



ON AN OPIAL INEQUALITY WITH A BOUNDARY CONDITION

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ABSTRACT. R.C. Brown conjectured (in 2001) that the Opial-type inequality

$$4 \int_0^1 |yy'| dx \leq \int_0^1 (y')^2 dx,$$

holds for all absolutely continuous functions $y : [0, 1] \rightarrow \mathbb{R}$ such that $y' \in L^2$ and $\int_0^1 y dx = 0$. This was subsequently proved by Denzler [3]. An alternative proof was given by Brown and Plum [2]. Here we give a shorter proof.

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1. INTRODUCTION

The classical Opial inequality asserts that

$$(1.1) \quad \int_a^b |y(x)y'(x)| dx \leq \frac{b-a}{4} \int_a^b y'^2 dx$$

for functions that satisfy $y(a) = y(b) = 0$, and $y' \in L^2$. Equality holds if and only if $y(x)$ is a constant multiple of the piecewise linear function $\hat{y}(x) = x - a$ in $[a, m]$ and $\hat{y}(x) = b - x$ in $[m, b]$, where m is the mid-point of the interval.

Beesack [1] observed that this result follows immediately from the half-interval inequality

$$(1.2) \quad \int_a^b |y(x)y'(x)| dx \leq \frac{b-a}{2} \int_a^b y'^2 dx,$$

with the boundary condition $y(a) = 0$ or $y(b) = 0$. Very short proofs of these results were discovered later. Notably, the proofs due to C. L. Mallow and R. N. Pedeson are each less than half a page long. See [3] and [2] for more references.

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Brown's conjecture is inspired by these results. The two known proofs ([3] and [2]) are comparable in length. A shorter proof (about half as long) is given in this paper. It is still lengthy and technical. Hence, it will be of interest if an even shorter proof can be found.

If y' were of a constant sign, the conjecture would just be a routine exercise in the calculus of variations. The extremal would be a quadratic function. In reality, we can only assert that in each subinterval in which y' remains of one sign, the extremal function is quadratic. In other words, the extremal is a piecewise quadratic spline, but it is difficult to predict how many cusps there are or where they appear.

Our approach is close in spirit to that of Denzler's, and it helps to first describe his proof from a high level. He starts out with an arbitrary function and goes through a sequence of steps, in each of which the previous function is replaced by another, using techniques of normalization, rearrangement, or some kind of "surgery." The new function "better" satisfies the inequality (in the sense that if one can establish the required inequality for the new function, then the same inequality must hold for the original function), and possesses some additional properties. After all the steps are carried out, he amasses enough properties for the final function to be able to prove that (1.1) must hold and the proof is complete.

Using the method of contradiction helps to streamline our arguments. Suppose that (1.1) is false for some function. We modify this function, in a number of steps. In each step, the function is replaced by another which also violates the inequality, but the new function has some additional properties. In the final step, we show that the newest function cannot satisfy the given integral constraint and we have a contradiction. Some of the more technical computations in the proof are done using the symbolic manipulation software Maple. The proof of the main result presented in Section 2 is self-contained, but keep in mind that some of the ideas can be traced back to [3] and [2].

Extremals are treated in Section 3.

2. PROOF OF THE MAIN RESULT

We use the notations

$$(2.1) \quad E(y, [a, b]) = \int_a^b (y'(x))^2 dx,$$

$$(2.2) \quad W(y, [a, b]) = \int_a^b |y(x)y'(x)| dx,$$

and

$$(2.3) \quad K(y, [a, b]) = E(y, [a, b]) - 4W(y, [a, b]).$$

When $[a, b]$ is $[0, 1]$, we simply write $E(y)$, $W(y)$, and $K(y)$. Applying Beesack's inequality to a function on an interval of length ≤ 0.5 , we have $K(y) \geq 0$. Likewise, Opial's inequality for an interval of length ≤ 1 gives $K(y) \geq 0$.

The Brown-Denzler-Plum result can be restated as $K(y) \geq 0$ for all y that satisfies

$$(2.4) \quad \int_0^1 y(x) dx = 0.$$

Suppose that the result is false and $K(\bar{y}) < 0$ for some \bar{y} . By a density argument, we can assume without loss of generality that \bar{y} has a finite number of local maxima and minima.

