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INEQUALITIES INVOLVING GENERALIZED BESSEL FUNCTIONS

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Abstract

Let u_p denote the normalized, generalized Bessel function of order p which depends on two parameters b and c and let $\lambda_p(x) = u_p(x^2)$, $x \geq 0$. It is proven that under some conditions imposed on p, b, and c the Askey inequality holds true for the function λ_p, i.e., that $\lambda_p(x) + \lambda_p(y) \leq 1 + \lambda_p(z)$, where $x, y \geq 0$ and $z^2 = x^2 + y^2$. The lower and upper bounds for the function λ_p are also established.

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Contents

1. Introduction

The Bessel function of the first kind of order p, denoted by $J_p(x)$, is defined as a particular solution of the second-order differential equation ([12, p. 38])

$$x^2y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0 \tag{1.1}$$

which is also called the Bessel equation. It is known ([12, p. 40]) that

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \qquad x \in \mathbb{R}. \tag{1.2}$$

R. Askey [2] has shown that for $\mathcal{J}_p(x) = \Gamma(p+1)(2/x)^p J_p(x)$ the following inequality

$$\mathcal{J}_p(x) + \mathcal{J}_p(y) \leq 1 + \mathcal{J}_p(z) \tag{1.3}$$

holds true for all $x, y, z, p \geq 0$ where $z^2 = x^2 + y^2$. Since $\mathcal{J}_0(x) = J_0(x)$, inequality (1.3) provides a generalization of Grünbaum's inequality ([6])

$$J_0(x) + J_0(y) \leq 1 + J_0(z). \tag{1.4}$$

Using Legendre polynomials Grünbaum has supplied another proof of (1.4) in [7].

Recently, E. Neuman ([9]) has obtained a different upper bound for $\mathcal{J}_p(x) + \mathcal{J}_p(y)$. In the same paper the lower and upper bounds for the function $\mathcal{J}_p(x)$ are established with the aid of Gegenbauer polynomials.

The purpose of this paper is to obtain similar results to those mentioned above for the function λ_p which is the transformed version of the normalized, generalized Bessel function u_p. Definitions of these functions together with the integral formula are contained in Section 2. An Askey type inequality for the function λ_p and the Grünbaum inequality for the modified Bessel functions of the first kind are derived in Section 3. The lower and upper bounds for the function λ_p are established in Section 4.

2. The Function λ_p

The following second-order differential equation (see [12, p. 77])

$$x^2y''(x) + xy'(x) - (x^2 + p^2)y(x) = 0 \tag{2.1}$$

frequently occurs in mathematical physics. A particular solution of (2.1), denoted by $I_p(x)$, is called the modified Bessel function of the first kind of order p and it is represented as the infinite series

$$I_p(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \qquad x \in \mathbb{R} \tag{2.2}$$

(see, e.g., [12, p. 77]).

A second order differential equation which reduces either to (1.1) or (2.1) reads as follows

$$x^2v''(x) + bxv'(x) + \left[cx^2 - p^2 + (1-b)p\right]v(x) = 0, \tag{2.3}$$

$b, c, p \in \mathbb{R}$. A particular solution v_p is

$$v_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma(p+n+(b+1)/2)} \left(\frac{x}{2}\right)^{2n+p} \tag{2.4}$$

and v_p is called the generalized Bessel function of the first kind of order p (see [4]). It is readily seen that for $b = 1$ and $c = 1$, v_p becomes J_p and for $b = 1$ and $c = -1$, v_p simplifies to I_p.

The normalized, generalized Bessel function of the first kind of order p, denoted by u_p, is defined as

$$u_p(x) = 2^p\Gamma\left(p+\frac{b+1}{2}\right)x^{-p/2}v_p(x^{1/2}). \tag{2.5}$$

Using the Pochhammer symbol $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)\cdot\dots\cdot(a+n-1)$ $(a \neq 0)$ we obtain the following formula

$$u_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{4^n\left(p+\frac{b+1}{2}\right)_n}\cdot\frac{x^n}{n!} \tag{2.6}$$

$(p+(b+1)/2 \neq 0, -1, \dots)$. For later use, let us write

$$u_p(x) = \sum_{n=0}^{\infty} b_n x^n,$$

where

$$b_n = \frac{1}{n!\left(p+\frac{b+1}{2}\right)_n}\left(-\frac{c}{4}\right)^n \tag{2.7}$$

$(n \geq 0)$.

