

# ON A DECOMPOSITION OF HILBERT'S INEQUALITY

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*Abstract:* By using the Euler-Maclaurin's summation formula and the weight coefficient, a pair of new inequalities is given, which is a decomposition of Hilbert's inequality. The equivalent forms and the extended inequalities with a pair of conjugate exponents are built.

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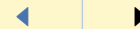
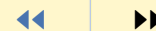
Hilbert's Inequality

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## 1. Introduction

In 1908, H. Weyl published the following Hilbert inequality: If  $\{a_n\}$ ,  $\{b_n\}$  are real sequences,  $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$  and  $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$ , then [1]

$$(1.1) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left( \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}},$$

where the constant factor  $\pi$  is the best possible. In 1925, G. H. Hardy gave an extension of (1.1) by introducing one pair of conjugate exponents  $(p, q)$  ( $\frac{1}{p} + \frac{1}{q} = 1$ ) as [2]: If  $p > 1$ ,  $a_n, b_n \geq 0$ ,  $0 < \sum_{n=1}^{\infty} a_n^p < \infty$  and  $0 < \sum_{n=1}^{\infty} b_n^q < \infty$ , then

$$(1.2) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left( \sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}},$$

where the constant factor  $\pi/\sin(\frac{\pi}{p})$  is the best possible. We refer to (1.2) as the Hardy-Hilbert inequality. In 1934, Hardy et al. [3] gave some applications of (1.1) and (1.2). By introducing a pair of non-conjugate exponents  $(p, q)$  in (1.1), Hardy et al. [3] gave: If  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} \geq 1$ ,  $0 < \lambda = 2 - (\frac{1}{p} + \frac{1}{q}) \leq 1$ , then

$$(1.3) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} \leq K(p, q) \left( \sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}},$$

where the constant factor  $K(p, q)$  is the best value only for  $\lambda = 1$ . In 1951, Bonsall [4] considered (1.3) in the case of a general kernel. In 1991, Mitrinović et al. [5] summarized the above method and results.



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In 1997-1998, by using weight coefficients, Yang and Gao [6], [7] gave a strengthened version of (1.2) as:

$$(1.4) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \left\{ \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/p}} \right] a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[ \frac{\pi}{\sin(\frac{\pi}{q})} - \frac{1-\gamma}{n^{1/q}} \right] b_n^q \right\}^{\frac{1}{q}},$$

where,  $1 - \gamma = 0.42278433^+$  ( $\gamma$  is the Euler constant). In 2001, Yang [8] gave an extension of (1.1) by introducing an independent parameter  $0 < \lambda \leq 4$  as

$$(1.5) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left( \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 \right)^{\frac{1}{2}},$$

where the constant factor  $B(\frac{\lambda}{2}, \frac{\lambda}{2})$  is the best possible ( $B(u, v)$  is the Beta function). In 2004, Yang [9] published the dual form of (1.2) as follows

$$(1.6) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left( \sum_{n=1}^{\infty} n^{p-2} a_n^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} n^{q-2} b_n^q \right)^{\frac{1}{q}}.$$

For  $p = q = 2$ , both (1.6) and (1.2) reduce to (1.1). It means that there are two different best extensions of (1.1). To generalize (1.2) and (1.6), in 2005, Yang [10] gave an extension of (1.2) and (1.6) with two pairs of conjugate exponents  $(p, q), (r, s) (p, r > 1)$  and parameters  $\alpha, \lambda > 0$  ( $\alpha\lambda \leq \min\{r, s\}$ ) as: If  $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\alpha\lambda}{r})-1} a_n^p < \infty$



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and  $0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\alpha\lambda}{s})-1} b_n^q < \infty$ , then

$$(1.7) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\alpha)^\lambda} < k_{\alpha\lambda}(r) \left\{ \sum_{n=1}^{\infty} n^{p(1-\frac{\alpha\lambda}{r})-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\frac{\alpha\lambda}{s})-1} b_n^q \right\}^{\frac{1}{q}},$$

where the constant factor  $k_{\alpha\lambda}(r) = \frac{1}{\alpha} B(\frac{\lambda}{r}, \frac{\lambda}{s})$  is the best possible. T. K. Pogány [11] also considered a best extension of (1.2) with the non-homogeneous kernel as  $\frac{1}{(\lambda_m + \rho_n)^\mu}$  ( $\mu, \lambda_m, \rho_n > 0$ ).

