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ON A DECOMPOSITION OF HILBERT'S INEQUALITY

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ABSTRACT. By using the Euler-Maclaurin's summation formula and the weight coefficient, a pair of new inequalities is given, which is a decomposition of Hilbert's inequality. The equivalent forms and the extended inequalities with a pair of conjugate exponents are built.

Key words and phrases: Hilbert's inequality; Weight coefficient; Equivalent form; Hilbert-type inequality.

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1. INTRODUCTION

In 1908, H. Weyl published the following Hilbert inequality: If $\{a_n\}$, $\{b_n\}$ are real sequences, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then [1]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. In 1925, G. H. Hardy gave an extension of (1.1) by introducing one pair of conjugate exponents (p,q) $(\frac{1}{p}+\frac{1}{q}=1)$ as [2]: If $p>1, a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\pi/\sin(\frac{\pi}{p})$ is the best possible. We refer to (1.2) as the Hardy-Hilbert inequality. In 1934, Hardy et al. [3] gave some applications of (1.1) and (1.2). By introducing a pair of non-conjugate exponents (p,q) in (1.1), Hardy et al. [3] gave: If $p,q>1, \frac{1}{p}+\frac{1}{q} \geq$

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$1, 0 < \lambda = 2 - (\frac{1}{p} + \frac{1}{q}) \leq 1$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leq K(p,q)\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $K(p,q)$ is the best value only for $\lambda = 1$. In 1951, Bonsall [4] considered (1.3) in the case of a general kernel. In 1991, Mitrinović et al. [5] summarized the above method and results.

In 1997-1998, by using weight coefficients, Yang and Gao [6], [7] gave a strengthened version of (1.2) as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/p}}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/q}}\right]b_n^q\right\}^{\frac{1}{q}}, \tag{1.4}$$

where, $1-\gamma = 0.42278433^{+}$ (γ is the Euler constant). In 2001, Yang [8] gave an extension of (1.1) by introducing an independent parameter $0 < \lambda \leq 4$ as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\sum_{n=1}^{\infty}n^{1-\lambda}a_n^2\sum_{n=1}^{\infty}n^{1-\lambda}b_n^2\right)^{\frac{1}{2}}, \tag{1.5}$$

where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best possible ($B(u,v)$ is the Beta function). In 2004, Yang [9] published the dual form of (1.2) as follows

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{n=1}^{\infty}n^{p-2}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{q-2}b_n^q\right)^{\frac{1}{q}}. \tag{1.6}$$

For $p = q = 2$, both (1.6) and (1.2) reduce to (1.1). It means that there are two different best extensions of (1.1). To generalize (1.2) and (1.6), in 2005, Yang [10] gave an extension of (1.2) and (1.6) with two pairs of conjugate exponents $(p,q),(r,s)(p,r > 1)$ and parameters $\alpha,\lambda > 0$ $(\alpha\lambda \leq \min\{r,s\})$ as: If $0 < \sum_{n=1}^{\infty}n^{p\left(1-\frac{\alpha\lambda}{r}\right)-1}a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty}n^{q\left(1-\frac{\alpha\lambda}{s}\right)-1}b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\alpha})^{\lambda}} < k_{\alpha\lambda}(r)\left\{\sum_{n=1}^{\infty}n^{p(1-\frac{\alpha\lambda}{r})-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1-\frac{\alpha\lambda}{s})-1}b_n^q\right\}^{\frac{1}{q}}, \tag{1.7}$$

where the constant factor $k_{\alpha\lambda}(r) = \frac{1}{\alpha}B(\frac{\lambda}{r},\frac{\lambda}{s})$ is the best possible. T. K. Pogány [11] also considered a best extension of (1.2) with the non-homogeneous kernel as $\frac{1}{(\lambda_m+\rho_n)^{\mu}}$ $(\mu,\lambda_m,\rho_n > 0)$.

