

BARICZ ÁRPÁD AND EDWARD NEUMAN

Faculty of Mathematics and Computer Science
"Babeş-Bolyai" University
Str. M. Kogălniceanu NR. 1
RO-400084 Cluj-Napoca, Romania.
EMail: bariczocsi@yahoo.com

Department of Mathematics
Mailcode 4408
Southern Illinois University
1245 Lincoln Drive
Carbondale, IL 62901, USA.
EMail: edneuman@math.siu.edu
URL: <http://www.math.siu.edu/neuman/personal.html>



volume 6, issue 4, article 126,
2005.

*Received 17 September, 2005;
accepted 22 September, 2005.*

Communicated by: A. Lupaş

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

Let u_p denote the normalized, generalized Bessel function of order p which depends on two parameters b and c and let $\lambda_p(x) = u_p(x^2)$, $x \geq 0$. It is proven that under some conditions imposed on p , b , and c the Askey inequality holds true for the function λ_p , i.e., that $\lambda_p(x) + \lambda_p(y) \leq 1 + \lambda_p(z)$, where $x, y \geq 0$ and $z^2 = x^2 + y^2$. The lower and upper bounds for the function λ_p are also established.

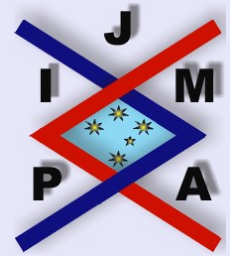
2000 Mathematics Subject Classification: 33C10, 26D20.

Key words: Askey's inequality, Grünbaum's inequality, Bessel functions, Gegenbauer polynomials.

The first author was partially supported by the Institute of Mathematics, University of Debrecen, Hungary. Thanks are due to Professor András Szilárd for his helpful suggestions and to Professor Péter T. Nagy for his support and encouragement.

Contents

1	Introduction	3
2	The Function λ_p	5
3	Askey's Inequality for the Function λ_p and Grünbaum's Inequality for Modified Bessel Functions of the First Kind	11
4	Lower and Upper Bounds for the Function λ_p	14
	References	



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 2 of 20

1. Introduction

The Bessel function of the first kind of order p , denoted by $J_p(x)$, is defined as a particular solution of the second-order differential equation ([12, p. 38])

$$(1.1) \quad x^2 y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0$$

which is also called the Bessel equation. It is known ([12, p. 40]) that

$$(1.2) \quad J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \quad x \in \mathbb{R}.$$

R. Askey [2] has shown that for $\mathcal{J}_p(x) = \Gamma(p+1)(2/x)^p J_p(x)$ the following inequality

$$(1.3) \quad \mathcal{J}_p(x) + \mathcal{J}_p(y) \leq 1 + \mathcal{J}_p(z)$$

holds true for all $x, y, z, p \geq 0$ where $z^2 = x^2 + y^2$. Since $\mathcal{J}_0(x) = J_0(x)$, inequality (1.3) provides a generalization of Grünbaum's inequality ([6])

$$(1.4) \quad J_0(x) + J_0(y) \leq 1 + J_0(z).$$

Using Legendre polynomials Grünbaum has supplied another proof of (1.4) in [7].

Recently, E. Neuman ([9]) has obtained a different upper bound for $\mathcal{J}_p(x) + \mathcal{J}_p(y)$. In the same paper the lower and upper bounds for the function $\mathcal{J}_p(x)$ are established with the aid of Gegenbauer polynomials.



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



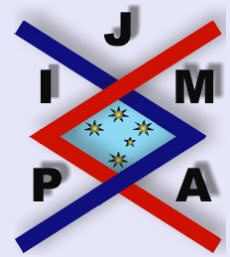
Go Back

Close

Quit

Page 3 of 20

The purpose of this paper is to obtain similar results to those mentioned above for the function λ_p which is the transformed version of the normalized, generalized Bessel function u_p . Definitions of these functions together with the integral formula are contained in Section 2. An Askey type inequality for the function λ_p and the Grünbaum inequality for the modified Bessel functions of the first kind are derived in Section 3. The lower and upper bounds for the function λ_p are established in Section 4.



