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A GENERALIZED OSTROWSKI-GRÜSS TYPE INEQUALITY FOR TWICE DIFFERENTIABLE MAPPINGS AND APPLICATIONS

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Abstract

A generalized Ostrowski type inequality for twice differentiable mappings in terms of the upper and lower bounds of the second derivative is established. The inequality is applied to numerical integration.

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1. Introduction

The integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality. The inequality is as follows:

Theorem 1.1. *Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be integrable functions such that $\Psi \leq f(x) \leq \varphi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$, where Ψ, φ, γ and Γ are constants. It follows that,*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\varphi-\Psi)(\Gamma-\gamma), \tag{1.1}$$

where the constant $\frac{1}{4}$ is sharp.

In [2], S.S. Dragomir and S. Wang proved the following Ostrowski type inequality in terms of lower and upper bounds of the first derivative.

Theorem 1.2. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and where the first derivative satisfies the condition,*

$$\gamma \leq f'(x) \leq \Gamma \qquad \text{for all } x \in [a,b],$$

then,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \leq \frac{1}{4}(b-a)(\Gamma-\gamma) \tag{1.2}$$

for all $x \in [a,b]$.

In [1], S.S. Dragomir and N.S. Barnett, proved the following inequality.

Theorem 1.3. *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be continuous on* $[a,b]$ *and twice differentiable on* (a,b), *where the second derivative* $f'' : (a,b) \longrightarrow \mathbb{R}$ *satisfies the condition,*

$$\varphi \leq f''(x) \leq \Phi \quad \text{for all } x \in (a,b),$$

then,

$$(1.3) \quad \left| f(x) + \left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right] \frac{f'(b) - f'(a)}{b-a} - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{8}(\Phi - \varphi)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2$$

for all $x \in [a,b]$.

In this paper we establish a more general form of (1.3) and apply the result to numerical integration.

2. Main Results

Theorem 2.1. *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a continuous mapping on* $[a,b]$, *and twice differentiable on* (a,b) *with second derivative* $f'' : (a,b) \longrightarrow \mathbb{R}$ *satisfying the condition:*

$$\varphi \leq f''(x) \leq \Phi, \qquad \text{for all } x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right].$$

It follows that,

$$\begin{aligned} &\Bigg| (1-h)\left[f(x) - \left(x - \frac{a+b}{2}\right) f'(x)\right] + h\frac{f(a)+f(b)}{2} \\ &\quad + \left[\frac{1}{2}(1-h)\left(x - \frac{a+b}{2}\right)^2 - \frac{(3h-1)(b-a)^2}{24}\right]\left(\frac{f'(b)-f'(a)}{b-a}\right) \\ &\quad - \frac{1}{b-a}\int_a^b f(t)dt \Bigg| \leq \frac{1}{8}(\Phi - \varphi)\left[\frac{1}{2}(b-a)(1-h) + \left|x - \frac{a+b}{2}\right|\right]^2, \end{aligned} \tag{2.1}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$ and $h \in [0,1]$.

Proof. The proof uses the following identity,

$$\begin{aligned} \int_a^b f(t)dt &= (b-a)(1-h) f(x) \\ &\quad - (b-a)(1-h)\left(x - \frac{a+b}{2}\right) f'(x) + h\frac{b-a}{2}(f(a)+f(b)) \\ &\quad - \frac{h^2(b-a)^2}{8}(f'(b) - f'(a)) + \int_a^b K(x,t) f''(t)\, dt. \end{aligned} \tag{2.2}$$

for all $x \in [a+h\frac{b-a}{2}, b-h\frac{b-a}{2}]$, where the kernel $K : [a,b]^2 \to \mathbb{R}$ is defined by

$$K(x,t) = \begin{cases} \frac{1}{2}\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2 & \text{if } t \in [a,x] \\ \frac{1}{2}\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2 & \text{if } t \in (x,b]. \end{cases} \tag{2.3}$$

This is a particular form of the identity given in [3, p. 59; Corollary 2.3].

Observe that the kernel K satisfies the estimation

$$0 \le K(x,t) \le \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right) \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]. \end{cases} \tag{2.4}$$

Applying the Grüss inequality for the mappings $f''(\cdot)$ and $K(x,\cdot)$ we get,

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b K(x,t) f''(t)\,dt - \frac{1}{b-a}\int_a^b K(x,t)\,dt \frac{1}{b-a}\int_a^b f''(t)\,dt \right| \\ &\qquad \le \frac{1}{4}(\Phi - \varphi) \times \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right) \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]. \end{cases} \end{aligned} \tag{2.5}$$

