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ON SOME INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS

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ABSTRACT. In this paper, we prove the open inequality $a^{ea} + b^{eb} \geq a^{eb} + b^{ea}$ for either $a \geq b \geq \frac{1}{e}$ or $\frac{1}{e} \geq a \geq b > 0$. In addition, other related results and conjectures are presented.

Key words and phrases: Power-exponential function, Convex function, Bernoulli's inequality, Conjecture.

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1. INTRODUCTION

In 2006, *A. Zeikii* posted and proved on the Mathlinks Forum [1] the following inequality

$$a^a + b^b \geq a^b + b^a, \tag{1.1}$$

where a and b are positive real numbers less than or equal to 1. In addition, he conjectured that the following inequality holds under the same conditions:

$$a^{2a} + b^{2b} \geq a^{2b} + b^{2a}. \tag{1.2}$$

Starting from this, we have conjectured in [1] that

$$a^{ea} + b^{eb} \geq a^{eb} + b^{ea} \tag{1.3}$$

for all positive real numbers a and b.

2. MAIN RESULTS

In what follows, we will prove some relevant results concerning the power-exponential inequality

$$a^{ra} + b^{rb} \geq a^{rb} + b^{ra} \tag{2.1}$$

for a, b and r positive real numbers. We will prove the following theorems.

Theorem 2.1. *Let r, a and b be positive real numbers. If (2.1) holds for $r = r_0$, then it holds for any $0 < r \leq r_0$.*

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Theorem 2.2. *If a and b are positive real numbers such that $\max\{a,b\} \geq 1$, then (2.1) holds for any positive real number r.*

Theorem 2.3. *If $0 < r \leq 2$, then (2.1) holds for all positive real numbers a and b.*

Theorem 2.4. *If a and b are positive real numbers such that either $a \geq b \geq \frac{1}{r}$ or $\frac{1}{r} \geq a \geq b$, then (2.1) holds for any positive real number $r \leq e$.*

Theorem 2.5. *If $r > e$, then (2.1) does not hold for all positive real numbers a and b.*

From the theorems above, it follows that the inequality (2.1) continues to be an open problem only for $2 < r \leq e$ and $0 < b < \frac{1}{r} < a < 1$. For the most interesting value of r, that is $r = e$, only the case $0 < b < \frac{1}{e} < a < 1$ is not yet proved.

3. Proofs of Theorems

Proof of Theorem 2.1. Without loss of generality, assume that $a \geq b$. Let $x = ra$ and $y = rb$, where $x \geq y$. The inequality (2.1) becomes

$$x^x - y^x \geq r^{x-y}(x^y - y^y). \tag{3.1}$$

By hypothesis,

$$x^x - y^x \geq r_0^{x-y}(x^y - y^y).$$

Since $x - y \geq 0$ and $x^y - y^y \geq 0$, we have $r_0^{x-y}(x^y - y^y) \geq r^{x-y}(x^y - y^y)$, and hence

$$x^x - y^x \geq r_0^{x-y}(x^y - y^y) \geq r^{x-y}(x^y - y^y).$$

□

Proof of Theorem 2.2. Without loss of generality, assume that $a \geq b$ and $a \geq 1$. From $a^{r(a-b)} \geq b^{r(a-b)}$, we get $b^{rb} \geq \frac{a^{rb}b^{ra}}{a^{ra}}$. Therefore,

$$\begin{aligned} a^{ra} + b^{rb} - a^{rb} - b^{ra} &\geq a^{ra} + \frac{a^{rb}b^{ra}}{a^{ra}} - a^{rb} - b^{ra} \\ &= \frac{(a^{ra} - a^{rb})(a^{ra} - b^{ra})}{a^{ra}} \geq 0, \end{aligned}$$

because $a^{ra} \geq a^{rb}$ and $a^{ra} \geq b^{ra}$. □

Proof of Theorem 2.3. By Theorem 2.1 and Theorem 2.2, it suffices to prove (2.1) for $r = 2$ and $1 > a > b > 0$. Setting $c = a^{2b}$, $d = b^{2b}$ and $s = \frac{a}{b}$ (where $c > d > 0$ and $s > 1$), the desired inequality becomes

