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GENERALIZATION OF AN IMPULSIVE NONLINEAR SINGULAR GRONWALL-BIHARI INEQUALITY WITH DELAY

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ABSTRACT. This paper generalizes a Tatar's result of an impulsive nonlinear singular Gronwall-Bihari inequality with delay [J. Inequal. Appl., 2006(2006), 1-12] to a new type of inequalities which includes n distinct nonlinear terms.

Key words and phrases: Gronwall-Bihari inequality, Nonlinear, Impulsive.

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1. INTRODUCTION

In order to investigate problems of the form

$$x' = f(t,x), \quad t \neq t_k,$$
$$\Delta x = I_k(x), \quad t = t_k,$$

Samoilenko and Perestyuk [6] first used the following impulsive integral inequality

$$u(t) \leq a + \int_c^t b(s)u(s)ds + \sum_{0<t_k<t} \eta_k u(t_k), \quad t \geq 0.$$

Then Bainov and Hristova [2] studied a similar inequality with constant delay. In 2004, Hristova [3] considered a more general inequality with nonlinear functions in u. All of these papers treated the functions (kernels) involved in the integrals which are regular. Recently, Tatar [7] investigated the following singular inequality

$$u(t) \leq a(t) + b(t)\int_0^t k_1(t,s)u^m(s)ds + c(t)\int_0^t k_2(t,s)u^n(s-\tau)ds$$
$$+d(t)\sum_{0<t_k<t}\eta_k u(t_k), \quad t \geq 0,$$

$$u(t) \leq \varphi(t), \quad t \in [-\tau, 0], \quad \tau > 0 \tag{1.1}$$

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where the kernels $k_i(t,s)$ are defined by $k_i(t,s) = (t-s)^{\beta_i - 1} s^{\gamma_i} F_i(s)$ for $\beta_i > 0$ and $\gamma_i > -1$, $i = 1, 2$, the points t_k (called "instants of impulse effect") are in increasing order and $\lim_{k\to\infty} t_k = +\infty$. This inequality was called the impulsive nonlinear singular version of the Gronwall inequality with delay by Tatar [7]. In this paper, we will consider an inequality

$$\begin{aligned} u(t) \le a(t) &+ \sum_{i=1}^{n} \int_0^{b_i(t)} (t-s)^{\beta_i - 1} s^{r_i} f_i(t,s) w_i(u(s)) ds \\ &+ \sum_{j=n+1}^{m+n} \int_0^{b_j(t)} (t-s)^{\beta_j - 1} s^{r_j} f_j(t,s) w_j(u(s-\tau)) ds \\ &+ d(t) \sum_{0<t_L<t} \eta_L u(t_L), \quad t \ge 0, \\ u(t) \le \varphi(t), &\quad t \in [-\tau, 0],\ \tau > 0, \end{aligned} \tag{1.2}$$

where n, m are positive integers, $\beta_l > 0$, $r_l > -1$ for $l = 1, \dots, n+m$ and $\eta_L \ge 0$ and other assumptions are given in Section 2. This inequality is more general than (1.1) since (1.2) has n nonlinear terms.

2. MAIN RESULTS

Notation: Following [1] and [5], we say $w_1 \propto w_2$ for $w_1, w_2 : A \subset \mathbb{R} \to \mathbb{R}\backslash\{0\}$ if $\frac{w_2}{w_1}$ is nondecreasing on A. This concept helps us to compare the monotonicity of different functions. Now we make the following assumptions:

- (H1) all w_i $(i = 1, \dots, n+m)$ are continuous and nondecreasing on $[0, \infty)$ and positive on $(0, \infty)$, and $w_1 \propto w_2 \propto \cdots \propto w_n$
- (H2) $a(t)$ and $d(t)$ are continuous and nonnegative on $[0, \infty)$;
- (H3) all $b_l : [0, \infty) \to [0, \infty)$ are continuously differentiable and nondecreasing such that $0 \le b_l(t) \le t$ on $[0, \infty)$, $t_L \le b_l(t) \le t_L + \tau$ for $t \in [t_L, t_L + \tau]$ and $t_L + \tau \le b_l(t) \le t_{L+1}$ for $t \in [t_L + \tau, t_{L+1}]$, $l = 1, \dots, n+m$ and $L = 0, 1, 2, \dots$ where $t_0 = 0$. The points t_L are called instants of impulse effect which are in increasing order, and $\lim_{L\to\infty} t_L = \infty$;
- (H4) all $f_l(t,s)$ $(l = 1, \dots, n+m)$ are continuous and nonnegative functions on $[0, \infty) \times [0, \infty)$;
- (H5) $\varphi(t)$ is nonnegative and continuous;
- (H6) $u(t)$ is a piecewise continuous function from $\mathbb{R} \to \mathbb{R}^+ = [0, \infty)$ with points of discontinuity of the first kind at the points $t_L \in \mathbb{R}$. It is also left continuous at the points t_L. This space is denoted by $PC(\mathbb{R}, \mathbb{R}^+)$.