We have a non-negative decomposition of kernel in (1.1):

$$\frac{1}{m+n} = \frac{\max\{m, n\}}{(m+n)^2} + \frac{\min\{m, n\}}{(m+n)^2} \quad (m, n \in \mathbb{N})$$

( $\mathbb{N}$  is the set of positive integer numbers). In this paper, by using the Euler-Maclaurin summation formula and the weight coefficient as in [8], we give a pair of new Hilbert-type inequalities as

$$(1.8) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} a_m b_n < \left(\frac{\pi}{2} + 1\right) \left( \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}};$$

$$(1.9) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\min\{m, n\}}{(m+n)^2} a_m b_n < \left(\frac{\pi}{2} - 1\right) \left( \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}},$$

where the sum of two best constant factors is  $\pi$ . The equivalent forms and extended inequalities with a pair of conjugate exponents are considered.



## 2. Some Lemmas

**Lemma 2.1 (Euler-Maclaurin's summation formula, cf. [8, 12, Lemma 1]).** If  $f(x) \in C^1[1, \infty)$ , then we have

$$(2.1) \quad \sum_{k=1}^{\infty} f(k) = \int_1^{\infty} f(x)dx + \frac{1}{2}f(1) + \int_1^{\infty} P_1(x)f'(x)dx,$$

where  $P_1(x) = x - [x] - \frac{1}{2}$  is the Bernoulli function of the first order; if  $g \in C^3[1, \infty)$ ,  $(-1)^i g^{(i)}(x) > 0$ ,  $g^{(i)}(\infty) = 0$ , ( $i = 0, 1, 2, 3$ ), then

$$(2.2) \quad \begin{aligned} \frac{1}{12}[g(n) - g(1)] &< \int_1^n P_1(x)g(x)dx < 0, \\ -\frac{1}{12}g(n) &< \int_n^{\infty} P_1(x)g(x)dx < 0. \end{aligned}$$

**Lemma 2.2.** If  $\frac{1}{2} \leq \alpha < 1$ , setting the weight coefficient  $\omega(\alpha, m)$  as

$$(2.3) \quad \omega(\alpha, m) := \sum_{n=1}^{\infty} \frac{\max\{m, n\}m^{\alpha}}{(m+n)^{2n^{\alpha}}} \quad (m \in \mathbb{N}),$$

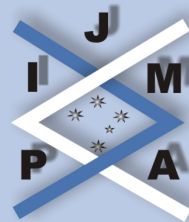
then we have

$$(2.4) \quad k(\alpha) = A_{\alpha}(m) + \omega(\alpha, m); \quad \omega\left(\frac{1}{2}, m\right) < k\left(\frac{1}{2}\right) = \frac{\pi}{2} + 1,$$

where

$$k(\alpha) := \frac{1}{\alpha} \int_0^{\infty} \frac{\max\{u^{1/\alpha}, 1\}}{(u^{1/\alpha} + 1)^2} u^{\frac{1}{\alpha}-2} du$$

and  $A_{\alpha}(m) = O(m^{\alpha-1})$ , ( $m \rightarrow \infty$ ).

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*Proof.* For fixed  $\frac{1}{2} \leq \alpha < 1$ ,  $m \in \mathbb{N}$ , setting  $f(x) := \frac{\max\{m,x\}}{(m+x)^2 x^\alpha}$ ,  $x \in (0, \infty)$ , then by (2.1), it follows that

$$\begin{aligned} (2.5) \quad \omega(\alpha, m) &= m^\alpha \sum_{n=1}^{\infty} f(n) \\ &= m^\alpha \left[ \int_1^{\infty} f(x) dx + \frac{1}{2} f(1) + \int_1^{\infty} P_1(x) f'(x) dx \right] \\ &= m^\alpha \int_0^{\infty} f(x) dx - m^\alpha \rho(\alpha, m), P_1(x) f'(x) dx. \end{aligned}$$

$$(2.6) \quad \rho(\alpha, m) := \int_0^1 f(x) dx - \frac{1}{2} f(1) - \int_1^{\infty} P_1(x) f'(x) dx.$$