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 4 of 20

2. The Function λ_p

The following second-order differential equation (see [12, p. 77])

$$(2.1) \quad x^2 y''(x) + xy'(x) - (x^2 + p^2)y(x) = 0$$

frequently occurs in mathematical physics. A particular solution of (2.1), denoted by $I_p(x)$, is called the modified Bessel function of the first kind of order p and it is represented as the infinite series

$$(2.2) \quad I_p(x) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}, \quad x \in \mathbb{R}$$

(see, e.g., [12, p. 77]).

A second order differential equation which reduces either to (1.1) or (2.1) reads as follows

$$(2.3) \quad x^2 v''(x) + bxv'(x) + [cx^2 - p^2 + (1-b)p]v(x) = 0,$$

$b, c, p \in \mathbb{R}$. A particular solution v_p is

$$(2.4) \quad v_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n! \Gamma(p+n+(b+1)/2)} \left(\frac{x}{2}\right)^{2n+p}$$

and v_p is called the generalized Bessel function of the first kind of order p (see [4]). It is readily seen that for $b = 1$ and $c = 1$, v_p becomes J_p and for $b = 1$ and $c = -1$, v_p simplifies to I_p .



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 5 of 20

The normalized, generalized Bessel function of the first kind of order p , denoted by u_p , is defined as

$$(2.5) \quad u_p(x) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) x^{-p/2} v_p(x^{1/2}).$$

Using the Pochhammer symbol $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ ($a \neq 0$) we obtain the following formula

$$(2.6) \quad u_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{4^n \left(p + \frac{b+1}{2}\right)_n} \cdot \frac{x^n}{n!}$$

($p + (b+1)/2 \neq 0, -1, \dots$). For later use, let us write

$$u_p(x) = \sum_{n=0}^{\infty} b_n x^n,$$

where

$$(2.7) \quad b_n = \frac{1}{n! \left(p + \frac{b+1}{2}\right)_n} \left(-\frac{c}{4}\right)^n$$

($n \geq 0$).

Finally, we define a function λ_p as follows

$$(2.8) \quad \lambda_p(x) = u_p(x^2).$$



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 6 of 20

Making use of (2.6) we obtain a series representation for the function in question

$$(2.9) \quad \lambda_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{\left(p + \frac{b+1}{2}\right)_n n!} \left(\frac{x}{2}\right)^{2n}.$$

The following lemma will be used in the sequel.

Lemma 2.1. *Let the numbers p and b be such $\operatorname{Re}(p + b/2) > 0$. Then for any $x \in \mathbb{R}$*

$$(2.10) \quad \lambda_p(x) = \begin{cases} \int_0^1 \cos(tx\sqrt{c}) d\mu(t), & c \geq 0 \\ \int_0^1 \cosh(tx\sqrt{-c}) d\mu(t), & c \leq 0, \end{cases}$$

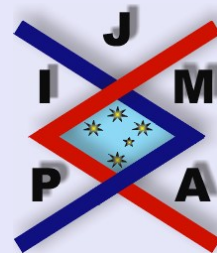
where $d\mu(t) = \mu(t) dt$ with

$$(2.11) \quad \mu(t) = \frac{2(1-t^2)^{p+(b-2)/2}}{B\left(p + \frac{b}{2}, \frac{1}{2}\right)}$$

being the probability measure on $[0, 1]$. Here $B(\cdot, \cdot)$ stands for the beta function.

Proof. We shall prove first that the function $\mu(t)$, defined in (2.11), is indeed the probability measure on $[0, 1]$. Clearly the function in question is nonnegative on the indicated interval. Moreover, with $A = 1/B(p + b/2, 1/2)$, we have

$$\begin{aligned} \int_0^1 d\mu(t) &= 2A \int_0^1 (1-t^2)^{p+(b-2)/2} dt \\ &= A \int_0^1 r^{-1/2} (1-r)^{p+(b-2)/2} dr = A \cdot A^{-1}. \end{aligned}$$



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 7 of 20

Here we have used the substitution $r = t^{1/2}$.