Without loss of generality, we will suppose that the t_L satisfy $\tau < t_{L+1} - t_L \le 2\tau$, $L = 0, 1, 2, \dots$. As in Remark 3.2 of [7], other cases can be reduced to this one.

Theorem 2.1. *Let the above assumptions hold. Suppose that u satisfies (1.2) and is in $PC([-\tau, \infty), [0, \infty))$. Then if $\beta_\alpha > -\frac{1}{p} + 1$ and $r_\alpha > -\frac{1}{p}$, it holds that*

$$u(t) \le \begin{cases} u_{L,0}(t), & t \in (t_L, t_L + \tau], \\ u_{L,1}(t), & t \in (t_L + \tau, t_{L+1}], \\ u_{k,0}(t), & t \in (t_k, t_k + \tau] \quad \text{if } t_k + \tau \le T, \\ u_{k,1}(t), & t \in (t_k + \tau, T] \quad \text{if } t_k + \tau < T, \\ u_{k,0}(t), & t \in (t_k, T] \quad \text{if } t_k + \tau > T, \end{cases}$$

where $t_k \le T < t_{k+1}$ *and*

$$u_{L,l}(t) = \left[W_n^{-1}\left(W_n(\gamma_{L,l,n}(t)) + \int_{t_L+l\tau}^{b_n(t)} (n+m+L+1)^{q-1} c_n^q(t) \tilde{f}_n^q(t,s)ds\right)\right]^{\frac{1}{q}},$$

$$\gamma_{L,l,j}(t) = W_{j-1}^{-1}\Bigg[W_{j-1}(\gamma_{L,l,j-1}(t)) + \int_{t_L+l\tau}^{b_{j-1}(t)} (n+m+L+1)^{q-1} c_{j-1}^q(t) \tilde{f}_{j-1}^q(t,s)ds\Bigg], \quad j \neq 1,$$

$$\gamma_{L,l,1}(t) = (n+m+L+1)^{q-1}\Bigg[\tilde{a}^q(t) + \sum_{i=1}^{n}\int_0^{t_L+l\tau} c_i^q(t)\tilde{f}_i^q(t,s)w_i^q(\phi(s))ds + \sum_{j=n+1}^{n+m}\int_0^{b_j(t)} c_j^q(t)\tilde{f}_j^q(t,s)w_j^q(\psi(s-\tau))ds + \sum_{e=1}^{L}\tilde{d}^q(t)\eta_e^q u_{e-1,1}^q(t_e)\Bigg],$$

$$\phi(t) = \begin{cases} u_{L,0}(t), & t\in(t_L,t_L+\tau], t\in(t_k,t_k+\tau] \quad \textit{if } t_k+\tau\le T, \\ & \quad \textit{and } t\in(t_k,T] \quad \textit{if } t_k+\tau>T, \\ u_{L,1}(t), & t\in(t_L+\tau,t_{L+1}] \textit{ and } t\in(t_k+\tau,T] \quad \textit{if } t_k+\tau<T, \end{cases}$$

$$\psi(t) = \begin{cases} \varphi(t), & t\in[-\tau,0], \\ u_{L,0}(t), & t\in(t_L,t_L+\tau], t\in(t_k,t_k+\tau] \quad \textit{if } t_k+\tau\le T, \\ & \quad \textit{and } t\in(t_k,T] \quad \textit{if } t_k+\tau>T, \\ u_{L,1}(t), & t\in(t_L+\tau,t_{L+1}] \textit{ and } t\in(t_k+\tau,T] \quad \textit{if } t_k+\tau<T, \end{cases}$$

$$\tilde{a}(t) = \max_{0\le x\le t} a(x), \quad \tilde{f}_\alpha(t,s) = \max_{0\le x\le t} f_\alpha(x,s), \quad \tilde{d}(t) = \max_{0\le x\le t} d(x),$$

$$W_i(u) = \int_{u_i}^{u} \frac{dv}{w_i^q(v^{\frac{1}{q}})}, \quad u>0, \; u_i>0,$$

$$c_\alpha(t) = t^{\frac{1}{p}+\beta_\alpha+r_\alpha-1}\left(\frac{\Gamma(1+p(\beta_\alpha-1))\Gamma(1+pr_\alpha)}{\Gamma(2+p(\beta_\alpha+r_\alpha-1))}\right)^{\frac{1}{p}},$$