Lemma 2.1. *Let r (s) be the smallest (largest) zero of $\bar{y}(x)$. Then either $r > 0.5$ or $s < 0.5$.*

Proof. Suppose $r \leq 0.5 \leq s$. Beesack's inequality for \bar{y} over $[0, r]$ and $[s, 1]$ implies $K(\bar{y}, [0, r]) \geq 0$ and $K(\bar{y}, [s, 1]) \geq 0$. Likewise, Opial's inequality for \bar{y} over $[r, s]$ implies $K(\bar{y}, [r, s]) \geq 0$. These inequalities contradict the assumption that $K(\bar{y}) < 0$. \square

By reflecting \bar{y} with respect to $x = 1/2$ and/or using $-\bar{y}$ instead, if necessary, we can assume without loss of generality that \bar{y} satisfies

$$(P1) \quad \bar{y}(s) = 0, \bar{y}(x) > 0 \text{ in } (s, 1], \text{ with } s < 0.5.$$

Note that for any y that satisfies (P1), $K(y, [0, s]) \geq 0$. If we scale down y on $[0, s]$ (i.e. replace it by λy on $[0, s]$ with $\lambda < 1$), we reduce its contribution to the entire $K(y)$. Likewise, if we scale up y on $[s, 1]$, we also decrease $K(y)$. This idea is used in the proof of Lemma 2.2, and later in Lemmas 2.4 and 2.6.

Lemma 2.2. *Suppose we have a function y_1 that satisfies $\int_0^1 y_1 dx < 0$ (instead of (2.4)), (P1) and $K(y_1) < 0$. Then there exists a y_2 satisfying (2.4) such that $K(y_2) < K(y_1) < 0$.*

Proof. By hypotheses

$$(2.5) \quad - \left(\int_s^1 y_1 dx \Big/ \int_0^s y_1 dx \right) = \lambda < 1.$$

Let us take $y_2 = \lambda y_1$ on $[0, s]$ and $y_2 = y_1$ on $[s, 1]$. Then $0 < K(y_2, [0, s]) = \lambda^2 K(y_1, [0, s]) < K(y_1, [0, s])$. It follows that $K(y_2) < K(y_1)$. \square

Lemma 2.3. *We may assume that \bar{y} satisfies*

$$(P2) \quad \bar{y}(x) < 0 \text{ in } [0, s) \text{ and is increasing.}$$

Proof. If (P2) is not satisfied, replace \bar{y} in $[0, s]$ by $y_1(x)$ such that $y_1'(x) = |\bar{y}'(x)|$, $y_1(s) = 0$. (The same idea was used by Mallow in [4].) In $[s, 1]$, $y_1 = \bar{y}$. Then y_1 satisfies both (P1) and (P2) and $-y_1(x) \geq |\bar{y}(x)|$ in $[0, s)$. It follows that $\int_0^1 y_1 dx < 0$. It is easy to verify that $E(y_1, [0, s]) = E(\bar{y}, [0, s])$ and $W(y_1, [0, s]) > W(\bar{y}, [0, s])$. As a consequence $K(y_1) < 0$. We can now apply Lemma 2.2 to complete the proof, taking y_2 to be the new \bar{y} . \square

Lemma 2.4.

$$(P3) \quad -\bar{y}(0) < \bar{y}(1).$$

Proof. Suppose (P3) is false. Then $\lambda = |\bar{y}(1)/\bar{y}(0)| \leq 1$. By scaling down \bar{y} in $[0, s)$ to $\lambda\bar{y}$, we get a new function y_2 such that $K(y_2) \leq K(\bar{y})$, and $y_2(0) = -y_2(1)$. By moving $|y_2|$ on $[0, s)$ to the right of y_2 on $[s, 1]$, we get a function that satisfies the classical Opial boundary conditions. It follows that $K(y_2) \geq 0$, contradicting the assumption that $K(\bar{y}) < 0$. \square

Using a suitable scaling, we can assume that $\int_0^s \bar{y} dx = -1$. It follows from (P2) that $W(\bar{y}, [0, s]) = \bar{y}^2(0)/2$. Our next step alters \bar{y} in $[0, s]$ to minimize $K(\bar{y}, [0, s])$, while preserving $\int_0^s \bar{y} dx$ (this guarantees that (2.4) is always satisfied). This is a classical variational problem, namely, to minimize $\int_0^s y'^2 dx - 2y^2(0)$ over the class \mathcal{K} of nonpositive, absolutely continuous, and monotonically increasing functions $y : [0, s) \rightarrow (-\infty, 0]$, such that $y(s) = 0$, and $\int_0^s y dx = -1$. The Euler equation for this problem has a very simple form, namely, $y'' = \text{constant}$. The optimizer must also satisfy the boundary condition $y'(0) = -2y(0)$. The solution is a quadratic function. These facts have been used in both [3] and [2]. Straightforward computation yields the following result. We include a direct proof.