In order to establish formula (2.10) we note that (2.9) implies $\lambda_p(0) = 1$ and also that $\lambda_p(-x) = \lambda(x)$. To this end, let $x > 0$. For the sake of brevity, let

$$I = \int_0^{\pi/2} (\sin \theta)^{2p+b-1} \cos(\sqrt{c} z \cos \theta) d\theta, \quad c \geq 0.$$

Using the Maclaurin expansion for the cosine function and integrating term by term we obtain

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{(2n)!} z^{2n} \int_0^{\pi/2} (\sin \theta)^{2p+b-1} (\cos \theta)^{2n} d\theta,$$

where the last integral converges uniformly provided $\operatorname{Re}(p + b/2) > 0$. Making use of the well-known formula

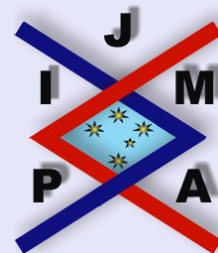
$$B(a, b) = 2 \int_0^{\pi/2} (\cos \theta)^{2a-1} (\sin \theta)^{2b-1} d\theta$$

($\operatorname{Re} a > 0, \operatorname{Re} b > 0$) we obtain

$$I = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{(2n)!} B\left(p + \frac{b}{2}, n + \frac{1}{2}\right) z^{2n}.$$

Application of

$$B\left(p + \frac{b}{2}, n + \frac{1}{2}\right) = \frac{\Gamma(p + b/2)\Gamma(n + 1/2)}{\Gamma(p + n + (b + 1)/2)}$$



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 8 of 20

and

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n}n!} \sqrt{\pi}$$

($n = 0, 1, \dots$) gives

$$\begin{aligned} I &= \frac{\sqrt{\pi}}{2} \Gamma\left(p + \frac{b}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n! \Gamma(p + n + (b + 1)/2)} \left(\frac{z}{2}\right)^{2n} \\ &= \frac{\sqrt{\pi}}{2} \Gamma\left(p + \frac{b}{2}\right) \left(\frac{2}{z}\right)^p v_p(z). \end{aligned}$$

Hence

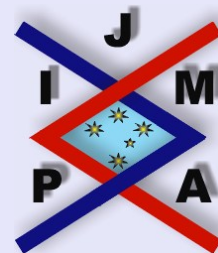
$$v_p(z) = 2 \left(\frac{z}{2}\right)^p \frac{1}{\sqrt{\pi} \Gamma\left(p + \frac{b}{2}\right)} \int_0^{\pi/2} (\sin \theta)^{2p+b-1} \cos(\sqrt{c} z \cos \theta) d\theta.$$

Utilizing (2.5) we obtain

$$u_p(z) = \frac{2}{B\left(p + \frac{b}{2}, \frac{1}{2}\right)} \int_0^{\pi/2} (\sin \theta)^{2p+b-1} \cos(\sqrt{c} z \cos \theta) d\theta.$$

Letting $z = x^2$ and making a substitution $t = \cos \theta$ we obtain, with the aid of (2.8) and (2.11), the first part of (2.10). When $c < 0$, the proof of the second part of (2.10) goes along the lines introduced above. We begin with a series expansion

$$\cosh(\sqrt{-c} z \cos \theta) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{(2n)!} z^{2n} (\cos \theta)^{2n}.$$



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 9 of 20

Application to the right side of

$$I := \int_0^{\pi/2} (\sin \theta)^{2p+b-1} \cosh(\sqrt{-c} z \cos \theta) d\theta$$

gives

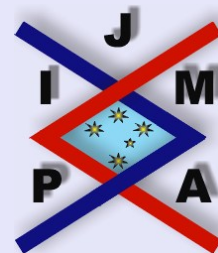
$$v_p(z) = 2 \left(\frac{z}{2}\right)^p \frac{1}{\sqrt{\pi} \Gamma\left(p + \frac{b}{2}\right)} \int_0^{\pi/2} (\sin \theta)^{2p+b-1} \cosh(\sqrt{-c} z \cos \theta) d\theta.$$

This in turn implies that

$$u_p(z) = 2A \int_0^{\pi/2} (\sin \theta)^{2p+b-1} \cosh(\sqrt{-cz} \cos \theta) d\theta.$$

Putting $z = x^2$ and making a substitution $t = \cos \theta$ we obtain, utilizing (2.8) and (2.11), the second part of (2.10). The proof is complete. \square

When $b = c = 1$, formula (2.10) simplifies to Eq. (9.1.20) in [1].