for $L=0,1,\dots,k-1$*,* $\alpha=1,2,\dots,n+m$*,* $l=0,1$*, and* $i,j=1,\dots,n$ *where* $\frac{1}{p}+\frac{1}{q}=1$ *for* $p>0$ *and* $q>1$*, and* T *is the largest number such that*

$$W_j(\gamma_{L,l,j}(t)) + \int_{t_L+l\tau}^{b_j(t)} (n+m+L+1)^{q-1} c_j(t)\tilde{f}_j^q(t,s)ds \le \int_{u_j}^{\infty}\frac{dz}{w_j^q(z^{1/q})}, \tag{2.1}$$

for all $t\in(t_L,t_L+\tau]$*, all* $t\in(t_k,t_k+\tau]$ *if* $t_k+\tau\le T$ *and all* $t\in(t_L,T]$ *if* $t_k+\tau>T$ *as* $l=0$*, or all* $t\in[t_L+\tau,t_{L+1}]$ *and all* $t\in[t_k+\tau,T)$ *if* $t_k+\tau<T$ *as* $l=1$ *where* $j=1,\dots,n$*,* $l=0,1$ *and* $L=0,1\dots,k-1$*.*

Before the proof, we introduce a lemma which will play a very important role.

Lemma 2.2 ([1]). *Suppose that*

(1) *all* w_i $(i=1,\dots,n)$ *are continuous and nondecreasing on* $[0,\infty)$ *and positive on* $(0,\infty)$*, and* $w_1 \propto w_2 \propto \cdots \propto w_n$*.*
(2) $a(t)$ *is continuously differentiable in* t *and nonnegative on* $[t_0,t_1)$*,*

(3) *all* b_l *are continuously differentiable and nondecreasing such that* $b_l(t) \leq t$ *for* $t \in [t_0, t_1)$

where t_0, t_1 *are constants and* $t_0 < t_1$*. If* $u(t)$ *is a continuous and nonnegative function on* $[t_0, t_1)$ *satisfying*

$$u(t) \leq a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} f_i(t,s) w_i(u(s)) ds, \quad t_0 \leq t < t_1,$$

then

$$u(t) \leq \tilde{W}_n^{-1} \left[\tilde{W}_n(\gamma_n(t)) + \int_{b_n(t_0)}^{b_n(t)} \tilde{f}_n(t,s) ds \right], \quad t_0 \leq t \leq T_1,$$

where

$$\gamma_i(t) = \tilde{W}_{i-1}^{-1} \left[\tilde{W}_{i-1}(\gamma_{i-1}(t)) + \int_{b_{i-1}(t_0)}^{b_{i-1}(t)} \tilde{f}_{i-1}(t,s) ds \right], \quad i = 2, 3, \ldots, n,$$

$$\gamma_1(t) = a(t_0) + \int_{t_0}^{t} |a'(s)| ds, \quad \tilde{W}_i(u) = \int_{u_i}^{u} \frac{dz}{w_i(z)}, \quad u_i > 0,$$

$T_1 < t_1$ *and* T_1 *is the largest number such that*

$$\tilde{W}_i(\gamma_i(T_1)) + \int_{b_i(t_0)}^{b_i(T_1)} \tilde{f}_i(T_1, s) ds \leq \int_{u_i}^{\infty} \frac{dz}{w_i(z)}, \quad i = 1, \ldots, n.$$

Proof of Theorem 2.1. Since $\beta_\alpha > -\frac{1}{p} + 1$ and $r_\alpha > -\frac{1}{p}$ for $\alpha = 1, \ldots, n+m$, by Hölder's inequality we obtain

$$\begin{aligned} u(t) \leq{}& a(t) + \sum_{i=1}^{n} \left(\int_0^t (t-s)^{p(\beta_i - 1)} s^{pr_i} ds \right)^{\frac{1}{p}} \left(\int_0^{b_i(t)} f_i^q(t,s) w_i^q(u(s)) ds \right)^{\frac{1}{q}} \\ &+ \sum_{j=n+1}^{m+n} \left(\int_0^t (t-s)^{p(\beta_j - 1)} s^{pr_j} ds \right)^{\frac{1}{p}} \left(\int_0^{b_j(t)} f_j^q(t,s) w_j^q(u(s-\tau)) ds \right)^{\frac{1}{q}} \\ &+ \sum_{0<t_L<t} d(t) \eta_L u(t_L) \\ \leq{}& a(t) + \sum_{i=1}^{n} c_i(t) \left(\int_0^{b_i(t)} f_i^q(t,s) w_i^q(u(s)) ds \right)^{\frac{1}{q}} \\ &+ \sum_{j=n+1}^{m+n} c_j(t) \left(\int_0^{b_j(t)} f_j^q(t,s) w_j^q(u(s-\tau)) ds \right)^{\frac{1}{q}} + \sum_{0<t_L<t} d(t) \eta_L u(t_L) \end{aligned}$$

where we use $b_\alpha(t) \leq t$ and the definition of $c_\alpha(t)$. Now we use the following result [4]:

If $A_1, \ldots, A_n$ are nonnegative for $n \in \mathbb{N}$, then for $q > 1$,

$$(A_1 + \cdots + A_n)^q \leq n^{q-1}(A_1^q + \cdots + A_n^q).$$

Since $t_k \le t \le T < t_{k+1}$, we have

$$u^q(t) \le (1+n+m+k)^{q-1} \left[a^q(t) + \sum_{i=1}^{n} c_i^q(t) \int_0^{b_i(t)} f_i^q(t,s) w_i^q(u(s))ds \right.$$
$$\left. + \sum_{j=n+1}^{m+n} c_j^q(t) \int_0^{b_j(t)} f_j^q(t,s) w_j^q(u(s-\tau))ds + \sum_{L=1}^{k} d^q(t)\eta_L^q u^q(t_L) \right].$$

We note that $\tilde{a}(t) \ge a(t)$, $\tilde{d}(t) \ge d(t)$ and $\tilde{f}_\alpha(t,s) \ge f_\alpha(t,s)$ and they are continuous and nondecreasing in t. The above inequality becomes

$$u^q(t) \le (1+n+m+k)^{q-1} \left[\tilde{a}^q(t) + \sum_{i=1}^{n} \left(\sum_{L=0}^{k-1} c_i^q(t) \int_{t_L}^{t_{L+1}} \tilde{f}_i^q(t,s) w_i^q(u(s))ds \right.\right.$$
$$\left. + c_i^q(t) \int_{t_k}^{b_i(t)} \tilde{f}_i^q(t,s) w_i^q(u(s))ds \right)$$
$$+ \sum_{j=n+1}^{m+n} \left(\sum_{L=0}^{k-1} c_j^q(t) \int_{t_L}^{t_{L+1}} \tilde{f}_j^q(t,s) w_j^q(u(s-\tau))ds \right.$$
$$\left.\left. + c_j^q(t) \int_{t_k}^{b_j(t)} \tilde{f}_j^q(t,s) w_j^q(u(s-\tau))ds \right) + \sum_{L=1}^{k} \tilde{d}^q(t)\eta_L^q u^q(t_L) \right]. \tag{2.2}$$

In the following, we apply mathematical induction with respect to k.

(1) $k = 0$. We note that $t_0 = 0$ and we have for any fixed $\tilde{t} \in [0, t_1]$

$$u^q(t) \le (n+m+1)^{q-1} \left[\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t}) \int_0^{b_i(t)} \tilde{f}_i^q(\tilde{t},s) w_i^q(u(s))ds \right.$$
$$\left. + \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \int_0^{b_j(t)} \tilde{f}_j^q(\tilde{t},s) w_j^q(u(s-\tau))ds \right] \tag{2.3}$$

for $t \in [0, \tilde{t}]$ since $c_\alpha(t)$ are nondecreasing.

Now we consider $\tilde{t} \in [0, \tau] \subset [0, t_1]$ and $t \in [0, \tilde{t}]$. Note that $0 \le b_j(t) \le t$ so $-\tau \le b_j(t) - \tau \le 0$ for $t \in [0, \tilde{t}]$. Since $u(t) \le \varphi(t)$ for $t \in [-\tau, 0]$, we have

$$u^q(t) \le z_{0,0}(t), \quad t \in [0, \tilde{t}],$$

where

$$z_{0,0}(t) = (n+m+1)^{q-1} \left[\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t}) \int_0^{b_i(t)} \tilde{f}_i^q(\tilde{t},s) w_i^q(u(s))ds \right.$$
$$\left. + \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \int_0^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t},s) w_j^q(\varphi(s-\tau))ds \right]. \tag{2.4}$$

It implies that

$$u(t) \le z_{0,0}(t)^{1/q}, \quad t \in [0, \tilde{t}]. \tag{2.5}$$

Thus, (2.4) becomes

$$z_{0,0}(t) \le (n+m+1)^{q-1}\left[\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t}) \int_0^{b_i(t)} \tilde{f}_i^q(\tilde{t},s) w_i^q(z_{0,0}^{1/q}(s))ds + \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \int_0^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t},s) w_j^q(\varphi(s-\tau))ds\right]. \tag{2.6}$$

