, where Ψ, φ, γ and Γ are constants. It follows that,*

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx \right| \leq \frac{1}{4}(\varphi - \Psi)(\Gamma - \gamma),$$

where the constant $\frac{1}{4}$ is sharp.

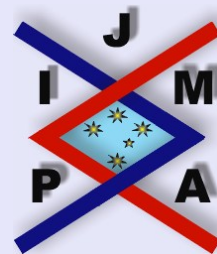
In [2], S.S. Dragomir and S. Wang proved the following Ostrowski type inequality in terms of lower and upper bounds of the first derivative.

Theorem 1.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and where the first derivative satisfies the condition,*

$$\gamma \leq f'(x) \leq \Gamma \quad \text{for all } x \in [a, b],$$

then,

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma)$$



A Generalized Ostrowski-Grüss Type Inequality for Twice Differentiable Mappings and Applications

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 3 of 15

for all $x \in [a, b]$.

In [1], S.S. Dragomir and N.S. Barnett, proved the following inequality.

Theorem 1.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and twice differentiable on (a, b) , where the second derivative $f'' : (a, b) \rightarrow \mathbb{R}$ satisfies the condition,

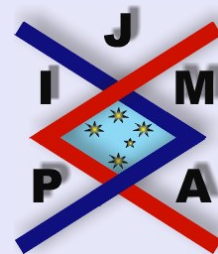
$$\varphi \leq f''(x) \leq \Phi \quad \text{for all } x \in (a, b),$$

then,

$$(1.3) \quad \left| f(x) + \left[\frac{(b-a)^2}{24} + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] \frac{f'(b) - f'(a)}{b-a} - \left(x - \frac{a+b}{2} \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (\Phi - \varphi) \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right]^2$$

for all $x \in [a, b]$.

In this paper we establish a more general form of (1.3) and apply the result to numerical integration.



A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 4 of 15

2. Main Results

Theorem 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous mapping on $[a, b]$, and twice differentiable on (a, b) with second derivative $f'' : (a, b) \rightarrow \mathbb{R}$ satisfying the condition:

$$\varphi \leq f''(x) \leq \Phi, \quad \text{for all } x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2} \right].$$

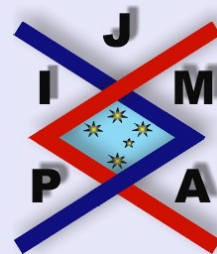
It follows that,

$$(2.1) \quad \left| (1-h) \left[f(x) - \left(x - \frac{a+b}{2} \right) f'(x) \right] + h \frac{f(a) + f(b)}{2} \right. \\ \left. + \left[\frac{1}{2} (1-h) \left(x - \frac{a+b}{2} \right)^2 - \frac{(3h-1)(b-a)^2}{24} \right] \left(\frac{f'(b) - f'(a)}{b-a} \right) \right. \\ \left. - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (\Phi - \varphi) \left[\frac{1}{2} (b-a)(1-h) + \left| x - \frac{a+b}{2} \right| \right]^2,$$

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$ and $h \in [0, 1]$.

Proof. The proof uses the following identity,

$$(2.2) \quad \int_a^b f(t) dt = (b-a)(1-h)f(x) \\ - (b-a)(1-h) \left(x - \frac{a+b}{2} \right) f'(x) + h \frac{b-a}{2} (f(a) + f(b)) \\ - \frac{h^2 (b-a)^2}{8} (f'(b) - f'(a)) + \int_a^b K(x, t) f''(t) dt.$$



A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 5 of 15

for all $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$, where the kernel $K : [a, b]^2 \rightarrow \mathbb{R}$ is defined by

$$(2.3) \quad K(x, t) = \begin{cases} \frac{1}{2} [t - (a + h\frac{b-a}{2})]^2 & \text{if } t \in [a, x] \\ \frac{1}{2} [t - (b - h\frac{b-a}{2})]^2 & \text{if } t \in (x, b]. \end{cases}$$

This is a particular form of the identity given in [3, p. 59; Corollary 2.3].