We have a non-negative decomposition of kernel in (1.1):

$$\frac{1}{m+n} = \frac{\max\{m,n\}}{(m+n)^2} + \frac{\min\{m,n\}}{(m+n)^2} \qquad (m,n \in \mathbb{N})$$

($\mathbb{N}$ is the set of positive integer numbers). In this paper, by using the Euler-Maclaurin summation formula and the weight coefficient as in [8], we give a pair of new Hilbert-type inequalities as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}a_m b_n < \left(\frac{\pi}{2}+1\right)\left(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}; \tag{1.8}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\min\{m,n\}}{(m+n)^2}a_m b_n < \left(\frac{\pi}{2}-1\right)\left(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}, \tag{1.9}$$

where the sum of two best constant factors is π. The equivalent forms and extended inequalities with a pair of conjugate exponents are considered.

2. Some Lemmas

Lemma 2.1 (Euler-Maclaurin's summation formula, cf. [8, 12, Lemma 1])**.** *If* $f(x) \in C^1[1,\infty)$, *then we have*

$$\sum_{k=1}^{\infty} f(k) = \int_1^{\infty} f(x)dx + \frac{1}{2}f(1) + \int_1^{\infty} P_1(x)f'(x)dx, \tag{2.1}$$

where $P_1(x) = x - [x] - \frac{1}{2}$ *is the Bernoulli function of the first order; if* $g \in C^3[1,\infty)$, $(-1)^i g^{(i)}(x) > 0$, $g^{(i)}(\infty) = 0$, $(i = 0,1,2,3)$*, then*

$$\frac{1}{12}[g(n) - g(1)] < \int_1^n P_1(x)g(x)dx < 0, \tag{2.2}$$
$$-\frac{1}{12}g(n) < \int_n^{\infty} P_1(x)g(x)dx < 0.$$

Lemma 2.2. *If* $\frac{1}{2} \leq \alpha < 1$, *setting the weight coefficient* $\omega(\alpha, m)$ *as*

$$\omega(\alpha, m) := \sum_{n=1}^{\infty} \frac{\max\{m,n\}m^{\alpha}}{(m+n)^2 n^{\alpha}} \qquad (m \in \mathbb{N}), \tag{2.3}$$

then we have

$$k(\alpha) = A_{\alpha}(m) + \omega(\alpha, m); \qquad \omega\left(\frac{1}{2}, m\right) < k\left(\frac{1}{2}\right) = \frac{\pi}{2} + 1, \tag{2.4}$$

where

$$k(\alpha) := \frac{1}{\alpha}\int_0^{\infty} \frac{\max\left\{u^{1/\alpha}, 1\right\}}{(u^{1/\alpha}+1)^2} u^{\frac{1}{\alpha}-2} du$$

and $A_{\alpha}(m) = O(m^{\alpha-1})$, $(m \to \infty)$.

Proof. For fixed $\frac{1}{2} \leq \alpha < 1, m \in \mathbb{N}$, setting $f(x) := \frac{\max\{m,x\}}{(m+x)^2 x^{\alpha}}$, $x \in (0,\infty)$, then by (2.1), it follows that

$$\begin{aligned} \omega(\alpha, m) &= m^{\alpha} \sum_{n=1}^{\infty} f(n) \\ &= m^{\alpha}\left[\int_1^{\infty} f(x)dx + \frac{1}{2}f(1) + \int_1^{\infty} P_1(x)f'(x)dx\right] \\ &= m^{\alpha}\int_0^{\infty} f(x)dx - m^{\alpha}\rho(\alpha, m), P_1(x)f'(x)dx. \end{aligned} \tag{2.5}$$

$$\rho(\alpha, m) := \int_0^1 f(x)dx - \frac{1}{2}f(1) - \int_1^{\infty} P_1(x)f'(x)dx. \tag{2.6}$$