By Lemma 2.2, (2.6) and (2.1), we have

$$z_{0,0}(t) \le W_n^{-1}\left[W_n(\tilde{\gamma}_{0,0,n}(t)) + \int_0^{b_n(t)} (n+m+1)^{q-1} c_n(\tilde{t}) \tilde{f}_n^q(\tilde{t},s)ds\right],$$

$$\tilde{\gamma}_{0,0,j}(t) = W_{j-1}^{-1}\Bigg[W_{j-1}(\tilde{\gamma}_{0,0,j-1}(t)) + \int_0^{b_{j-1}(t)} (n+m+1)^{q-1} c_{j-1}(\tilde{t}) \tilde{f}_{j-1}^q(\tilde{t},s)ds\Bigg], \quad j \ne 1,$$

$$\tilde{\gamma}_{0,0,1}(t) = (n+m+1)^{q-1}\left[\tilde{a}^q(\tilde{t}) + \sum_{j=n+1}^{n+m} \int_0^{b_j(\tilde{t})} c_j^q(\tilde{t}) \tilde{f}_j^q(\tilde{t},s) w_j^q(\psi(s-\tau))ds\right]$$

since $\psi(t) = \varphi(t)$ for $t \in [-\tau, 0]$.

Since (2.5) is true for any $t \in [0, \tilde{t}]$ and $\tilde{\gamma}_{0,0,j}(\tilde{t}) = \gamma_{0,0,j}(\tilde{t})$, we have

$$u(\tilde{t}) \le z_{0,0}(\tilde{t})^{1/q} \le u_{0,0}(\tilde{t}).$$

We know that $\tilde{t} \in [0, \tau]$ is arbitrary so we replace $\tilde{t}$ by t and get

$$u(t) \le u_{0,0}(t), \quad \text{for } t \in [0, \tau]. \tag{2.7}$$

This implies that the theorem is true for $t \in [0, \tau]$ and $k = 0$.

For $t \in [\tau, \tilde{t}]$ and $\tilde{t} \in [\tau, t_1]$, use the assumption (H3) and then we know that $b_\alpha(\tau) = \tau$ and $\tau \le b_\alpha(t) \le t_1$ for $t \in [\tau, t_1]$ and $\alpha = 1, \dots, n+m$. Thus,

$$0 \le b_\alpha(t) - \tau \le t_1 - \tau \le \tau$$

since $\tau < t_1 - t_0 = t_1 \le 2\tau$. Using this fact, (2.3) and (2.7), we get

$$\begin{aligned} u^q(t) \le (n+m+1)^{q-1}\Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t}) \int_0^{\tau} \tilde{f}_i^q(\tilde{t},s) w_i^q(u(s))ds \\ &+ \sum_{i=1}^{n} c_i^q(\tilde{t}) \int_\tau^{b_i(t)} \tilde{f}_i^q(\tilde{t},s) w_i^q(u(s))ds \\ &+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \int_0^{\tau} \tilde{f}_j^q(\tilde{t},s) w_j^q(\psi(s-\tau))ds \\ &+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \int_\tau^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t},s) w_j^q(u(s-\tau))ds\Bigg] \end{aligned}$$

$$\begin{aligned}
&\le (n+m+1)^{q-1}\Bigg[\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t})\int_0^{\tau} \tilde{f}_i^q(\tilde{t},s)w_i^q(u_{0,0}(s))ds \\
&\quad + \sum_{i=1}^{n} c_i^q(\tilde{t})\int_{\tau}^{b_i(t)} \tilde{f}_i^q(\tilde{t},s)w_i^q(u(s))ds \\
&\quad + \sum_{j=n+1}^{m+n} c_j^q(\tilde{t})\int_0^{\tau} \tilde{f}_j^q(\tilde{t},s)w_j^q(\psi(s-\tau))ds \\
&\quad + \sum_{j=n+1}^{m+n} c_j^q(\tilde{t})\int_{\tau}^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t},s)w_j^q(u_{0,0}(s-\tau))ds\Bigg] \\
&\le (n+m+1)^{q-1}\Bigg[\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t})\int_0^{\tau} \tilde{f}_i^q(\tilde{t},s)w_i^q(\phi(s))ds \\
&\quad + \sum_{i=1}^{n} c_i^q(\tilde{t})\int_{\tau}^{b_i(t)} \tilde{f}_i^q(\tilde{t},s)w_i^q(u(s))ds \\
&\quad + \sum_{j=n+1}^{m+n} c_j^q(\tilde{t})\int_0^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t},s)w_j^q(\psi(s-\tau))ds\Bigg] \\
&:= z_{0,1}(t),
\end{aligned}$$

where we use the definitions of ϕ and ψ. Thus,

$$u(t) \le z_{0,1}^{1/q}(t), \quad t \in [\tau, \tilde{t}]. \tag{2.8}$$

Therefore,

$$\begin{aligned}
z_{0,1} \le (n+m+1)^{q-1}\Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t})\int_0^{\tau} \tilde{f}_i^q(\tilde{t},s)w_i^q(\phi(s))ds \\
&+ \sum_{i=1}^{n} c_i^q(\tilde{t})\int_{\tau}^{b_i(t)} \tilde{f}_i^q(\tilde{t},s)w_i^q(z_{0,1}^{1/q}(s))ds \\
&+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t})\int_0^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t},s)w_j^q(\psi(s-\tau))ds\Bigg].
\end{aligned}$$