Observe that the kernel K satisfies the estimation

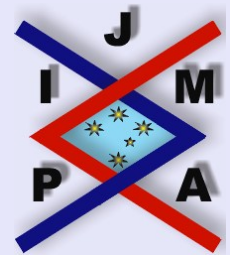
$$(2.4) \quad 0 \leq K(x, t) \leq \begin{cases} \frac{1}{2} [(b - h\frac{b-a}{2}) - x]^2, & x \in [a + h\frac{b-a}{2}, \frac{a+b}{2}) \\ \frac{1}{2} [x - (a + h\frac{b-a}{2})]^2, & x \in [\frac{a+b}{2}, b - h\frac{b-a}{2}]. \end{cases}$$

Applying the Grüss inequality for the mappings $f''(\cdot)$ and $K(x, \cdot)$ we get,

$$(2.5) \quad \left| \frac{1}{b-a} \int_a^b K(x, t) f''(t) dt - \frac{1}{b-a} \int_a^b K(x, t) dt \frac{1}{b-a} \int_a^b f''(t) dt \right| \leq \frac{1}{4} (\Phi - \varphi) \times \begin{cases} \frac{1}{2} [(b - h\frac{b-a}{2}) - x]^2, & x \in [a + h\frac{b-a}{2}, \frac{a+b}{2}) \\ \frac{1}{2} [x - (a + h\frac{b-a}{2})]^2, & x \in [\frac{a+b}{2}, b - h\frac{b-a}{2}]. \end{cases}$$

Observe that,

$$(2.6) \quad \int_a^b K(x, t) dt = \int_a^x \frac{[t - (a + h\frac{b-a}{2})]^2}{2} dt + \int_x^b \frac{[t - (b - h\frac{b-a}{2})]^2}{2} dt$$



**A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications**

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 6 of 15

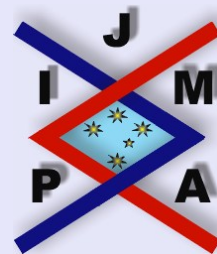
$$\begin{aligned}
&= \frac{1}{6} \left[\left(x - \left(a + h \frac{b-a}{2} \right) \right)^3 + \left(\left(b - h \frac{b-a}{2} \right) - x \right)^3 + \frac{h^3 (b-a)^3}{4} \right] \\
&= (b-a)(1-h) \left[\frac{(b-a)^2 (1-h)^2}{24} + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] + \frac{h^3 (b-a)^3}{24}.
\end{aligned}$$

Using (2.6) in (2.5), we get

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b K(x,t) f''(t) dt - \left[\frac{(b-a)^2 (1-h)^3}{24} + \frac{1}{2} (1-h) \left(x - \frac{a+b}{2} \right)^2 \right. \right. \\
&\quad \left. \left. + \frac{h^3 (b-a)^2}{24} \right] \left(\frac{f'(b) - f'(a)}{b-a} \right) \right| \\
&\leq \frac{1}{4} (\Phi - \varphi) \times \begin{cases} \frac{1}{2} \left[(b - h \frac{b-a}{2}) - x \right]^2, & x \in [a + h \frac{b-a}{2}, \frac{a+b}{2}] \\ \frac{1}{2} \left[x - (a + h \frac{b-a}{2}) \right]^2, & x \in [\frac{a+b}{2}, b - h \frac{b-a}{2}]. \end{cases}
\end{aligned}$$

Also, by using identity (2.2), the above inequality reduces to,

$$\begin{aligned}
&\left| (1-h) \left[f(x) - \left(x - \frac{a+b}{2} \right) f'(x) \right] + h \frac{f(a) + f(b)}{2} \right. \\
&\quad \left. + \left[\frac{1}{2} (1-h) \left(x - \frac{a+b}{2} \right)^2 - \frac{(3h-1)(b-a)^2}{24} \right] \left(\frac{f'(b) - f'(a)}{b-a} \right) \right. \\
&\quad \left. - \frac{1}{b-a} \int_a^b f(t) dt \right|
\end{aligned}$$



**A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications**

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 7 of 15

$$\leq \frac{1}{4} (\Phi - \varphi) \times \begin{cases} \frac{1}{2} [(b - h\frac{b-a}{2}) - x]^2, & x \in [a + h\frac{b-a}{2}, \frac{a+b}{2}); \\ \frac{1}{2} [x - (a + h\frac{b-a}{2})]^2, & x \in [\frac{a+b}{2}, b - h\frac{b-a}{2}]. \end{cases}$$

Since,

$$\begin{aligned} \max \left\{ \frac{[(b - h\frac{b-a}{2}) - x]^2}{2}, \frac{[x - (a + h\frac{b-a}{2})]^2}{2} \right\} \\ = \begin{cases} \frac{1}{2} [(b - h\frac{b-a}{2}) - x]^2, & x \in [a + h\frac{b-a}{2}, \frac{a+b}{2}) \\ \frac{1}{2} [x - (a + h\frac{b-a}{2})]^2, & x \in [\frac{a+b}{2}, b - h\frac{b-a}{2}], \end{cases} \end{aligned}$$

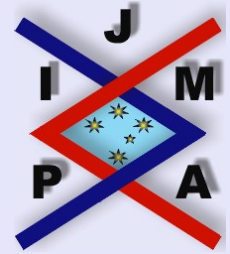
but on the other hand,

$$\begin{aligned} \max \left\{ \frac{[(b - h\frac{b-a}{2}) - x]^2}{2}, \frac{[x - (a + h\frac{b-a}{2})]^2}{2} \right\} \\ = \frac{1}{2} \left[\frac{1}{2} (b - a) (1 - h) + \left| x - \frac{a + b}{2} \right| \right]^2, \end{aligned}$$

inequality (2.1) is proved. \square

Remark 1. For $h = 0$ in (2.1), we obtain (1.3).