Lemma 2.5. *We may assume that, in $[0, s]$,*

$$(P4) \quad \bar{y}(x) = \frac{3(s-x)(2sx-s-x)}{s^3(2-s)}.$$

Proof. The expression appears complicated, but all we need to know about \bar{y} for the proof are the following facts (indeed, these facts are sufficient to recover the expression for \bar{y}): it is a quadratic function in x , $\bar{y}(s) = 0$, $\int_0^s \bar{y} dx = -1$, and $\bar{y}'(0) = -2\bar{y}(0)$.

Let $y(x) \in \mathcal{K}$ be another function. Denote $\phi = y - \bar{y}$. Integrating over $[0, s]$, we have

$$(2.6) \quad \begin{aligned} K(y, [0, s]) &= K(\bar{y}, [0, s]) + K(\phi, [0, s]) + \int_0^s 2\phi'(x)\bar{y}'(x) dx - 4\phi(0)\bar{y}(0) \\ &= K(\bar{y}, [0, s]) + K(\phi, [0, s]) \\ &\geq K(\bar{y}, [0, s]). \end{aligned}$$

The last two terms on the right-hand side of (2.6) cancel out after we apply integration by parts and use the facts that \bar{y}'' is a constant, $\int_0^s \phi dx = 0$ and $\bar{y}'(0) = -2\bar{y}(0)$. The last inequality follows from $K(\phi, [0, s]) \geq 0$ (Beesack's inequality for intervals of length < 0.5). \square

Unfortunately, we cannot minimize $K(\bar{y}, [s, 1])$ in a similar way, because we do not have the a priori knowledge that \bar{y} is monotonic in $[s, 1]$ and the variational technique fails.

By assumption, we can divide $[s, 1]$ into a finite number of subintervals with points

$$(2.7) \quad s = s_0 < s_1 < s_2 < \cdots < s_n = 1$$

such that each even(odd)-indexed s_i is a local minimum (maximum), so that in each subinterval $[s_{i-1}, s_i]$, \bar{y} is monotonic. For convenience, we call the points s_i ($i = 1, \dots, n-1$) cusps of the function \bar{y} . By (P4), $\bar{y}(0) = -3/s(2-s)$. From the set of local minima, we select the subset of those points at which \bar{y} is less than $|\bar{y}(0)|$:

$$(2.8) \quad \mathcal{M} = \{s_i : i = 1, 3, \dots, \text{ at which } \bar{y}(s_i) < |\bar{y}(0)|\}.$$

It has only a finite number of points. We will construct a procedure that replaces \bar{y} by a new function with the property that the number of points in the corresponding set \mathcal{M} is reduced by at least one. Thus, after a finite number of steps, the set \mathcal{M} for the newest function is empty. This leads to the next lemma.

Lemma 2.6. *We may assume that*

$$(P5) \quad \bar{y}(s_i) \geq |\bar{y}(0)| \text{ at each local minimum } s_i.$$

This implies that $\bar{y}(x) \geq |\bar{y}(0)|$ for all $x > s_1$.

Proof. The following technique of arranging the function \bar{y} to decrease $K(\bar{y})$ is borrowed from Denzler [3]. Suppose ρ is the first local minimum in \mathcal{M} . At this point; $\bar{y}(\rho) < |\bar{y}(0)|$. Take the largest neighborhood $(\alpha, \beta) \ni \rho$ such that $\bar{y}(x) \leq \bar{y}(\alpha) = \bar{y}(\beta)$ for all $x \in (\alpha, \beta)$, and $\bar{y}(\alpha) \leq |\bar{y}(0)|$. Following Denzler, we remove the graph of \bar{y} over the interval (α, β) (pushing the graph over $[0, \alpha]$ to the right to close up the gap) and splice its negative copy (i.e. reflect it with respect to the x -axis) into the graph of y to the left of α , at the point where $\bar{y}(x) = -\bar{y}(\alpha)$. In this way, we get a new function z that has a zero $\sigma > s$, $K(z) = K(\bar{y})$, $\int_0^\sigma z dx < -1$ and $\int_\sigma^1 z dx < 1$.