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 10 of 20

3. Askey's Inequality for the Function λ_p and Grünbaum's Inequality for Modified Bessel Functions of the First Kind

We begin with the following.

Theorem 3.1. *Let the real numbers p , b , and c be such that $p + b/2 > 1/2$ and let $x, y, z \geq 0$ with $z^2 = x^2 + y^2$. Then the following inequality*

$$(3.1) \quad \lambda_p(x) + \lambda_p(y) \leq 1 + \lambda_p(z)$$

holds true.

Proof. There is nothing to prove when $c = 0$, because in this case $\lambda_p(x) = 1$. Assume that $c > 0$. It follows from (1.2) and (2.9) that

$$(3.2) \quad J_{p+(b-1)/2}(x\sqrt{c}) = \lambda_p(x).$$

Making use of (1.3) with x replaced by $x\sqrt{c}$, y replaced by $y\sqrt{c}$, and p replaced by $p + (b - 1)/2$ together with application of (3.2) gives the desired result. Now let $c < 0$. Then the inequality (3.1) can be written as

$$u_p(x^2) + u_p(y^2) \leq 1 + u_p(z^2)$$

or after replacing x^2 by x , y^2 by y , and z^2 by z , as

$$(3.3) \quad u_p(x) + u_p(y) \leq 1 + u_p(z).$$



Inequalities Involving
Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 11 of 20

Let us note that in order for the inequality (3.3) to be valid it suffices to show that a function $f(x) = u_p(x) - 1$ is superadditive, i.e., that $f(x + y) \geq f(x) + f(y)$ for $x, y \geq 0$. We shall prove that if the function $g(x) = f(x)/x$ is increasing, then $f(x)$ is superadditive. We have $g(x) = (u_p(x) - 1)/x$. Hence $g'(x) = [xu'_p(x) - (u_p(x) - 1)]/x^2$. In order for $g(x)$ to be increasing it is necessary and sufficient that $xu'_p(x) \geq u_p(x) - 1$. Since

$$u_p(x) = \sum_{n=0}^{\infty} b_n x^n$$

with the coefficients b_n ($n \geq 0$) defined in (2.7), the last inequality can be written as

$$\sum_{n=1}^{\infty} (n-1)b_n x^n \geq 0.$$

Making use of (2.7) we see that $b_n \geq 0$ for all $n \geq 1$. This in turn implies that the function $g(x) = f(x)/x$ is increasing. Using this one can prove easily the superadditivity of $f(x)$. We have

$$f(x + y) = x \frac{f(x + y)}{x + y} + y \frac{f(x + y)}{x + y} \geq x \frac{f(x)}{x} + y \frac{f(y)}{y} = f(x) + f(y).$$

This completes the proof of (3.3). Letting $x := x^2$, $y := y^2$, and $z := z^2$ in (3.3) and utilizing (2.8) we obtain the assertion. \square

Before we state the next theorem, let us introduce more notation. Let $\mathcal{I}_p(x) = (2/x)^p \Gamma(p+1) I_p(x)$. Let us note that $\mathcal{I}_p = \lambda_p$ when $b = 1$ and $c = -1$.



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 12 of 20

Theorem 3.2. Let $p, x, y, z \geq 0$ with $z^2 = x^2 + y^2$. Then

$$(3.4) \quad \mathcal{I}_p(x) + \mathcal{I}_p(y) \leq 1 + \mathcal{I}_p(z).$$

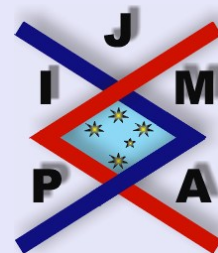
Proof. Let $p > 0$. Then the inequality (3.4) is a special case of (3.1). When $p = 0$, $\mathcal{I}_0 = I_0$. In order to prove Grünbaum's inequality for the modified Bessel functions of the first kind of order zero:

$$(3.5) \quad I_0(x) + I_0(y) \leq 1 + I_0(z)$$

we may proceed as in the proof of Theorem 3.1, case of negative value of c . We need Petrović's theorem for convex functions (see [10], [8, Theorem 1, p. 22]). This result states that if ϕ is a convex function on the domain which contains $0, x_1, x_2, \dots, x_n \geq 0$, then

$$\phi(x_1) + \phi(x_2) + \dots + \phi(x_n) \leq \phi(x_1 + \dots + x_n) + (n - 1)\phi(0).$$