Using Lemma 2.2, (2.1) and $b_\alpha(\tau) = \tau$, we obtain for $t \in [\tau, \tilde{t}]$

$$z_{0,1}(t) \le W_n^{-1}\left[W_n(\tilde{\gamma}_{0,1,n}(t)) + \int_{\tau}^{b_n(t)} (n+m+1)^{q-1} c_n^q(\tilde{t})\tilde{f}_n^q(\tilde{t},s)ds\right],$$

$$\tilde{\gamma}_{0,1,j}(t) = W_{j-1}^{-1}\left[W_{j-1}(\tilde{\gamma}_{0,1,j-1}(t)) + \int_{\tau}^{b_{j-1}(t)} (n+m+1)^{q-1} c_{j-1}^q(\tilde{t})\tilde{f}_{j-1}^q(\tilde{t},s)ds\right], \ j \ne 1,$$

$$\begin{aligned}
\tilde{\gamma}_{0,1,1}(t) = (n+m+1)^{q-1}\Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n}\int_0^{\tau} c_i^q(\tilde{t})\tilde{f}_i^q(\tilde{t},s)w_i^q(\phi(s))ds \\
&+ \sum_{j=n+1}^{n+m}\int_0^{b_j(\tilde{t})} c_j^q(\tilde{t})\tilde{f}_j^q(\tilde{t},s)w_j^q(\psi(s-\tau))ds\Bigg].
\end{aligned}$$

Since (2.8) is true for any $t \in [\tau, t_1]$ and $\tilde{\gamma}_{0,1,1}(\tilde{t}) = \gamma_{0,1,1}(\tilde{t})$, we have

$$u(\tilde{t}) \le z_{0,1}^{1/q}(\tilde{t}) \le u_{0,1}(\tilde{t}).$$

We know that $\tilde{t} \in [\tau, t_1]$ is arbitrary so we replace $\tilde{t}$ by t and get

$$u(t) \le u_{0,1}(t), \qquad t \in [\tau, t_1]. \tag{2.9}$$

This implies that the theorem is valid for $t \in [\tau, t_1]$ and $L = 0$.

(2) $L = 1$. First we consider $t \in (t_1, \tilde{t}]$, where $\tilde{t} \in (t_1, t_1+\tau]$ is arbitrary. Note that $\tau < t_2 - t_1 \le 2\tau$. (H3) gives $b_\alpha(t_1) = t_1$ and $t_1 \le b_\alpha(t) \le t_1 + \tau$ for $t \in (t_1, t_1+\tau]$ so $t_1 - \tau \le b_\alpha(t) - \tau \le t_1$ for $t \in (t_1, t_1+\tau]$. By (2.7) and (2.9), (2.2) can be written as

$$\begin{aligned} u^q(t) \le (n+m+2)^{q-1} \Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^n c_i^q(\tilde{t}) \left(\int_0^\tau + \int_\tau^{t_1} \right) \tilde{f}_i^q(\tilde{t}, s) w_i^q(u(s)) ds \\ &+ \sum_{i=1}^n c_i^q(\tilde{t}) \int_{t_1}^{b_i(t)} \tilde{f}_i^q(\tilde{t}, s) w_i^q(u(s)) ds \\ &+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \left(\int_0^\tau + \int_\tau^{t_1} \right) \tilde{f}_j^q(\tilde{t}, s) w_j^q(u(s-\tau)) ds \\ &+ \sum_{j=1}^n c_j^q(\tilde{t}) \int_{t_1}^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t}, s) w_j^q(u(s-\tau)) ds + \tilde{d}^q(\tilde{t}) \eta_1^q u^q(t_1) \Bigg] \\ \le (n+m+2)^{q-1} \Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^n c_i^q(\tilde{t}) \int_0^{t_1} \tilde{f}_i^q(\tilde{t}, s) w_i^q(\phi(s)) ds \\ &+ \sum_{i=1}^n c_i^q(\tilde{t}) \int_{t_1}^{b_i(t)} \tilde{f}_i^q(\tilde{t}, s) w_i^q(u(s)) ds \\ &+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \int_0^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t}, s) w_j^q(\psi(s-\tau)) ds + \tilde{d}^q(\tilde{t}) \eta_1^q u_{0,1}^q(t_1) \Bigg] \\ :=& z_{1,0}(t), \end{aligned}$$