Finally, we define a function λ_p as follows

$$\lambda_p(x) = u_p(x^2). \tag{2.8}$$

Making use of (2.6) we obtain a series representation for the function in question

$$\lambda_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{\left(p + \frac{b+1}{2}\right)_n n!} \left(\frac{x}{2}\right)^{2n}. \tag{2.9}$$

The following lemma will be used in the sequel.

Lemma 2.1. *Let the numbers* p *and* b *be such* $\operatorname{Re}(p + b/2) > 0$*. Then for any* $x \in \mathbb{R}$

$$\lambda_p(x) = \begin{cases} \int_0^1 \cos(tx\sqrt{c})\, d\mu(t), & c \geq 0 \\ \int_0^1 \cosh(tx\sqrt{-c})\, d\mu(t), & c \leq 0, \end{cases} \tag{2.10}$$

where $d\mu(t) = \mu(t)\, dt$ *with*

$$\mu(t) = \frac{2(1-t^2)^{p+(b-2)/2}}{B\left(p + \frac{b}{2}, \frac{1}{2}\right)} \tag{2.11}$$

being the probability measure on $[0, 1]$*. Here* $B(\cdot, \cdot)$ *stands for the beta function.*

Proof. We shall prove first that the function $\mu(t)$, defined in (2.11), is indeed the probability measure on $[0, 1]$. Clearly the function in question is nonnegative on the indicated interval. Moreover, with $A = 1/B(p + b/2, 1/2)$, we have

$$\begin{aligned} \int_0^1 d\mu(t) &= 2A \int_0^1 (1-t^2)^{p+(b-2)/2} dt \\ &= A \int_0^1 r^{-1/2}(1-r)^{p+(b-2)/2} dr = A \cdot A^{-1}. \end{aligned}$$

Here we have used the substitution $r = t^{1/2}$.

In order to establish formula (2.10) we note that (2.9) implies $\lambda_p(0) = 1$ and also that $\lambda_p(-x) = \lambda(x)$. To this end, let $x > 0$. For the sake of brevity, let

$$I = \int_0^{\pi/2} (\sin\theta)^{2p+b-1} \cos(\sqrt{c}\, z \cos\theta)\, d\theta, \qquad c \geq 0.$$

Using the Maclaurin expansion for the cosine function and integrating term by term we obtain

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{(2n)!} z^{2n} \int_0^{\pi/2} (\sin\theta)^{2p+b-1} (\cos\theta)^{2n} d\theta,$$

where the last integral converges uniformly provided $\mathrm{Re}(p + b/2) > 0$. Making use of the well-known formula

$$B(a,b) = 2\int_0^{\pi/2} (\cos\theta)^{2a-1} (\sin\theta)^{2b-1} d\theta$$

($\mathrm{Re}\, a > 0$, $\mathrm{Re}\, b > 0$) we obtain

$$I = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{(2n)!} B\left(p + \frac{b}{2}, n + \frac{1}{2}\right) z^{2n}.$$

Application of

$$B\left(p + \frac{b}{2}, n + \frac{1}{2}\right) = \frac{\Gamma(p + b/2)\Gamma(n + 1/2)}{\Gamma(p + n + (b+1)/2)}$$

and

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$$

$(n=0,1,\dots)$ gives

$$\begin{aligned}I&=\frac{\sqrt{\pi}}{2}\Gamma\left(p+\frac{b}{2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{n!\,\Gamma(p+n+(b+1)/2)}\left(\frac{z}{2}\right)^{2n}\\&=\frac{\sqrt{\pi}}{2}\Gamma\left(p+\frac{b}{2}\right)\left(\frac{2}{z}\right)^{p}v_p(z).\end{aligned}$$

Hence

$$v_p(z)=2\left(\frac{z}{2}\right)^p\frac{1}{\sqrt{\pi}\,\Gamma\left(p+\frac{b}{2}\right)}\int_0^{\pi/2}(\sin\theta)^{2p+b-1}\cos(\sqrt{c}\,z\cos\theta)\,d\theta.$$

Utilizing (2.5) we obtain

$$u_p(z)=\frac{2}{B\left(p+\frac{b}{2},\frac{1}{2}\right)}\int_0^{\pi/2}(\sin\theta)^{2p+b-1}\cos(\sqrt{c}\,z\cos\theta)\,d\theta.$$

Letting $z=x^2$ and making a substitution $t=\cos\theta$ we obtain, with the aid of (2.8) and (2.11), the first part of (2.10). When $c<0$, the proof of the second part of (2.10) goes along the lines introduced above. We begin with a series expansion

$$\cosh(\sqrt{-c}\,z\cos\theta)=\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{(2n)!}z^{2n}(\cos\theta)^{2n}.$$