$$c^s - d^s \geq c - d.$$

In order to prove this inequality, we show that

$$c^s - d^s > s(cd)^{\frac{s-1}{2}}(c - d) > c - d. \tag{3.2}$$

The left side of the inequality in (3.2) is equivalent to $f(c) > 0$, where $f(c) = c^s - d^s - s(cd)^{\frac{s-1}{2}}(c - d)$. We have $f'(c) = \frac{1}{2}sc^{\frac{s-3}{2}}g(c)$, where

$$g(c) = 2c^{\frac{s+1}{2}} - (s+1)cd^{\frac{s-1}{2}} + (s-1)d^{\frac{s+1}{2}}.$$

Since

$$g'(c) = (s+1)\left(c^{\frac{s-1}{2}} - d^{\frac{s-1}{2}}\right) > 0,$$

$g(c)$ is strictly increasing, $g(c) > g(d) = 0$, and hence $f'(c) > 0$. Therefore, $f(c)$ is strictly increasing, and then $f(c) > f(d) = 0$.

The right side of the inequality in (3.2) is equivalent to

$$\frac{a}{b}(ab)^{a-b} > 1.$$

Write this inequality as $f(b) > 0$, where

$$f(b) = \frac{1+a-b}{1-a+b}\ln a - \ln b.$$

In order to prove that $f(b) > 0$, it suffices to show that $f'(b) < 0$ for all $b \in (0, a)$; then $f(b)$ is strictly decreasing, and hence $f(b) > f(a) = 0$. Since

$$f'(b) = \frac{-2}{(1-a+b)^2}\ln a - \frac{1}{b},$$

the inequality $f'(b) < 0$ is equivalent to $g(a) > 0$, where

$$g(a) = 2\ln a + \frac{(1-a+b)^2}{b}.$$

Since $0 < b < a < 1$, we have

$$g'(a) = \frac{2}{a} - \frac{2(1-a+b)}{b} = \frac{2(a-1)(a-b)}{ab} < 0.$$

Thus, $g(a)$ is strictly decreasing on $[b, 1]$, and therefore $g(a) > g(1) = b > 0$. This completes the proof. Equality holds if and only if $a = b$. □

Proof of Theorem 2.4. Without loss of generality, assume that $a \geq b$. Let $x = ra$ and $y = rb$, where either $x \geq y \geq 1$ or $1 \geq x \geq y$. The inequality (2.1) becomes

$$x^x - y^x \geq r^{x-y}(x^y - y^y).$$

Since $x \geq y$, $x^y - y^y \geq 0$ and $r \leq e$, it suffices to show that

$$x^x - y^x \geq e^{x-y}(x^y - y^y). \tag{3.3}$$

For the nontrivial case $x > y$, using the substitutions $c = x^y$ and $d = y^y$ (where $c > d$), we can write (3.3) as

$$c^{\frac{x}{y}} - d^{\frac{x}{y}} \geq e^{x-y}(c - d).$$

In order to prove this inequality, we will show that

$$c^{\frac{x}{y}} - d^{\frac{x}{y}} > \frac{x}{y}(cd)^{\frac{x-y}{2y}}(c-d) > e^{x-y}(c-d).$$

The left side of the inequality is just the left hand inequality in (3.2) for $s = \frac{x}{y}$, while the right side of the inequality is equivalent to

$$\frac{x}{y}(xy)^{\frac{x-y}{2}} > e^{x-y}.$$

We write this inequality as $f(x) > 0$, where

$$f(x) = \ln x - \ln y + \frac{1}{2}(x-y)(\ln x + \ln y) - x + y.$$

We have

$$f'(x) = \frac{1}{x} + \frac{\ln(xy)}{2} - \frac{y}{2x} - \frac{1}{2}$$

and

$$f''(x) = \frac{x+y-2}{2x^2}.$$

Case $x > y \geq 1$. Since $f''(x) > 0$, $f'(x)$ is strictly increasing, and hence

$$f'(x) > f'(y) = \frac{1}{y} + \ln y - 1.$$

Let $g(y) = \frac{1}{y} + \ln y - 1$. From $g'(y) = \frac{y-1}{y^2} > 0$, it follows that $g(y)$ is strictly increasing, $g(y) \geq g(1) = 0$, and hence $f'(x) > 0$. Therefore, $f(x)$ is strictly increasing, and then $f(x) > f(y) = 0$.