We scale down z over $[0, \sigma]$ (we use the same notation z to denote new functions) to make $\int_0^\sigma z dx = -1$, and this decreases $K(z)$ (same argument as in the proof of Lemma 2.2). Over $[\sigma, 1]$, we scale up z to make $\int_\sigma^1 z dx = 1$, so that the new z again satisfies (2.4). This also decreases $K(z)$. Finally, we use Lemma 2.5 to change z over $[0, \sigma]$ to further decrease $K(z)$. From (P4), $z(0) = -3/\sigma(2-\sigma)$. Since $\sigma > s$, $|z(0)| < |\bar{y}(0)|$. It follows easily from this and the fact that z is scaled up from (a portion of) \bar{y} in $[\sigma, 1]$ that the set \mathcal{M} corresponding to z has strictly (at least ρ is not in the new set) fewer points than the original \mathcal{M} corresponding to \bar{y} . As usual, we rename the new function z to be \bar{y} for the next step. \square

Now we assume that (P5) holds. By our construction, s_1 is either the first cusp of \bar{y} or 1 (if \bar{y} has no cusp).

Lemma 2.7. *We may assume that*

(P6) *the expression for $\bar{y}(x)$ in (P4) can be extended to $x \in [0, s_1]$.*

Proof. Assume the contrary. Since \bar{y} is increasing in $[0, s_1]$, we can use the variational technique to replace \bar{y} in $[0, s_1]$ by a quadratic function z to minimize K , while preserving the integral of z over $[0, s_1]$ (this guarantees that the integral condition (2.4) is preserved). Now z may have a zero different from s , and $\int_0^s z dx$ may no longer be -1 . It is easy to see that after renaming the new zero to s and applying a proper scaling, the new function satisfies (P6). \square

If \bar{y} satisfies (P6), and hence (P4), then $\int_0^s \bar{y} dx = -1$. We claim that (P5) and (P6) imply that $\int_s^1 \bar{y} dx > 1$, as long as $s \in (0, 1/2)$. This contradicts (2.4) and is the final step we need to complete the proof of the main result.

Let $\tau > s$ be the point at which $\bar{y}(\tau) = |\bar{y}(0)| = 3/s(2 - s)$. (It follows from Lemma 2.7 that $\tau < s_1$.) Substituting this into the expression in (P4) and solving for τ , we get

$$(2.9) \quad \tau = \frac{s(\sqrt{s^2 - 4s + 2} - s)}{1 - 2s}.$$

There are actually two roots, but the other root is discarded because it is negative.

(P5) implies that $\bar{y}(x) > |\bar{y}(0)|$ for $x > \tau$. Hence,

$$(2.10) \quad \int_s^1 \bar{y}(x) dx \geq \int_s^\tau \bar{y}(x) dx + (1 - \tau)|\bar{y}(0)|.$$

With the help of the symbolic manipulation software Maple, the above inequality becomes

$$(2.11) \quad \int_s^1 \bar{y}(x) dx \geq \frac{3 - 10s + 9s^2 - 2s^4 - (4s - 8s^2 + 2s^3)R}{s(2 - s)(1 - 2s)^2},$$

where $R = \sqrt{s^2 - 4s + 2}$. We have our desired contradiction if we can show that the difference between the numerator and the denominator of the above fraction is nonnegative for $s \in [0, 1/2]$. After simplification, this difference is the function $f(s)$ in our final lemma.

Lemma 2.8. *For $s \in [0, 1/2]$,*

$$(2.12) \quad f(s) = (3 - 12s + 18s^2 - 12s^3 + 2s^4) - (4s - 8s^2 + 2s^3)R \geq 0.$$

Proof. Since $f(0) = 3$ and $f(1/2) = 0$, if we can show that f has no other zeros than $s = 1/2$ in $[0, 1/2]$, then (2.12) holds. To solve for $f(s) = 0$, we move all the terms not involving R to one side of the equation, square both sides, and then simplify. We end up with a polynomial equation that can be factored as

$$(2.13) \quad (4s^2 - 18s + 9)(1 - 2s)^3 = 0.$$

The only real solution to this equation in $[0, 1/2]$ is $s = 1/2$. \square

3. EXTREMALS

Let \hat{y} be an extremal. Then $K(\hat{y}) = 0$. We want to show that \hat{y} is linear and $\hat{y}(0.5) = 0$.

With slight modifications, all the arguments in Section 2 can be applied to \hat{y} . First, Lemma 2.1 and (P1) hold for \hat{y} with the modification that $s \leq 0.5$, instead of the strict inequality $s < 0.5$. Lemmas 2.3, 2.4, and 2.5, together with (P2), (P3) and (P4), are true with the understanding that s may be equal to 0.5, and $<$ in Lemma 2.4 is replaced by \leq . Lemma 2.8 and (P6) need no changes. The contradiction derived in Lemma 2.8 is now interpreted to mean that s cannot be < 0.5 . Hence we conclude that $s = 0.5$. Using (P4) and (P6), we see that \hat{y} must be linear.

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