Application to the right side of

$$I := \int_0^{\pi/2} (\sin\theta)^{2p+b-1} \cosh(\sqrt{-c}\, z\cos\theta)\, d\theta$$

gives

$$v_p(z) = 2\left(\frac{z}{2}\right)^p \frac{1}{\sqrt{\pi}\,\Gamma\left(p+\frac{b}{2}\right)} \int_0^{\pi/2} (\sin\theta)^{2p+b-1} \cosh(\sqrt{-c}\, z\cos\theta)\, d\theta.$$

This in turn implies that

$$u_p(z) = 2A \int_0^{\pi/2} (\sin\theta)^{2p+b-1} \cosh(\sqrt{-cz}\cos\theta)\, d\theta.$$

Putting $z = x^2$ and making a substitution $t = \cos\theta$ we obtain, utilizing (2.8) and (2.11), the second part of (2.10). The proof is complete. □

When $b = c = 1$, formula (2.10) simplifies to Eq. (9.1.20) in [1].

3. Askey's Inequality for the Function λ_p and Grünbaum's Inequality for Modified Bessel Functions of the First Kind

We begin with the following.

Theorem 3.1. *Let the real numbers p, b, and c be such that $p + b/2 > 1/2$ and let $x, y, z \geq 0$ with $z^2 = x^2 + y^2$. Then the following inequality*

$$\lambda_p(x) + \lambda_p(y) \leq 1 + \lambda_p(z) \tag{3.1}$$

holds true.

Proof. There is nothing to prove when $c = 0$, because in this case $\lambda_p(x) = 1$. Assume that $c > 0$. It follows from (1.2) and (2.9) that

$$J_{p+(b-1)/2}(x\sqrt{c}) = \lambda_p(x). \tag{3.2}$$

Making use of (1.3) with x replaced by $x\sqrt{c}$, y replaced by $y\sqrt{c}$, and p replaced by $p+(b-1)/2$ together with application of (3.2) gives the desired result. Now let $c < 0$. Then the inequality (3.1) can be written as

$$u_p(x^2) + u_p(y^2) \leq 1 + u_p(z^2)$$

or after replacing x^2 by x, y^2 by y, and z^2 by z, as

$$u_p(x) + u_p(y) \leq 1 + u_p(z). \tag{3.3}$$

Let us note that in order for the inequality (3.3) to be valid it suffices to show that a function $f(x) = u_p(x) - 1$ is superadditive, i.e., that $f(x+y) \geq f(x) + f(y)$ for $x, y \geq 0$. We shall prove that if the function $g(x) = f(x)/x$ is increasing, then $f(x)$ is superadditive. We have $g(x) = \big(u_p(x) - 1\big)/x$. Hence $g'(x) = \big[xu_p'(x) - (u_p(x) - 1)\big]/x^2$. In order for $g(x)$ to be increasing it is necessary and sufficient that $xu_p'(x) \geq u_p(x) - 1$. Since

$$u_p(x) = \sum_{n=0}^{\infty} b_n x^n$$

with the coefficients b_n $(n \geq 0)$ defined in (2.7), the last inequality can be written as

$$\sum_{n=1}^{\infty} (n-1) b_n x^n \geq 0.$$

Making use of (2.7) we see that $b_n \geq 0$ for all $n \geq 1$. This in turn implies that the function $g(x) = f(x)/x$ is increasing. Using this one can prove easily the superadditivity of $f(x)$. We have

$$f(x+y) = x\frac{f(x+y)}{x+y} + y\frac{f(x+y)}{x+y} \geq x\frac{f(x)}{x} + y\frac{f(y)}{y} = f(x) + f(y).$$

This completes the proof of (3.3). Letting $x := x^2$, $y := y^2$, and $z := z^2$ in (3.3) and utilizing (2.8) we obtain the assertion. □

Before we state the next theorem, let us introduce more notation. Let $\mathcal{I}_p(x) = (2/x)^p \Gamma(p+1) I_p(x)$. Let us note that $\mathcal{I}_p = \lambda_p$ when $b = 1$ and $c = -1$.