Corollary 2.2. If f is as in Theorem 2.1, then we have the following perturbed



**A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications**

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 8 of 15

midpoint inequality:

$$(2.7) \quad \left| (1-h) f\left(\frac{a+b}{2}\right) + h \frac{f(a)+f(b)}{2} - \frac{(3h-1)(b-a)}{24} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{32} (\Phi - \varphi) (b-a)^2 (1-h)^2,$$

giving,

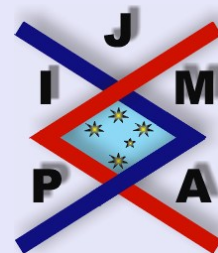
$$(2.8) \quad \left| f\left(\frac{a+b}{2}\right) + \frac{(b-a)}{24} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{32} (\Phi - \varphi) (b-a)^2,$$

for $h = 0$.

Remark 2. The classical midpoint inequality states that

$$(2.9) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{24} (b-a)^2 \|f''\|_{\infty}.$$

If $\Phi - \varphi \leq \frac{4}{3} \|f''\|_{\infty}$, then the estimation provided by (2.7) is better than the estimation in the classical midpoint inequality (2.9). A sufficient condition for $\Phi - \varphi \leq \frac{4}{3} \|f''\|_{\infty}$ to be true is $0 \leq \varphi \leq \Phi$. Indeed, if $0 \leq \varphi \leq \Phi$, then $\Phi - \varphi \leq \|f''\|_{\infty} < \frac{4}{3} \|f''\|_{\infty}$.



A Generalized Ostrowski-Grüss Type Inequality for Twice Differentiable Mappings and Applications

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 9 of 15

Corollary 2.3. Let f be as in Theorem 2.1, then,

$$(2.10) \quad \left| \frac{f(a) + f(b)}{2} - \frac{(b-a)}{12} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{32} (\Phi - \varphi) (2-h)^2 (b-a)^2.$$

Proof. Put $x = a$ and $x = b$ in turn in (2.1) and use the triangle inequality. \square

Corollary 2.4. Let f be as in Theorem 2.1, then we have the following perturbed Trapezoid inequality:

$$(2.11) \quad \left| \frac{f(a) + f(b)}{2} - \frac{(b-a)}{12} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{32} (\Phi - \varphi) (b-a)^2.$$

Proof. Put $h = 1$ in (2.10). \square

Remark 3. The classical Trapezoid inequality states that

$$(2.12) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{12} (b-a)^2 \|f''\|_{\infty}.$$

If we assume that $\Phi - \varphi \leq \frac{2}{3} \|f''\|_{\infty}$, then the estimation provided by (2.10) is better than the estimation in the classical Trapezoid inequality (2.12).



A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 10 of 15

3. Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$, ($i = 0, 1, \dots, n-1$) a sequence of intermediate points and $h_i := x_{i+1} - x_i$ ($i = 0, 1, \dots, n-1$). Following the approach taken in [1] we have the following:

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and a twice differentiable function on (a, b) , whose second derivative, $f'' : (a, b) \rightarrow \mathbb{R}$ satisfies:*

$$\varphi \leq f''(x) \leq \Phi, \quad \text{for all } x \in (a, b),$$

then,

$$(3.1) \quad \int_a^b f(t)dt = A(f, f', I_n, \xi, \delta) + R(f, f', I_n, \xi, \delta),$$

where

$$(3.2) \quad A(f, f', I_n, \xi, \delta) = (1 - \delta) \sum_{i=0}^{n-1} h_i f(\xi_i) - (1 - \delta) \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_i + x_{i-1}}{2} \right) f'(\xi_i) + \delta \sum_{i=0}^{n-1} h_i \left(\frac{f(x_i) + f(x_{i+1})}{2} \right) + \sum_{i=0}^{n-1} \left[\frac{1}{2} (1 - \delta) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 - \frac{(3\delta - 1) h_i^2}{24} \right] (f'(x_{i+1}) - f'(x_i))$$



A Generalized Ostrowski-Grüss Type Inequality for Twice Differentiable Mappings and Applications