We find

$$-\frac{1}{2}f(1) = \frac{-m}{2(m+1)^2} = \frac{-1}{2(m+1)} + \frac{1}{2(m+1)^2},$$

and

$$\int_0^1 f(x)dx = \int_0^1 \frac{m}{(m+x)^2x^\alpha}dx \geq \int_0^1 \frac{m}{(m+x)^2}dx = \frac{1}{m+1};$$
$$\int_0^1 f(x)dx \leq \int_0^1 \frac{m}{m^2x^\alpha}dx = \frac{1}{(1-\alpha)m}.$$

For $x \in (0,m)$, $f(x) = \frac{m}{(m+x)^2x^\alpha}$, it follows $f'(x) = \frac{-2m}{(m+x)^3x^\alpha} - \frac{\alpha m}{(m+x)^2x^{\alpha+1}}$; for $x \in (m,\infty)$, $f(x) = \frac{x^{1-\alpha}}{(m+x)^2}$, we find

$$\begin{aligned} f'(x) &= \frac{-2x^{1-\alpha}}{(m+x)^3} + \frac{1-\alpha}{(m+x)^2x^\alpha} \\ &= \frac{-2(x+m-m)}{(m+x)^3x^\alpha} + \frac{1-\alpha}{(m+x)^2x^\alpha} \\ &= \frac{-2}{(m+x)^2x^\alpha} + \frac{2m}{(m+x)^3x^\alpha} + \frac{1-\alpha}{(m+x)^2x^\alpha}. \end{aligned}$$

In the following, it is obvious that $g_1(x) = \frac{1}{(m+x)^3x^\alpha}$, $g_2(x) = \frac{1}{(m+x)^2x^{\alpha+1}}$ and $g_3(x) = \frac{1}{(m+x)^2x^\alpha}$ are suited to apply in (2.2). Then by (2.2), we obtain

$$\begin{aligned} (2.7) \qquad & -\int_1^m P_1(x)f'(x)dx \\ &= \int_1^m \frac{2mP_1(x)dx}{(m+x)^3x^\alpha} + \int_1^m \frac{\alpha m P_1(x)dx}{(m+x)^2x^{\alpha+1}} \\ &> \frac{2m}{12}\left[\frac{1}{8m^{3+\alpha}} - \frac{1}{(m+1)^3}\right] + \frac{\alpha m}{12}\left[\frac{1}{4m^{3+\alpha}} - \frac{1}{(m+1)^2}\right] \\ &= \frac{\alpha+1}{48m^{2+\alpha}} - \frac{\alpha}{12(m+1)} - \frac{2-\alpha}{12(m+1)^2} + \frac{1}{6(m+1)^3}; \end{aligned}$$

$$\begin{aligned} (2.8) \qquad & -\int_m^\infty P_1(x)f'(x)dx \\ &= \int_m^\infty \frac{2P_1(x)dx}{(m+x)^2x^\alpha} - \int_m^\infty \frac{2mP_1(x)dx}{(m+x)^3x^\alpha} - (1-\alpha)\int_m^\infty \frac{P_1(x)dx}{(m+x)^2x^\alpha} \\ &> \frac{-1}{24m^{2+\alpha}} - \int_1^\infty P_1(x)f'(x)dx \\ &= -\int_1^m P_1(x)f'(x)dx - \int_m^\infty P_1(x)f'(x)dx \\ &> \frac{\alpha-1}{48m^{2+\alpha}} - \frac{\alpha}{12(m+1)} - \frac{2-\alpha}{12(m+1)^2} + \frac{1}{6(m+1)^3}. \end{aligned}$$

Hence by (2.6), for $\alpha = \frac{1}{2}$, it follows that

$$\begin{aligned} (2.9) \qquad \rho\left(\frac{1}{2},m\right) &> \frac{-1}{2(m+1)} + \frac{1}{2(m+1)^2} + \frac{1}{m+1} \\ &\quad + \frac{\frac{1}{2}-1}{48m^{2+1/2}} - \frac{\frac{1}{2}}{12(m+1)} - \frac{2-\frac{1}{2}}{12(m+1)^2} + \frac{1}{6(m+1)^3} \end{aligned}$$

$$= \frac{11}{24(m+1)} + \frac{9}{24(m+1)^2} + \frac{-1}{96m^{2+1/2}} + \frac{1}{6(m+1)^3}$$
$$\geq \frac{11}{24(m+1)} + \left[\frac{9}{96m^2} + \frac{-1}{96m^2}\right] + \frac{1}{6(m+1)^3} > 0.$$