Theorem 3.2. *Let $p, x, y, z \geq 0$ with $z^2 = x^2 + y^2$. Then*

$$\mathcal{I}_p(x) + \mathcal{I}_p(y) \leq 1 + \mathcal{I}_p(z). \tag{3.4}$$

Proof. Let $p > 0$. Then the inequality (3.4) is a special case of (3.1). When $p = 0$, $\mathcal{I}_0 = I_0$. In order to prove Grünbaum's inequality for the modified Bessel functions of the first kind of order zero:

$$I_0(x) + I_0(y) \leq 1 + I_0(z) \tag{3.5}$$

we may proceed as in the proof of Theorem 3.1, case of negative value of c. We need Petrović's theorem for convex functions (see [10], [8, Theorem 1, p. 22]). This result states that if ϕ is a convex function on the domain which contains $0, x_1, x_2, \ldots, x_n \geq 0$, then

$$\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n) \leq \phi(x_1 + \cdots + x_n) + (n-1)\phi(0).$$

If $n = 2$ and $\phi(0) = 0$, then the last inequality shows that ϕ is a superadditive function. Let $f(x) = u_0(x) - 1$. Using (2.6) with $b = 1$ and $c = -1$ we see that $f(x)$ is a convex function and also that $f(0) = 0$. Using Petrović's result we conclude that the function $f(x)$ is superadditive. This in turn implies inequality (3.5). □

4. Lower and Upper Bounds for the Function λ_p

In the recent paper (see [5, Theorem 1.22]) Á. Baricz has shown that for $x, y \in (0,1)$ and under some assumptions on the parameters p, b, and c, the following inequality

$$\lambda_p(x) + \lambda_p(y) \le 2\lambda_p(z)$$

holds true provided $z^2 = 1 - \sqrt{(1-x^2)(1-y^2)}$.

We are in a position to prove the following.

Theorem 4.1. *Let the real numbers p, b, and c be such that $p + b/2 > 0$. Then for arbitrary real numbers x and y the inequality*

$$\left[\lambda_p(x) + \lambda_p(y)\right]^2 \le \left[1 + \lambda_p(x+y)\right]\left[1 + \lambda_p(x-y)\right] \tag{4.1}$$

is valid. Equality holds in (4.1) if $c = 0$.

Proof. There is nothing to prove when $c = 0$. In this case $\lambda_p(x) = 1$ (see (2.9), (2.10)). Assume that $c > 0$. Theorem 2.1 in [9] states that (4.1) is satisfied when $b = c = 1$, i.e., when $\lambda_p = \mathcal{J}_p$. Replacing x by $x\sqrt{c}$, y by $y\sqrt{c}$, and p by $p + (b-1)/2$ we obtain the desired result (4.1). Assume now that $c < 0$. It follows from Lemma 2.1 that

$$\lambda_p(x) = \int_0^1 \cosh(tx\sqrt{-c})\, d\mu(t).$$

Using the identities

$$\cosh\alpha + \cosh\beta = 2\cosh\left(\frac{\alpha+\beta}{2}\right)\cosh\left(\frac{\alpha-\beta}{2}\right),$$

$$2\cosh^2\left(\frac{\alpha}{2}\right) = 1 + \cosh\alpha,$$

and the Cauchy-Schwarz inequality for integrals, we obtain

$$\begin{aligned}
&\lambda_p(x)+\lambda_p(y)\\
&=\int_0^1\left[\cosh(tx\sqrt{-c})+\cosh(ty\sqrt{-c})\right]d\mu(t)\\
&=2\int_0^1\cosh\frac{t(x+y)\sqrt{-c}}{2}\cosh\frac{t(x-y)\sqrt{-c}}{2}\,d\mu(t)\\
&\le 2\left[\int_0^1\cosh^2\frac{t(x+y)\sqrt{-c}}{2}\,d\mu(t)\right]^{\frac{1}{2}}\left[\int_0^1\cosh^2\frac{t(x-y)\sqrt{-c}}{2}\,d\mu(t)\right]^{\frac{1}{2}}\\
&=\left[\int_0^1\left(1+\cosh(t(x+y)\sqrt{-c})\right)d\mu(t)\int_0^1\left(1+\cosh(t(x-y)\sqrt{-c})\right)d\mu(t)\right]^{\frac{1}{2}}\\
&=\left[\left(1+\lambda_p(x+y)\right)\left(1+\lambda_p(x-y)\right)\right]^{\frac{1}{2}}.
\end{aligned}$$

Hence the assertion follows. □

When $x=y$, inequality (4.1) reduces to $2\lambda_p^2(x)\le 1+\lambda_p(2x)$ which resembles the double-angle formulas for the cosine and the hyperbolic cosine functions, i.e., $2\cos^2 x=1+\cos(2x)$ and $2\cosh^2 x=1+\cosh(2x)$, respectively.