We find

$$-\frac{1}{2} f(1) = \frac{-m}{2(m+1)^2} = \frac{-1}{2(m+1)} + \frac{1}{2(m+1)^2},$$

and

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \frac{m}{(m+x)^2 x^\alpha} dx \geq \int_0^1 \frac{m}{(m+x)^2} dx = \frac{1}{m+1}; \\ \int_0^1 f(x) dx &\leq \int_0^1 \frac{m}{m^2 x^\alpha} dx = \frac{1}{(1-\alpha)m}. \end{aligned}$$

For  $x \in (0, m)$ ,  $f(x) = \frac{m}{(m+x)^2 x^\alpha}$ , it follows  $f'(x) = \frac{-2m}{(m+x)^3 x^\alpha} - \frac{\alpha m}{(m+x)^2 x^{\alpha+1}}$ ; for

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$x \in (m, \infty)$ ,  $f(x) = \frac{x^{1-\alpha}}{(m+x)^2}$ , we find

$$\begin{aligned} f'(x) &= \frac{-2x^{1-\alpha}}{(m+x)^3} + \frac{1-\alpha}{(m+x)^2 x^\alpha} \\ &= \frac{-2(x+m-m)}{(m+x)^3 x^\alpha} + \frac{1-\alpha}{(m+x)^2 x^\alpha} \\ &= \frac{-2}{(m+x)^2 x^\alpha} + \frac{2m}{(m+x)^3 x^\alpha} + \frac{1-\alpha}{(m+x)^2 x^\alpha}. \end{aligned}$$

In the following, it is obvious that  $g_1(x) = \frac{1}{(m+x)^3 x^\alpha}$ ,  $g_2(x) = \frac{1}{(m+x)^2 x^{\alpha+1}}$  and  $g_3(x) = \frac{1}{(m+x)^2 x^\alpha}$  are suited to apply in (2.2). Then by (2.2), we obtain

$$\begin{aligned} (2.7) \quad & - \int_1^m P_1(x) f'(x) dx \\ &= \int_1^m \frac{2mP_1(x) dx}{(m+x)^3 x^\alpha} + \int_1^m \frac{\alpha m P_1(x) dx}{(m+x)^2 x^{\alpha+1}} \\ &> \frac{2m}{12} \left[ \frac{1}{8m^{3+\alpha}} - \frac{1}{(m+1)^3} \right] + \frac{\alpha m}{12} \left[ \frac{1}{4m^{3+\alpha}} - \frac{1}{(m+1)^2} \right] \\ &= \frac{\alpha+1}{48m^{2+\alpha}} - \frac{\alpha}{12(m+1)} - \frac{2-\alpha}{12(m+1)^2} + \frac{1}{6(m+1)^3}; \end{aligned}$$

$$\begin{aligned} (2.8) \quad & - \int_m^\infty P_1(x) f'(x) dx \\ &= \int_m^\infty \frac{2P_1(x) dx}{(m+x)^2 x^\alpha} - \int_m^\infty \frac{2mP_1(x) dx}{(m+x)^3 x^\alpha} - (1-\alpha) \int_m^\infty \frac{P_1(x) dx}{(m+x)^2 x^\alpha} \end{aligned}$$



$$\begin{aligned}
&> \frac{-1}{24m^{2+\alpha}} - \int_1^\infty P_1(x)f'(x)dx \\
&= - \int_1^m P_1(x)f'(x)dx - \int_m^\infty P_1(x)f'(x)dx \\
&> \frac{\alpha - 1}{48m^{2+\alpha}} - \frac{\alpha}{12(m + 1)} - \frac{2 - \alpha}{12(m + 1)^2} + \frac{1}{6(m + 1)^3}.
\end{aligned}$$

Hence by (2.6), for  $\alpha = \frac{1}{2}$ , it follows that

$$\begin{aligned}
(2.9) \quad \rho\left(\frac{1}{2}, m\right) &> \frac{-1}{2(m + 1)} + \frac{1}{2(m + 1)^2} + \frac{1}{m + 1} \\
&\quad + \frac{\frac{1}{2} - 1}{48m^{2+1/2}} - \frac{\frac{1}{2}}{12(m + 1)} - \frac{2 - \frac{1}{2}}{12(m + 1)^2} + \frac{1}{6(m + 1)^3} \\
&= \frac{11}{24(m + 1)} + \frac{9}{24(m + 1)^2} + \frac{-1}{96m^{2+1/2}} + \frac{1}{6(m + 1)^3} \\
&\geq \frac{11}{24(m + 1)} + \left[\frac{9}{96m^2} + \frac{-1}{96m^2}\right] + \frac{1}{6(m + 1)^3} > 0.
\end{aligned}$$