By (2.7) and (2.8), we obtain

$$-\int_1^\infty P_1(x)f'(x)dx = -\int_1^m P_1(x)f'(x)dx - \int_m^\infty P_1(x)f'(x)dx$$
$$< \frac{1}{48m^{2+\alpha}} + \frac{1-\alpha}{48m^{2+\alpha}}$$
$$= \frac{2-\alpha}{48m^{2+\alpha}}.$$

Then by (2.6), it follows

$$0 < m^{1-\alpha}[m^\alpha \rho(\alpha, m)] \tag{2.10}$$
$$< \frac{-m}{2(m+1)} + \frac{m}{2(m+1)^2} + \frac{1}{1-\alpha} + \frac{2-\alpha}{48m^{1+\alpha}}$$
$$\to \frac{1}{1-\alpha} - \frac{1}{2} \qquad (m \to \infty).$$

Setting $u = (x/m)^\alpha$, we find

$$m^\alpha \int_0^\infty f(x)dx = m^\alpha \int_0^\infty \frac{\max\{m, x\}}{(m+x)^2 x^\alpha} dx \tag{2.11}$$
$$= \frac{1}{\alpha} \int_0^\infty \frac{\max\{u^{1/\alpha}, 1\}}{(u^{1/\alpha}+1)^2} u^{\frac{1}{\alpha}-2} du = k(\alpha),$$

$$k\left(\frac{1}{2}\right) = 2\int_0^\infty \frac{\max\{u^2, 1\}}{(u^2+1)^2} du = 4\int_0^1 \frac{du}{(u^2+1)^2} \tag{2.12}$$
$$= 4\int_0^{\frac{\pi}{4}} \cos^2\theta d\theta = \frac{\pi}{2} + 1.$$

Hence by (2.5), (2.9), (2.10) and (2.11), (2.4) is valid and the lemma is proved. □

Similar to Lemma 2.2, we still have

Lemma 2.3. *If $\frac{1}{2} \leq \alpha < 1$, setting the weight coefficient $\varpi(\alpha, m)$ as*

$$\varpi(\alpha, m) := \sum_{n=1}^\infty \frac{\min\{m, n\} m^\alpha}{(m+n)^2 n^\alpha} \qquad (m \in \mathbb{N}), \tag{2.13}$$

then we have

$$\widetilde{k}(\alpha) = B_\alpha(m) + \varpi(\alpha, m); \qquad \varpi\left(\frac{1}{2}, m\right) < \widetilde{k}\left(\frac{1}{2}\right) = \frac{\pi}{2} - 1, \tag{2.14}$$

where

$$\widetilde{k}(\alpha) = \frac{1}{\alpha}\int_0^\infty \frac{\min\{u^{1/\alpha}, 1\}}{(u^{1/\alpha}+1)^2} u^{\frac{1}{\alpha}-2} du$$

and $B_\alpha(m) = O(m^{\alpha-2}),\ (m \to \infty)$.

3. MAIN RESULTS AND THEIR EQUIVALENT FORMS

Theorem 3.1. *If* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, a_n, b_n \geq 0, 0 < \sum_{n=1}^{\infty} n^{\frac{p}{2}-1}a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{\frac{q}{2}-1}b_n^q < \infty$, *then we have the following equivalent inequalities*

$$I := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}a_m b_n < \left(\frac{\pi}{2}+1\right)\left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1}b_n^q\right)^{\frac{1}{q}}; \tag{3.1}$$

$$J := \sum_{n=1}^{\infty} n^{\frac{p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}a_m\right]^p < \left(\frac{\pi}{2}+1\right)^p \sum_{n=1}^{\infty} n^{\frac{q}{2}-1}a_n^p, \tag{3.2}$$

where the constant factors $\frac{\pi}{2}+1$ *and* $\left(\frac{\pi}{2}+1\right)^p$ *are the best possible.*