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 11 of 15

and the remainder $R(f, f', I_n, \xi, \delta)$ satisfies the estimation:

$$\begin{aligned}
 & |R(f, f', I_n, \xi, \delta)| \\
 & \leq \frac{1}{8} (\Phi - \varphi) \sum_{i=0}^{n-1} h_i \left[\frac{(1-\delta)}{2} h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^2 \\
 (3.3) \quad & \leq \frac{1}{32} (\Phi - \varphi) (1-\delta)^2 \sum_{i=0}^{n-1} h_i^3,
 \end{aligned}$$

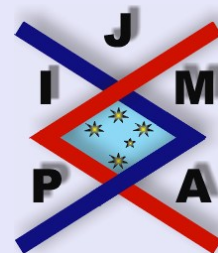
where $\delta \in [0, 1]$ and $x_i + \delta \frac{h_i}{2} \leq \xi_i \leq x_{i+1} - \delta \frac{h_i}{2}$.

Proof. Applying Theorem 2.1 on the interval $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) gives:

$$\begin{aligned}
 & \left| (1-\delta) \left[h_i f(\xi_i) - h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) \right] + \delta h_i \left(\frac{f(x_i) + f(x_{i+1}))}{2} \right) \right. \\
 & \quad \left. + \left[\frac{1}{2} (1-\delta) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 - \frac{(3\delta-1) h_i^2}{24} \right] (f'(x_{i+1}) - f'(x_i)) \right. \\
 & \quad \left. - \int_{x_i}^{x_{i+1}} f(t) dt \right| \\
 & \leq \frac{1}{8} (\Phi - \varphi) h_i \left[\frac{1}{2} (1-\delta) h_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^2, \\
 & \leq \frac{1}{8} (\Phi - \varphi) (1-\delta)^2 h_i^3
 \end{aligned}$$

as

$$\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \leq (1-\delta) \frac{h_i}{2} \quad \text{for all } i \in \{0, 1, \dots, n-1\}$$



**A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications**

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 12 of 15

for any choice ξ_i of the intermediate points.

Summing the above inequalities over i from 0 to $n - 1$, and using the generalized triangle inequality, we get the desired estimation (3.3). \square

Corollary 3.2. *The following perturbed midpoint rule holds:*

$$(3.4) \quad \int_a^b f(x) dx = M(f, f', I_n) + R_M(f, f', I_n),$$

where

$$(3.5) \quad M(f, f', I_n) = \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{24} \sum_{i=0}^{n-1} h_i^2 (f'(x_{i+1}) - f'(x_i))$$

and the remainder term $R_M(f, f', I_n)$ satisfies the estimation:

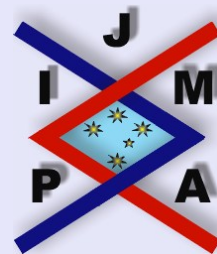
$$(3.6) \quad |R_M(f, f', I_n)| \leq \frac{1}{32} (\Phi - \varphi) \sum_{i=0}^{n-1} h_i^3.$$

Corollary 3.3. *The following perturbed trapezoid rule holds*

$$(3.7) \quad \int_a^b f(x) dx = T(f, f', I_n) + R_T(f, f', I_n),$$

where

$$(3.8) \quad T(f, f', I_n) = \sum_{i=0}^{n-1} h_i \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{12} \sum_{i=0}^{n-1} h_i^2 (f'(x_{i+1}) - f'(x_i))$$



**A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications**

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

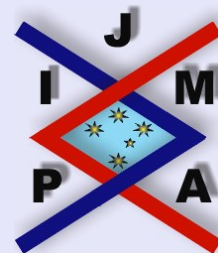
Quit

Page 13 of 15

and the remainder term $R_T(f, f', I_n)$ satisfies the estimation:

$$(3.9) \quad |R_T(f, f', I_n)| \leq \frac{1}{32} (\Phi - \varphi) \sum_{i=0}^{n-1} h_i^3.$$

Remark 4. Note that the above mentioned perturbed midpoint formula (3.5) and perturbed trapezoid formula (3.8) can offer better approximations of the integral $\int_a^b f(x) dx$ for general classes of mappings as discussed in Remarks 1 and 2.



**A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications**

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

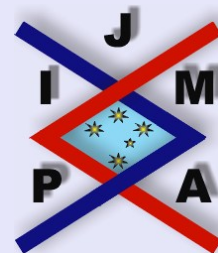
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Quit

Page 14 of 15

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**A Generalized Ostrowski-Grüss
Type Inequality for Twice
Differentiable Mappings and
Applications**

A. Rafiq, N.A. Mir and Fiza Zafar

Title Page

Contents



Go Back

Close

Quit

Page 15 of 15