Our next goal is to establish computable lower and upper bounds for the function λ_p. For the reader's convenience, we recall some facts about Gegenbauer polynomials G_k^p ($p>-\frac{1}{2}$, $k\in\mathbb{N}$) and the Gauss-Gegenbauer quadrature formulas. The polynomials in question are orthogonal on the interval $[-1,1]$ with the weight function $t\to(1-t^2)^{p-(1/2)}$. The explicit formula for G_k^p is ([1, 22.3.4])

$$G_k^p(t) = \sum_{n=0}^{[k/2]} (-1)^n \frac{\Gamma(p+k-n)}{\Gamma(p)n!(k-2n)!}(2t)^{k-2n}. \tag{4.2}$$

In particular,

$$G_2^p(t) = 2p(p+1)t^2 - p. \tag{4.3}$$

The classical Gauss-Gegenbauer quadrature formula with the remainder reads as follows [3]

$$\int_{-1}^{1} (1-t^2)^{p-\frac{1}{2}} f(t)dt = \sum_{i=1}^{k} w_i f(t_i) + \gamma_k f^{(2k)}(\alpha), \tag{4.4}$$

where $f \in C^{2k}([-1,1])$, γ_k is a positive number which does not depend on f, α is an intermediate point in $(-1,1)$. The nodes t_i ($i = 1, 2, \dots, k$) are the roots of G_k^p and the weights w_i are given explicitly by [11, (15.3.2)]

$$w_i = \pi \left(\frac{2^{1-p}}{\Gamma(p)} \right)^2 \frac{\Gamma(2p+k)}{k!(1-t_i^2)} \left[(G_k^p)'(t_i) \right]^{-2} \tag{4.5}$$

$(1 \le i \le k)$.

The last result of this paper is contained in the following.

Theorem 4.2. *For* $p, b \in \mathbb{R}$, *let* $\kappa := p + (b+1)/2 > 1/2$.

(i) If $c \in [0,1]$ and $|x| \le \frac{\pi}{2}$, then

$$\cos\left(\sqrt{\frac{c}{2\kappa}}x\right) \le \lambda_p(x) \le \frac{1}{3\kappa}\left[2\kappa - 1 + (\kappa+1)\cos\left(\sqrt{\frac{3c}{2(\kappa+1)}}x\right)\right]. \tag{4.6}$$

(ii) If $c \le 0$ and $x \in \mathbb{R}$, then

$$\cosh\left(\sqrt{\frac{-c}{2\kappa}}x\right) \le \lambda_p(x). \tag{4.7}$$

Equalities hold in (4.6) and (4.7) if $c = 0$ or $x = 0$.

Proof. Utilizing Theorem 2.2 in [9] we see that the inequalities (4.6) are valid when $b = c = 1$, i.e., when $\lambda_p = \mathcal{J}_p$:

$$\cos\left(\frac{x}{\sqrt{2(p+1)}}\right) \le \mathcal{J}_p(x) \le \frac{1}{3(p+1)}\left[2p + 1 + (p+2)\cos\left(\sqrt{\frac{3}{2(p+2)}}x\right)\right].$$

Let $0 \le c \le 1$. Replacing x by $x\sqrt{c}$, y by $y\sqrt{c}$, p by $p + (b-1)/2$, and utilizing (3.2) we obtain the desired result. Assume now that $c \le 0$. In order to establish the lower bound in (4.7) we use the Gauss-Gegenbauer quadrature formula (4.4) with $k = 2$ and $f(t) = \cosh(tx\sqrt{-c})$. Since $f^{(4)}(t) = x^4c^2\cosh(tx\sqrt{-c}) \geq 0$ for $|t| \leq 1$, (4.4) yields

$$w_1 f(t_1) + w_2 f(t_2) \leq \int_{-1}^{1} (1-t^2)^{p-\frac{1}{2}} \cosh\left(tx\sqrt{-c}\right) dt. \tag{4.8}$$

Using formulas (4.3) and (4.5), with p replaced by $p + (b-1)/2$, we obtain

$$-t_1 = t_2 = \frac{1}{\sqrt{2\kappa}},$$
$$w_1 = w_2 = \frac{1}{2}B\left(\kappa - \frac{1}{2}, \frac{1}{2}\right).$$

This, in conjuction with (4.8), gives

$$B\left(\kappa - \frac{1}{2}, \frac{1}{2}\right)\cosh\left(\sqrt{\frac{-c}{2\kappa}}x\right) \leq \int_{-1}^{1}(1-t^2)^{\kappa-\frac{3}{2}}\cosh(tx\sqrt{-c})\,dt$$
$$= 2\int_{0}^{1}(1-t^2)^{\kappa-\frac{3}{2}}\cosh(tx\sqrt{-c})\,dt.$$

Application of Lemma 2.1 gives the desired result (4.7). The proof is complete. □

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