Proof. By Hölder's inequality and (2.3) – (2.4), we find

$$\begin{aligned}\left[\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}a_m\right]^p &= \left\{\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}\left[\frac{m^{1/(2q)}}{n^{1/(2p)}}a_m\right]\left[\frac{n^{1/(2p)}}{m^{1/(2q)}}\right]\right\}^p \\ &\leq \left[\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}\frac{m^{p/(2q)}}{n^{1/2}}a_m^p\right] \\ &\quad\times\left[\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}\frac{n^{q/(2p)}}{m^{1/2}}\right]^{p-1} \\ &= \omega^{p-1}\left(\frac{1}{2},n\right)n^{1-\frac{p}{2}}\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}\frac{m^{p/(2q)}}{n^{1/2}}a_m^p \\ &\leq \left(\frac{\pi}{2}+1\right)^{p-1} n^{1-\frac{p}{2}}\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}\frac{m^{p/(2q)}}{n^{1/2}}a_m^p;\end{aligned} \tag{3.3}$$

$$\begin{aligned}J &\leq \left(\frac{\pi}{2}+1\right)^{p-1}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}\frac{m^{p/(2q)}}{n^{1/2}}a_m^p \\ &= \left(\frac{\pi}{2}+1\right)^{p-1}\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}\frac{m^{p/(2q)}}{n^{1/2}}\right]a_m^p \\ &= \left(\frac{\pi}{2}+1\right)^{p-1}\sum_{m=1}^{\infty}\omega\left(\frac{1}{2},m\right)m^{\frac{p}{2}-1}a_m^p < \left(\frac{\pi}{2}+1\right)^p\sum_{m=1}^{\infty}m^{\frac{p}{2}-1}a_m^p.\end{aligned}$$

Therefore (3.2) is valid. By Hölder's inequality, we find that

$$I = \sum_{n=1}^{\infty}\left[n^{\frac{1}{2}-\frac{1}{p}}\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}a_m\right]\left[n^{\frac{-1}{2}+\frac{1}{p}}b_n\right] \leq J^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{q}{2}-1}b_n^q\right)^{\frac{1}{q}}. \tag{3.4}$$

Then by (3.2), we have (3.1). On the other hand, suppose that (3.1) is valid. Setting

$$b_n := n^{\frac{p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}a_m\right]^{p-1}, \qquad n \in \mathbb{N},$$

then it follows $\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q = J$. By (3.3), we confirm that $J < \infty$. If $J = 0$, then (3.2) is naturally valid; if $0 < J < \infty$, then by (3.1), we find

$$\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q = J = I < \left(\frac{\pi}{2}+1\right)\left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q\right)^{\frac{1}{q}};$$

$$\left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q\right)^{\frac{1}{p}} = J^{\frac{1}{p}} < \left(\frac{\pi}{2}+1\right)\left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p\right)^{\frac{1}{p}},$$

and inequality (3.2) is valid, which is equivalent to (3.1).

For $0 < \varepsilon < \frac{q}{2}$, setting $\widetilde{a} = \{\widetilde{a}_n\}_{n=1}^{\infty}$, $\widetilde{b} = \{\widetilde{b}_n\}_{n=1}^{\infty}$ as $\widetilde{a}_n^{\frac{-1}{2}-\frac{\varepsilon}{p}}$, $\widetilde{b}_n^{\frac{-1}{2}-\frac{\varepsilon}{q}}$, for $n \in \mathbb{N}$, if there exists a constant $0 < k \le \frac{\pi}{2}+1$, such that (3.1) is still valid when we replace $\frac{\pi}{2}+1$ by k, then we find

$$\widetilde{I} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\max\{m,n\}\widetilde{a}_m\widetilde{b}_n}{(m+n)^2} < k\left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1}\widetilde{a}_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1}\widetilde{b}_n^q\right)^{\frac{1}{q}} = k\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}};$$