If $n = 2$ and $\phi(0) = 0$, then the last inequality shows that ϕ is a superadditive function. Let $f(x) = u_0(x) - 1$. Using (2.6) with $b = 1$ and $c = -1$ we see that $f(x)$ is a convex function and also that $f(0) = 0$. Using Petrović's result we conclude that the function $f(x)$ is superadditive. This in turn implies inequality (3.5). \square



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 13 of 20

4. Lower and Upper Bounds for the Function λ_p

In the recent paper (see [5, Theorem 1.22]) Á. Baricz has shown that for $x, y \in (0, 1)$ and under some assumptions on the parameters p, b , and c , the following inequality

$$\lambda_p(x) + \lambda_p(y) \leq 2\lambda_p(z)$$

holds true provided $z^2 = 1 - \sqrt{(1-x^2)(1-y^2)}$.

We are in a position to prove the following.

Theorem 4.1. *Let the real numbers p, b , and c be such that $p + b/2 > 0$. Then for arbitrary real numbers x and y the inequality*

$$(4.1) \quad [\lambda_p(x) + \lambda_p(y)]^2 \leq [1 + \lambda_p(x+y)][1 + \lambda_p(x-y)]$$

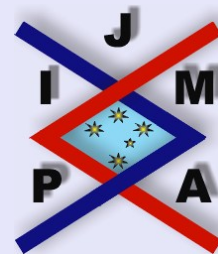
is valid. Equality holds in (4.1) if $c = 0$.

Proof. There is nothing to prove when $c = 0$. In this case $\lambda_p(x) = 1$ (see (2.9), (2.10)). Assume that $c > 0$. Theorem 2.1 in [9] states that (4.1) is satisfied when $b = c = 1$, i.e., when $\lambda_p = \mathcal{J}_p$. Replacing x by $x\sqrt{c}$, y by $y\sqrt{c}$, and p by $p + (b-1)/2$ we obtain the desired result (4.1). Assume now that $c < 0$. It follows from Lemma 2.1 that

$$\lambda_p(x) = \int_0^1 \cosh(tx\sqrt{-c}) d\mu(t).$$

Using the identities

$$\begin{aligned} \cosh \alpha + \cosh \beta &= 2 \cosh \left(\frac{\alpha + \beta}{2} \right) \cosh \left(\frac{\alpha - \beta}{2} \right), \\ 2 \cosh^2 \left(\frac{\alpha}{2} \right) &= 1 + \cosh \alpha, \end{aligned}$$



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 14 of 20

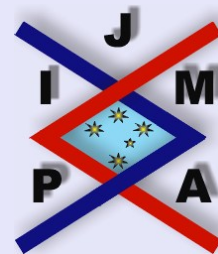
and the Cauchy-Schwarz inequality for integrals, we obtain

$$\begin{aligned}
 & \lambda_p(x) + \lambda_p(y) \\
 &= \int_0^1 [\cosh(tx\sqrt{-c}) + \cosh(ty\sqrt{-c})] d\mu(t) \\
 &= 2 \int_0^1 \cosh \frac{t(x+y)\sqrt{-c}}{2} \cosh \frac{t(x-y)\sqrt{-c}}{2} d\mu(t) \\
 &\leq 2 \left[\int_0^1 \cosh^2 \frac{t(x+y)\sqrt{-c}}{2} d\mu(t) \right]^{\frac{1}{2}} \left[\int_0^1 \cosh^2 \frac{t(x-y)\sqrt{-c}}{2} d\mu(t) \right]^{\frac{1}{2}} \\
 &= \left[\int_0^1 (1 + \cosh(t(x+y)\sqrt{-c})) d\mu(t) \int_0^1 (1 + \cosh(t(x-y)\sqrt{-c})) d\mu(t) \right]^{\frac{1}{2}} \\
 &= [(1 + \lambda_p(x+y))(1 + \lambda_p(x-y))]^{\frac{1}{2}}.
 \end{aligned}$$