By (2.7) and (2.8), we obtain

$$\begin{aligned}
- \int_1^\infty P_1(x)f'(x)dx &= - \int_1^m P_1(x)f'(x)dx - \int_m^\infty P_1(x)f'(x)dx \\
&< \frac{1}{48m^{2+\alpha}} + \frac{1 - \alpha}{48m^{2+\alpha}} \\
&= \frac{2 - \alpha}{48m^{2+\alpha}}.
\end{aligned}$$



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Then by (2.6), it follows

$$(2.10) \quad \begin{aligned} 0 &< m^{1-\alpha}[m^\alpha \rho(\alpha, m)] \\ &< \frac{-m}{2(m+1)} + \frac{m}{2(m+1)^2} + \frac{1}{1-\alpha} + \frac{2-\alpha}{48m^{1+\alpha}} \\ &\rightarrow \frac{1}{1-\alpha} - \frac{1}{2} \quad (m \rightarrow \infty). \end{aligned}$$

Setting  $u = (x/m)^\alpha$ , we find

$$(2.11) \quad \begin{aligned} m^\alpha \int_0^\infty f(x) dx &= m^\alpha \int_0^\infty \frac{\max\{m, x\}}{(m+x)^2 x^\alpha} dx \\ &= \frac{1}{\alpha} \int_0^\infty \frac{\max\{u^{1/\alpha}, 1\}}{(u^{1/\alpha} + 1)^2} u^{\frac{1}{\alpha}-2} du = k(\alpha), \end{aligned}$$

$$(2.12) \quad \begin{aligned} k\left(\frac{1}{2}\right) &= 2 \int_0^\infty \frac{\max\{u^2, 1\}}{(u^2 + 1)^2} du = 4 \int_0^1 \frac{du}{(u^2 + 1)^2} \\ &= 4 \int_0^{\pi/4} \cos^2 \theta d\theta = \frac{\pi}{2} + 1. \end{aligned}$$

Hence by (2.5), (2.9), (2.10) and (2.11), (2.4) is valid and the lemma is proved.  $\square$

Similar to Lemma 2.2, we still have

**Lemma 2.3.** If  $\frac{1}{2} \leq \alpha < 1$ , setting the weight coefficient  $\varpi(\alpha, m)$  as

$$(2.13) \quad \varpi(\alpha, m) := \sum_{n=1}^{\infty} \frac{\min\{m, n\} m^\alpha}{(m+n)^{2n\alpha}} \quad (m \in \mathbb{N}),$$

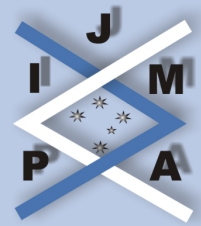
then we have

$$(2.14) \quad \tilde{k}(\alpha) = B_\alpha(m) + \varpi(\alpha, m); \quad \varpi\left(\frac{1}{2}, m\right) < \tilde{k}\left(\frac{1}{2}\right) = \frac{\pi}{2} - 1,$$

where

$$\tilde{k}(\alpha) = \frac{1}{\alpha} \int_0^\infty \frac{\min\{u^{1/\alpha}, 1\}}{(u^{1/\alpha} + 1)^2} u^{\frac{1}{\alpha}-2} du$$

and  $B_\alpha(m) = O(m^{\alpha-2})$ ,  $(m \rightarrow \infty)$ .



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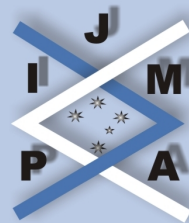
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### 3. Main Results and their Equivalent Forms

**Theorem 3.1.** *If  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $a_n, b_n \geq 0$ ,  $0 < \sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p < \infty$  and  $0 < \sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q < \infty$ , then we have the following equivalent inequalities*

$$(3.1) \quad I := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} a_m b_n < \left(\frac{\pi}{2} + 1\right) \left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q\right)^{\frac{1}{q}};$$

$$(3.2) \quad J := \sum_{n=1}^{\infty} n^{\frac{p}{2}-1} \left[\sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} a_m\right]^p < \left(\frac{\pi}{2} + 1\right)^p \sum_{n=1}^{\infty} n^{\frac{q}{2}-1} a_n^p,$$

where the constant factors  $\frac{\pi}{2} + 1$  and  $\left(\frac{\pi}{2} + 1\right)^p$  are the best possible.