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2} m^{\frac{-1}{2}-\frac{\varepsilon}{p}}\right] n^{\frac{-1}{2}-\frac{\varepsilon}{q}} \\ &= \sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\sum_{n=1}^{\infty}\frac{\max\{m,n\} m^{\frac{1}{2}+\frac{\varepsilon}{q}}}{(m+n)^2 n^{\frac{1}{2}+\frac{\varepsilon}{q}}} = \sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\omega\left(\frac{1}{2}+\frac{\varepsilon}{q}, m\right).\end{aligned}$$

And then by (2.4) and the above results, it follows that

$$\begin{aligned} k\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} &> k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right)\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\left[1 - \frac{1}{k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right)} A_{\frac{1}{2}+\frac{\varepsilon}{q}}(m)\right] \\ &= k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right)\left[\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}} - \frac{1}{k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right)}\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}} A_{\frac{1}{2}+\frac{\varepsilon}{q}}(m)\right] \\ &= k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right)\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}\left[1 - \frac{\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}} A_{\frac{1}{2}+\frac{\varepsilon}{q}}(m)]}{k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right)\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}}\right]; \\ k &> k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right)\left[1 - \frac{\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}} O\left(\left(\frac{1}{m}\right)^{\frac{1}{2}-\frac{\varepsilon}{q}}\right)}{k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right)\sum_{m=1}^{\infty}\frac{1}{m^{1+\varepsilon}}}\right]. \end{aligned} \tag{3.5}$$

Setting $\alpha = \frac{1}{2}+\frac{\varepsilon}{q}$, by Fatou's Lemma, it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0^+} k\left(\frac{1}{2}+\frac{\varepsilon}{q}\right) &= \lim_{\alpha\to\frac{1}{2}^+}\frac{1}{\alpha}\int_0^{\infty}\frac{\max\{u^{1/\alpha},1\}}{(u^{1/\alpha}+1)^2} u^{\frac{1}{\alpha}-2}\,du \\ &\ge 2\int_0^{\infty}\lim_{\alpha\to\frac{1}{2}^+}\frac{\max\{u^{1/\alpha},1\}}{(u^{1/\alpha}+1)^2} u^{\frac{1}{\alpha}-2}\,du = k\left(\frac{1}{2}\right) = \frac{\pi}{2}+1.\end{aligned}$$

Then by (3.5), we have $k \ge \frac{\pi}{2}+1$ $(\varepsilon \to 0^+)$. Hence $k = \frac{\pi}{2}+1$ is the best value of (3.1). We confirm that the constant factor in (3.2) is the best, otherwise we would obtain a contradiction by (3.4) that the constant factor in (3.1) is not the best possible. The theorem is proved. □

In the same manner, by Lemma 2.3, we have:

Theorem 3.2. *If* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, a_n, b_n \geq 0, 0 < \sum_{n=1}^{\infty} n^{\frac{p}{2}-1}a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{\frac{q}{2}-1}b_n^q < \infty$, *then we have the following equivalent inequalities*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\min\{m,n\}}{(m+n)^2}a_m b_n < \left(\frac{\pi}{2}-1\right)\left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1}b_n^q\right)^{\frac{1}{q}}; \tag{3.6}$$

$$\sum_{n=1}^{\infty} n^{\frac{p}{2}-1}\left[\sum_{m=1}^{\infty}\frac{\min\{m,n\}}{(m+n)^2}a_m\right]^p < \left(\frac{\pi}{2}-1\right)^p \sum_{n=1}^{\infty} n^{\frac{q}{2}-1}a_n^p,$$

where the constant factors $\frac{\pi}{2}-1$ *and* $(\frac{\pi}{2}-1)^p$ *are the best possible.*

Remark 1. For $p = q = 2$, (3.1) reduces to (1.8) and (3.6) reduces to (1.9).

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