Hence the assertion follows. \square

When $x = y$, inequality (4.1) reduces to $2\lambda_p^2(x) \leq 1 + \lambda_p(2x)$ which resembles the double-angle formulas for the cosine and the hyperbolic cosine functions, i.e., $2\cos^2 x = 1 + \cos(2x)$ and $2\cosh^2 x = 1 + \cosh(2x)$, respectively.

Our next goal is to establish computable lower and upper bounds for the function λ_p . For the reader's convenience, we recall some facts about Gegenbauer polynomials G_k^p ($p > -\frac{1}{2}$, $k \in \mathbb{N}$) and the Gauss-Gegenbauer quadrature formulas. The polynomials in question are orthogonal on the interval $[-1, 1]$ with the weight function $t \rightarrow (1 - t^2)^{p-(1/2)}$. The explicit formula for G_k^p is ([1,



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 15 of 20

22.3.4)

$$(4.2) \quad G_k^p(t) = \sum_{n=0}^{\lfloor k/2 \rfloor} (-1)^n \frac{\Gamma(p+k-n)}{\Gamma(p)n!(k-2n)!} (2t)^{k-2n}.$$

In particular,

$$(4.3) \quad G_2^p(t) = 2p(p+1)t^2 - p.$$

The classical Gauss-Gegenbauer quadrature formula with the remainder reads as follows [3]

$$(4.4) \quad \int_{-1}^1 (1-t^2)^{p-\frac{1}{2}} f(t) dt = \sum_{i=1}^k w_i f(t_i) + \gamma_k f^{(2k)}(\alpha),$$

where $f \in C^{2k}([-1, 1])$, γ_k is a positive number which does not depend on f , α is an intermediate point in $(-1, 1)$. The nodes t_i ($i = 1, 2, \dots, k$) are the roots of G_k^p and the weights w_i are given explicitly by [11, (15.3.2)]

$$(4.5) \quad w_i = \pi \left(\frac{2^{1-p}}{\Gamma(p)} \right)^2 \frac{\Gamma(2p+k)}{k!(1-t_i^2)} [(G_k^p)'(t_i)]^{-2}$$

($1 \leq i \leq k$).

The last result of this paper is contained in the following.

Theorem 4.2. For $p, b \in \mathbb{R}$, let $\kappa := p + (b+1)/2 > 1/2$.



**Inequalities Involving
Generalized Bessel Functions**

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 16 of 20

(i) If $c \in [0, 1]$ and $|x| \leq \frac{\pi}{2}$, then

$$(4.6) \quad \cos \left(\sqrt{\frac{c}{2\kappa}} x \right) \leq \lambda_p(x) \\ \leq \frac{1}{3\kappa} \left[2\kappa - 1 + (\kappa + 1) \cos \left(\sqrt{\frac{3c}{2(\kappa + 1)}} x \right) \right].$$

(ii) If $c \leq 0$ and $x \in \mathbb{R}$, then

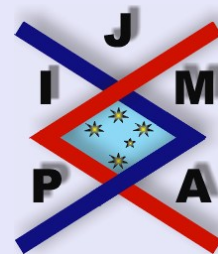
$$(4.7) \quad \cosh \left(\sqrt{\frac{-c}{2\kappa}} x \right) \leq \lambda_p(x).$$

Equalities hold in (4.6) and (4.7) if $c = 0$ or $x = 0$.