*Proof.* By Hölder's inequality and (2.3) – (2.4), we find

$$(3.3) \quad \left[\sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} a_m\right]^p = \left\{ \sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} \left[\frac{m^{1/(2q)}}{n^{1/(2p)}} a_m\right] \left[\frac{n^{1/(2p)}}{m^{1/(2q)}}\right] \right\}^p \leq \left[\sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} \frac{m^{p/(2q)}}{n^{1/2}} a_m^p\right] \left[\sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} \frac{n^{q/(2p)}}{m^{1/2}}\right]^{p-1}$$



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$$\begin{aligned}
 &= \omega^{p-1} \left( \frac{1}{2}, n \right) n^{1-\frac{p}{2}} \sum_{m=1}^{\infty} \frac{\max\{m, n\} m^{p/(2q)}}{(m+n)^2} \frac{m^{p/(2q)}}{n^{1/2}} a_m^p \\
 &\leq \left( \frac{\pi}{2} + 1 \right)^{p-1} n^{1-\frac{p}{2}} \sum_{m=1}^{\infty} \frac{\max\{m, n\} m^{p/(2q)}}{(m+n)^2} \frac{m^{p/(2q)}}{n^{1/2}} a_m^p;
 \end{aligned}$$

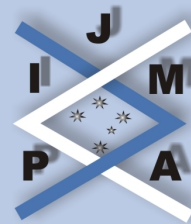
$$\begin{aligned}
 J &\leq \left( \frac{\pi}{2} + 1 \right)^{p-1} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\max\{m, n\} m^{p/(2q)}}{(m+n)^2} \frac{m^{p/(2q)}}{n^{1/2}} a_m^p \\
 &= \left( \frac{\pi}{2} + 1 \right)^{p-1} \sum_{m=1}^{\infty} \left[ \sum_{n=1}^{\infty} \frac{\max\{m, n\} m^{p/(2q)}}{(m+n)^2} \frac{m^{p/(2q)}}{n^{1/2}} \right] a_m^p \\
 &= \left( \frac{\pi}{2} + 1 \right)^{p-1} \sum_{m=1}^{\infty} \omega \left( \frac{1}{2}, m \right) m^{\frac{p}{2}-1} a_m^p < \left( \frac{\pi}{2} + 1 \right)^p \sum_{m=1}^{\infty} m^{\frac{p}{2}-1} a_m^p.
 \end{aligned}$$

Therefore (3.2) is valid. By Hölder's inequality, we find that

$$\begin{aligned}
 (3.4) \quad I &= \sum_{n=1}^{\infty} \left[ n^{\frac{1}{2}-\frac{1}{p}} \sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} a_m \right] \left[ n^{-\frac{1}{2}+\frac{1}{p}} b_n \right] \\
 &\leq J^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q \right)^{\frac{1}{q}}.
 \end{aligned}$$

Then by (3.2), we have (3.1). On the other hand, suppose that (3.1) is valid. Setting

$$b_n := n^{\frac{p}{2}-1} \left[ \sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} a_m \right]^{p-1}, \quad n \in \mathbb{N},$$



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then it follows  $\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q = J$ . By (3.3), we confirm that  $J < \infty$ . If  $J = 0$ , then (3.2) is naturally valid; if  $0 < J < \infty$ , then by (3.1), we find

$$\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q = J = I < \left(\frac{\pi}{2} + 1\right) \left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q\right)^{\frac{1}{q}};$$

$$\left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q\right)^{\frac{1}{p}} = J^{\frac{1}{p}} < \left(\frac{\pi}{2} + 1\right) \left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p\right)^{\frac{1}{p}},$$

and inequality (3.2) is valid, which is equivalent to (3.1).