where we use the definitions of ϕ and ψ so

$$u(t) \le z_{1,0}^{1/q}(t), \quad t \in (t_1, \tilde{t}]. \tag{2.10}$$

Thus,

$$\begin{aligned} z_{1,0}(t) \le (n+m+2)^{q-1} \Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^n c_i^q(\tilde{t}) \int_0^{t_1} \tilde{f}_i^q(\tilde{t}, s) w_i^q(\phi(s)) ds \\ &+ \sum_{i=1}^n c_i^q(\tilde{t}) \int_{t_1}^{b_i(t)} \tilde{f}_i^q(\tilde{t}, s) w_i^q(z_{1,0}^{1/q}(s)) ds \\ &+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \int_0^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t}, s) w_j^q(\psi(s-\tau)) ds + \tilde{d}^q(\tilde{t}) \eta_1^q u_{0,1}^q(t_1) \Bigg]. \end{aligned}$$

By Lemma 2.2, (2.1) and $b_\alpha(t_1) = t_1$, we obtain for $t \in (t_1, \tilde{t}]$

$$z_{1,0}(t) \le W_n^{-1}\left[W_n(\tilde{\gamma}_{1,0,n}(t)) + \int_{t_1}^{b_n(t)} (n+m+2)^{q-1} c_n^q(\tilde{t}) \tilde{f}_n^q(\tilde{t}, s)ds\right],$$

$$\tilde{\gamma}_{1,0,j}(t) = W_{j-1}^{-1}\left[W_{j-1}(\tilde{\gamma}_{1,0,j-1}(t)) + \int_{t_1}^{b_{j-1}(t)} (n+m+2)^{q-1} c_{j-1}^q(\tilde{t}) \tilde{f}_{j-1}^q(\tilde{t}, s)ds\right],\ j \ne 1,$$

$$\begin{aligned}\tilde{\gamma}_{1,0,1}(t) = (n+m+2)^{q-1}\Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n}\int_0^{t_1} c_i^q(\tilde{t})\tilde{f}_i^q(\tilde{t}, s) w_i^q(\phi(s))ds \\ &+ \sum_{j=n+1}^{n+m}\int_0^{b_j(\tilde{t})} c_j^q(\tilde{t})\tilde{f}_j^q(\tilde{t}, s) w_j^q(\psi(s-\tau))ds + \tilde{d}^q(\tilde{t})\eta_1^q u_{0,1}^q(t_1)\Bigg].\end{aligned}$$

Since (2.10) is true for any $t \in (t_1, \tilde{t}]$ and $\tilde{\gamma}_{1,0,1}(\tilde{t}) = \gamma_{1,0,1}(\tilde{t})$, we have

$$u(\tilde{t}) \le z_{1,0}^{1/q}(\tilde{t}) \le u_{1,0}(\tilde{t}).$$

We know that $\tilde{t} \in (t_1, t_1+\tau]$ is arbitrary so we replace $\tilde{t}$ by t and get

$$u(t) \le u_{1,0}(t), \qquad t \in (t_1, t_1+\tau]. \tag{2.11}$$

This implies that the theorem is valid for $t \in (t_1, t_1+\tau]$ and $L = 1$.

We now consider $t \in [t_1+\tau, \tilde{t}]$, where $\tilde{t} \in [t_1+\tau, t_2]$ is arbitrary. Again, by (H3) we have $t_1+\tau \le b_\alpha(t) \le t_2$ for $t \in [t_1+\tau, t_2]$ and $b_\alpha(t_1+\tau) = t_1+\tau$ so $t_1 \le b_\alpha(t)-\tau \le t_2-\tau \le t_1+\tau$ since $\tau < t_2 - t_1 \le 2\tau$. Obviously, by (2.7), (2.9) and (2.11), (2.2) becomes