Proof. Utilizing Theorem 2.2 in [9] we see that the inequalities (4.6) are valid when $b = c = 1$, i.e., when $\lambda_p = \mathcal{J}_p$:

$$\cos \left(\frac{x}{\sqrt{2(p+1)}} \right) \leq \mathcal{J}_p(x) \\ \leq \frac{1}{3(p+1)} \left[2p + 1 + (p + 2) \cos \left(\sqrt{\frac{3}{2(p+2)}} x \right) \right].$$

Let $0 \leq c \leq 1$. Replacing x by $x\sqrt{c}$, y by $y\sqrt{c}$, p by $p + (b-1)/2$, and utilizing (3.2) we obtain the desired result. Assume now that $c \leq 0$. In order to establish the lower bound in (4.7) we use the Gauss-Gegenbauer quadrature formula (4.4)



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 17 of 20

with $k = 2$ and $f(t) = \cosh(tx\sqrt{-c})$. Since $f^{(4)}(t) = x^4 c^2 \cosh(tx\sqrt{-c}) \geq 0$ for $|t| \leq 1$, (4.4) yields

$$(4.8) \quad w_1 f(t_1) + w_2 f(t_2) \leq \int_{-1}^1 (1-t^2)^{p-\frac{1}{2}} \cosh(tx\sqrt{-c}) dt.$$

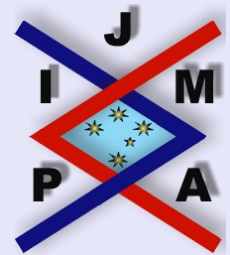
Using formulas (4.3) and (4.5), with p replaced by $p + (b-1)/2$, we obtain

$$\begin{aligned} -t_1 = t_2 &= \frac{1}{\sqrt{2\kappa}}, \\ w_1 = w_2 &= \frac{1}{2} B\left(\kappa - \frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

This, in conjunction with (4.8), gives

$$\begin{aligned} B\left(\kappa - \frac{1}{2}, \frac{1}{2}\right) \cosh\left(\sqrt{\frac{-c}{2\kappa}}x\right) &\leq \int_{-1}^1 (1-t^2)^{\kappa-\frac{3}{2}} \cosh(tx\sqrt{-c}) dt \\ &= 2 \int_0^1 (1-t^2)^{\kappa-\frac{3}{2}} \cosh(tx\sqrt{-c}) dt. \end{aligned}$$

Application of Lemma 2.1 gives the desired result (4.7). The proof is complete. \square



**Inequalities Involving
Generalized Bessel Functions**

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 18 of 20

References

- [1] M. ABRAMOWITZ AND I.A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publications, Inc., New York, 1965.
- [2] R. ASKEY, Grünbaum's inequality for Bessel functions, *J. Math. Anal. Appl.* **41** (1973), 122–124.
- [3] K.E. ATKINSON, *An Introduction to Numerical Analysis*, 2nd ed., John Wiley and Sons, New York 1989.
- [4] Á. BARICZ, Geometric properties of generalized Bessel functions, *J. Math. Anal. Appl.*, submitted.
- [5] Á. BARICZ, Functional inequalities involving power series II, *J. Math. Anal. Appl.*, submitted.
- [6] F.A. GRÜNBAUM, A property of Legendre polynomials, *Proc. Nat. Acad. Sci., USA* **67** (1970), 959–960.
- [7] F.A. GRÜNBAUM, A new kind of inequality for Bessel functions, *J. Math. Anal. Appl.* **41** (1973), 115–121.
- [8] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.
- [9] E. NEUMAN, Inequalities involving Bessel functions of the first kind, *J. Ineq. Pure and Appl. Math.* **5**(4) (2004), Article 94. [ONLINE] Available online at <http://jipam.vu.edu.au/article.php?sid=449>



Inequalities Involving Generalized Bessel Functions

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



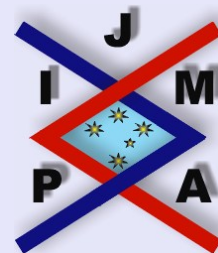
Go Back

Close

Quit

Page 19 of 20

- [10] M. PETROVIĆ, Sur une fonctionnelle, *Publ. Math. Univ. Belgrade* **1** (1932), 149–156.
- [11] G. SZEGÖ, *Orthogonal Polynomials, Colloquium Publications*, vol. 23, 4th ed., American Mathematical Society, Providence, RI, 1975.
- [12] G.N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1962.



**Inequalities Involving
Generalized Bessel Functions**

Jorma K. Merikoski and
Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 20 of 20