Observe that,

$$\int_a^b K(x,t)dt = \int_a^x \frac{\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2}{2}dt + \int_x^b \frac{\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2}{2}dt \tag{2.6}$$

$$= \frac{1}{6}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^3+\left(\left(b-h\frac{b-a}{2}\right)-x\right)^3+\frac{h^3(b-a)^3}{4}\right]$$

$$= (b-a)(1-h)\left[\frac{(b-a)^2(1-h)^2}{24}+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]+\frac{h^3(b-a)^3}{24}.$$

Using (2.6) in (2.5), we get

$$\left|\frac{1}{b-a}\int_a^b K(x,t)f''(t)\,dt-\left[\frac{(b-a)^2(1-h)^3}{24}+\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2\right.\right.$$
$$\left.\left.+\frac{h^3(b-a)^2}{24}\right]\left(\frac{f'(b)-f'(a)}{b-a}\right)\right|$$
$$\leq \frac{1}{4}(\Phi-\varphi)\times\begin{cases}\frac{1}{2}\left[\left(b-h\frac{b-a}{2}\right)-x\right]^2, & x\in\left[a+h\frac{b-a}{2},\frac{a+b}{2}\right)\\ \frac{1}{2}\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2, & x\in\left[\frac{a+b}{2},b-h\frac{b-a}{2}\right].\end{cases}$$

Also, by using identity (2.2), the above inequality reduces to,

$$\left|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right]+h\frac{f(a)+f(b)}{2}\right.$$
$$+\left[\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)^2-\frac{(3h-1)(b-a)^2}{24}\right]\left(\frac{f'(b)-f'(a)}{b-a}\right)$$
$$\left.-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{1}{4}(\Phi - \varphi) \times \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right); \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]. \end{cases}$$

Since,

$$\max\left\{\frac{\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2}{2}, \frac{\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2}{2}\right\}$$
$$= \begin{cases} \frac{1}{2}\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2, & x \in \left[a + h\frac{b-a}{2}, \frac{a+b}{2}\right) \\ \frac{1}{2}\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2, & x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right], \end{cases}$$

but on the other hand,

$$\max\left\{\frac{\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2}{2}, \frac{\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2}{2}\right\}$$
$$= \frac{1}{2}\left[\frac{1}{2}(b-a)(1-h) + \left|x - \frac{a+b}{2}\right|\right]^2,$$

inequality (2.1) is proved. □

Remark 1. *For $h = 0$ in (2.1), we obtain (1.3).*

Corollary 2.2. *If f is as in Theorem 2.1, then we have the following perturbed* *midpoint inequality:*

$$\begin{aligned}(2.7)\quad &\left|(1-h)\,f\left(\frac{a+b}{2}\right)+h\frac{f(a)+f(b)}{2}\right.\\ &\left.-\frac{(3h-1)(b-a)}{24}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ &\qquad\le\frac{1}{32}(\Phi-\varphi)(b-a)^2(1-h)^2,\end{aligned}$$

giving,

$$\begin{aligned}(2.8)\quad &\left|f\left(\frac{a+b}{2}\right)+\frac{(b-a)}{24}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ &\qquad\le\frac{1}{32}(\Phi-\varphi)(b-a)^2,\end{aligned}$$

for $h=0$.

Remark 2. *The classical midpoint inequality states that*

$$(2.9)\qquad \left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{24}(b-a)^2\left\|f''\right\|_\infty.$$

If $\Phi-\varphi\le\frac{4}{3}\left\|f''\right\|_\infty$, *then the estimation provided by* (2.7) *is better than the estimation in the classical midpoint inequality* (2.9). *A sufficient condition for* $\Phi-\varphi\le\frac{4}{3}\left\|f''\right\|_\infty$ *to be true is* $0\le\varphi\le\Phi$*.Indeed, if* $0\le\varphi\le\Phi$, *then* $\Phi-\varphi\le\left\|f''\right\|_\infty<\frac{4}{3}\left\|f''\right\|_\infty$.

Corollary 2.3. *Let f be as in Theorem 2.1, then,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{12}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{32}(\Phi-\varphi)(2-h)^2(b-a)^2. \tag{2.10}$$

Proof. Put $x=a$ and $x=b$ in turn in (2.1) and use the triangle inequality. □

Corollary 2.4. *Let f be as in Theorem 2.1, then we have the following perturbed Trapezoid inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{12}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{32}(\Phi-\varphi)(b-a)^2. \tag{2.11}$$

Proof. Put $h=1$ in (2.10). □

Remark 3. *The classical Trapezoid inequality states that*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{12}(b-a)^2\left\|f''\right\|_\infty. \tag{2.12}$$

If we assume that $\Phi-\varphi\leq\frac{2}{3}\left\|f''\right\|_\infty$, then the estimation provided by (2.10) is better than the estimation in the classical Trapezoid inequality (2.12).

3. Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$, $(i = 0, 1, \ldots, n-1)$ a sequence of intermediate points and $h_i := x_{i+1} - x_i$ $(i = 0, 1, \ldots, n-1)$. Following the approach taken in [1] we have the following:

Theorem 3.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and a twice differentiable function on (a, b), whose second derivative, $f'' : (a, b) \longrightarrow \mathbb{R}$ satisfies:*

$$\varphi \leq f''(x) \leq \Phi, \qquad \text{for all } x \in (a, b),$$

then,

$$\int_a^b f(t)dt = A(f, f', I_n, \xi, \delta) + R(f, f', I_n, \xi, \delta), \tag{3.1}$$

where

$$\begin{aligned} A(f, f', I_n, \xi, \delta) &= (1-\delta) \sum_{i=0}^{n-1} h_i f(\xi_i) - (1-\delta) \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_i + x_{i-1}}{2} \right) f'(\xi_i) \\ &\quad + \delta \sum_{i=0}^{n-1} h_i \left(\frac{f(x_i) + f(x_{i+1})}{2} \right) + \sum_{i=0}^{n-1} \left[\frac{1}{2}(1-\delta) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right. \\ &\quad \left. - \frac{(3\delta - 1) h_i^2}{24} \right] (f'(x_{i+1}) - f'(x_i)) \end{aligned} \tag{3.2}$$

and the remainder $R(f,f',I_n,\xi,\delta)$ *satisfies the estimation:*

$$\begin{aligned}|R(f,f',I_n,\xi,\delta)| &\\ &\le \frac{1}{8}(\Phi-\varphi)\sum_{i=0}^{n-1}h_i\left[\frac{(1-\delta)}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2\\ &\le \frac{1}{32}(\Phi-\varphi)(1-\delta)^2\sum_{i=0}^{n-1}h_i^3, \qquad (3.3)\end{aligned}$$

where $\delta\in[0,1]$ *and* $x_i+\delta\frac{h_i}{2}\le\xi_i\le x_{i+1}-\delta\frac{h_i}{2}$.

Proof. Applying Theorem 2.1 on the interval $[x_i,x_{i+1}]$ $(i=0,\dots,n-1)$ gives:

$$\begin{aligned}&\Bigg|(1-\delta)\left[h_if(\xi_i)-h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)f'(\xi_i)\right]+\delta h_i\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)\\ &\quad+\left[\frac{1}{2}(1-\delta)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2-\frac{(3\delta-1)h_i^2}{24}\right](f'(x_{i+1})-f'(x_i))\\ &\quad-\int_{x_i}^{x_{i+1}}f(t)\,dt\Bigg|\\ &\le \frac{1}{8}(\Phi-\varphi)h_i\left[\frac{1}{2}(1-\delta)h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2,\\ &\le \frac{1}{8}(\Phi-\varphi)(1-\delta)^2h_i^3\end{aligned}$$

as

$$\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\le(1-\delta)\frac{h_i}{2}\quad\text{for all } i\in\{0,1,...,n-1\}$$

for any choice ξ_i of the intermediate points.

Summing the above inequalities over i from 0 to $n-1$, and using the generalized triangle inequality, we get the desired estimation (3.3) . □

Corollary 3.2. *The following perturbed midpoint rule holds:*

$$\int_a^b f(x)\,dx = M(f, f', I_n) + R_M(f, f', I_n), \tag{3.4}$$

where

$$M(f, f', I_n) = \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{24}\sum_{i=0}^{n-1} h_i^2 \left(f'(x_{i+1}) - f'(x_i)\right) \tag{3.5}$$

and the remainder term $R_M(f, f', I_n)$ satisfies the estimation:

$$|R_M(f, f', I_n)| \le \frac{1}{32}(\Phi - \varphi)\sum_{i=0}^{n-1} h_i^3. \tag{3.6}$$

Corollary 3.3. *The following perturbed trapezoid rule holds*

$$\int_a^b f(x)\,dx = T(f, f', I_n) + R_T(f, f', I_n), \tag{3.7}$$

where

$$T(f, f', I_n) = \sum_{i=0}^{n-1} h_i \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{12}\sum_{i=0}^{n-1} h_i^2 \left(f'(x_{i+1}) - f'(x_i)\right) \tag{3.8}$$

and the remainder term $R_T(f, f', I_n)$ satisfies the estimation:

$$|R_T(f, f', I_n)| \leq \frac{1}{32}(\Phi - \varphi) \sum_{i=0}^{n-1} h_i^3. \tag{3.9}$$

Remark 4. *Note that the above mentioned perturbed midpoint formula (3.5) and perturbed trapezoid formula (3.8) can offer better approximations of the integral $\int_a^b f(x)\,dx$ for general classes of mappings as discussed in Remarks 1 and 2.*

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