For  $0 < \varepsilon < \frac{q}{2}$ , setting  $\tilde{a} = \{\tilde{a}_n\}_{n=1}^{\infty}$ ,  $\tilde{b} = \{\tilde{b}_n\}_{n=1}^{\infty}$  as  $\tilde{a}_n^{\frac{-1}{2}-\frac{\varepsilon}{p}}$ ,  $\tilde{b}_n^{\frac{-1}{2}-\frac{\varepsilon}{q}}$ , for  $n \in \mathbb{N}$ , if there exists a constant  $0 < k \leq \frac{\pi}{2} + 1$ , such that (3.1) is still valid when we replace  $\frac{\pi}{2} + 1$  by  $k$ , then we find

$$\tilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\max\{m, n\} \tilde{a}_m \tilde{b}_n}{(m+n)^2}$$

$$< k \left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} \tilde{a}_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} \tilde{b}_n^q\right)^{\frac{1}{q}} = k \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}};$$

$$\tilde{I} = \sum_{n=1}^{\infty} \left[ \sum_{m=1}^{\infty} \frac{\max\{m, n\}}{(m+n)^2} m^{\frac{-1}{2}-\frac{\varepsilon}{p}} \right] n^{\frac{-1}{2}-\frac{\varepsilon}{q}}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \sum_{n=1}^{\infty} \frac{\max\{m, n\} m^{\frac{1}{2}+\frac{\varepsilon}{q}}}{(m+n)^2 n^{\frac{1}{2}+\frac{\varepsilon}{q}}} = \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \omega\left(\frac{1}{2} + \frac{\varepsilon}{q}, m\right).$$



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And then by (2.4) and the above results, it follows that

$$\begin{aligned}
 (3.5) \quad k \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} &> k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right) \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \left[ 1 - \frac{1}{k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right)} A_{\frac{1}{2} + \frac{\varepsilon}{q}}(m) \right] \\
 &= k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right) \left[ \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} - \frac{1}{k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right)} \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} A_{\frac{1}{2} + \frac{\varepsilon}{q}}(m) \right] \\
 &= k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right) \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} \left[ 1 - \frac{\sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} A_{\frac{1}{2} + \frac{\varepsilon}{q}}(m)}{k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right) \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}}} \right]; \\
 k &> k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right) \left[ 1 - \frac{\sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}} O \left( \left( \frac{1}{m} \right)^{\frac{1}{2} - \frac{\varepsilon}{q}} \right)}{k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right) \sum_{m=1}^{\infty} \frac{1}{m^{1+\varepsilon}}} \right].
 \end{aligned}$$

Setting  $\alpha = \frac{1}{2} + \frac{\varepsilon}{q}$ , by Fatou's Lemma, it follows that

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0^+} k \left( \frac{1}{2} + \frac{\varepsilon}{q} \right) &= \lim_{\alpha \rightarrow \frac{1}{2}^+} \frac{1}{\alpha} \int_0^{\infty} \frac{\max\{u^{1/\alpha}, 1\}}{(u^{1/\alpha} + 1)^2} u^{\frac{1}{\alpha} - 2} du \\
 &\geq 2 \int_0^{\infty} \lim_{\alpha \rightarrow \frac{1}{2}^+} \frac{\max\{u^{1/\alpha}, 1\}}{(u^{1/\alpha} + 1)^2} u^{\frac{1}{\alpha} - 2} du = k \left( \frac{1}{2} \right) = \frac{\pi}{2} + 1.
 \end{aligned}$$

Then by (3.5), we have  $k \geq \frac{\pi}{2} + 1$  ( $\varepsilon \rightarrow 0^+$ ). Hence  $k = \frac{\pi}{2} + 1$  is the best value of (3.1). We confirm that the constant factor in (3.2) is the best, otherwise we would obtain a contradiction by (3.4) that the constant factor in (3.1) is not the best possible. The theorem is proved.  $\square$



In the same manner, by Lemma 2.3, we have:

**Theorem 3.2.** *If  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $a_n, b_n \geq 0$ ,  $0 < \sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p < \infty$  and  $0 < \sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q < \infty$ , then we have the following equivalent inequalities*

$$(3.6) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\min\{m, n\}}{(m+n)^2} a_m b_n < \left(\frac{\pi}{2} - 1\right) \left(\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{\frac{q}{2}-1} b_n^q\right)^{\frac{1}{q}};$$
$$\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} \left[\sum_{m=1}^{\infty} \frac{\min\{m, n\}}{(m+n)^2} a_m\right]^p < \left(\frac{\pi}{2} - 1\right)^p \sum_{n=1}^{\infty} n^{\frac{q}{2}-1} a_n^p,$$

where the constant factors  $\frac{\pi}{2} - 1$  and  $(\frac{\pi}{2} - 1)^p$  are the best possible.

*Remark 1.* For  $p = q = 2$ , (3.1) reduces to (1.8) and (3.6) reduces to (1.9).



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