$$\begin{aligned}u^q(t) \le (n+m+2)^{q-1}\Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t})\int_0^{t_1+\tau} \tilde{f}_i^q(\tilde{t}, s) w_i^q(u(s))ds \\ &+ \sum_{i=1}^{n} c_i^q(\tilde{t})\int_{t_1+\tau}^{b_i(t)} \tilde{f}_i^q(\tilde{t}, s) w_i^q(u(s))ds \\ &+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t})\int_0^{t_1+\tau} \tilde{f}_j^q(\tilde{t}, s) w_j^q(u(s-\tau))ds \\ &+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t})\int_{t_1+\tau}^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t}, s) w_j^q(u(s-\tau))ds + \tilde{d}^q(\tilde{t})\eta_1^q u_{1,0}^q(t_1)\Bigg] \\ \le (n+m+2)^{q-1}\Bigg[&\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t})\int_0^{t_1+\tau} \tilde{f}_i^q(\tilde{t}, s) w_i^q(\phi(s))ds \\ &+ \sum_{i=1}^{n} c_i^q(\tilde{t})\int_{t_1+\tau}^{b_i(t)} \tilde{f}_i^q(\tilde{t}, s) w_i^q(u(s))ds \\ &+ \sum_{j=n+1}^{m+n} c_j^q(\tilde{t})\int_0^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t}, s) w_j^q(\psi(s-\tau))ds + \tilde{d}^q(\tilde{t})\eta_1^q u_{0,1}^q(t_1)\Bigg] \\ := z_{1,1}(t),&\end{aligned}$$

that is,

$$u(t) \le z_{1,1}^{1/q}(t), \quad t \in [t_1+\tau, \tilde{t}]. \tag{2.12}$$

Thus,

$$z_{1,1}(t) \leq (n+m+2)^{q-1} \left[\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} c_i^q(\tilde{t}) \int_0^{t_1+\tau} \tilde{f}_i^q(\tilde{t}, s) w_i^q(\phi(s)) ds \right.$$
$$+ c_i^q(\tilde{t}) \int_{t_1+\tau}^{b_i(t)} \tilde{f}_i^q(\tilde{t}, s) w_i^q(z_{1,1}^{1/q}(s)) ds$$
$$\left. + \sum_{j=n+1}^{m+n} c_j^q(\tilde{t}) \int_0^{b_j(\tilde{t})} \tilde{f}_j^q(\tilde{t}, s) w_j^q(\psi(s-\tau)) ds + \tilde{d}^q(\tilde{t}) \eta_1^q u_{0,1}^q(t_1) \right].$$

Using Lemma 2.2, (2.1) and $b_\alpha(t_1+\tau) = t_1+\tau$, we obtain for $t \in (t_1, \tilde{t}]$

$$z_{1,1}(t) \leq W_n^{-1} \left[W_n(\tilde{\gamma}_{1,1,n}(t)) + \int_{t_1+\tau}^{b_n(t)} (n+m+2)^{q-1} c_n^q(\tilde{t}) \tilde{f}_n^q(\tilde{t}, s) ds \right],$$

$$\tilde{\gamma}_{1,1,j}(t) = W_{j-1}^{-1} \Big[W_{j-1}(\tilde{\gamma}_{1,1,j-1}(t)) + \int_{t_1+\tau}^{b_{j-1}(t)} (n+m+2)^{q-1} c_{j-1}^q(\tilde{t}) \tilde{f}_{j-1}^q(\tilde{t}, s) ds \Big], \ j \neq 0,$$

$$\tilde{\gamma}_{1,1,1}(t) = (n+m+2)^{q-1} \left[\tilde{a}^q(\tilde{t}) + \sum_{i=1}^{n} \int_0^{t_1+\tau} c_i^q(\tilde{t}) \tilde{f}_i^q(\tilde{t}, s) w_i^q(\phi(s)) ds \right.$$
$$\left. + \sum_{j=n+1}^{n+m} \int_0^{b_j(\tilde{t})} c_j^q(\tilde{t}) \tilde{f}_j^q(\tilde{t}, s) w_j^q(\psi(s-\tau)) ds + \tilde{d}^q(\tilde{t}) \eta_1^q u_{0,1}^q(t_1) \right].$$

Since (2.12) is true for any $t \in (t_1, \tilde{t}]$ and $\tilde{\gamma}_{1,1,1}(\tilde{t}) = \gamma_{1,1,1}(\tilde{t})$, we have

$$u(\tilde{t}) \leq z_{1,1}^{1/q}(\tilde{t}) \leq u_{1,1}(\tilde{t}).$$

We know that $\tilde{t} \in [t_1+\tau, t_2]$ is arbitrary so we replace $\tilde{t}$ by t and get

$$u(t) \leq u_{1,1}(t), \qquad t \in [t_1+\tau, t_2].$$

This implies that the theorem is valid for $t \in [t_1+\tau, t_2]$ and $L = 1$.

(3) Finally, suppose that the theorem is valid for k, then for $k+1$ we redefine ϕ and ψ by replacing k with $k+1$. In a similar manner as in steps (1) and (2), we can see that the theorem holds for $t \in (t_{k+1}, T] \subset (t_{k+1}, t_{k+2}]$.

The proof is now completed. □

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