Case $1 \geq x > y$. Since $f''(x) < 0$, $f(x)$ is strictly concave on $[y, 1]$. Then, it suffices to show that $f(y) \geq 0$ and $f(1) > 0$. The first inequality is trivial, while the second inequality is equivalent to $g(y) > 0$ for $0 < y < 1$, where

$$g(y) = \frac{2(y-1)}{y+1} - \ln y.$$

From

$$g'(y) = \frac{-(y-1)^2}{y(y+1)^2} < 0,$$

it follows that $g(y)$ is strictly decreasing, and hence $g(y) > g(1) = 0$. This completes the proof.

Equality holds if and only if $a = b$. □

Proof of Theorem 2.5. (after an idea of *Wolfgang Berndt* [1]). We will show that

$$a^{ra} + b^{rb} < a^{rb} + b^{ra}$$

for $r = (x+1)e$, $a = \frac{1}{e}$ and $b = \frac{1}{r} = \frac{1}{(x+1)e}$, where $x > 0$; that is

$$xe^x + \frac{1}{(x+1)^x} > x + 1.$$

Since $e^x > 1 + x$, it suffices to prove that

$$\frac{1}{(x+1)^x} > 1 - x^2.$$

For the nontrivial case $0 < x < 1$, this inequality is equivalent to $f(x) < 0$, where

$$f(x) = \ln(1 - x^2) + x\ln(x+1).$$

We have

$$f'(x) = \ln(x+1) - \frac{x}{1-x}$$

and

$$f''(x) = \frac{x(x-3)}{(1+x)(1-x)^2}.$$

Since $f''(x) < 0$, $f'(x)$ is strictly decreasing for $0 < x < 1$, and then $f'(x) < f'(0) = 0$. Therefore, $f(x)$ is strictly decreasing, and hence $f(x) < f(0) = 0$. □

4. OTHER RELATED INEQUALITIES

Proposition 4.1. *If a and b are positive real numbers such that $\min\{a,b\} \le 1$, then the inequality*

$$a^{-ra} + b^{-rb} \le a^{-rb} + b^{-ra} \tag{4.1}$$

holds for any positive real number r.

Proof. Without loss of generality, assume that $a \le b$ and $a \le 1$. From $a^{r(b-a)} \le b^{r(b-a)}$ we get $b^{-rb} \le \frac{a^{-rb}b^{-ra}}{a^{-ra}}$, and

$$\begin{aligned} a^{-ra} + b^{-rb} - a^{-rb} - b^{-ra} &\le a^{-ra} + \frac{a^{-rb}b^{-ra}}{a^{-ra}} - a^{-rb} - b^{-ra} \\ &= \frac{(a^{-ra} - a^{-rb})(a^{-ra} - b^{-ra})}{a^{-ra}} \le 0, \end{aligned}$$

because $b^{-ra} \le a^{-ra} \le a^{-rb}$. □

Proposition 4.2. *If a, b, c are positive real numbers, then*

$$a^a + b^b + c^c \ge a^b + b^c + c^a. \tag{4.2}$$

This inequality, with $a, b, c \in (0,1)$, was posted as a conjecture on the Mathlinks Forum by Zeikii [1].

Proof. Without loss of generality, assume that $a = \max\{a,b,c\}$. There are three cases to consider: $a \ge 1$, $c \le b \le a < 1$ and $b \le c \le a < 1$.

Case $a \ge 1$. By Theorem 2.3, we have $b^b + c^c \ge b^c + c^b$. Thus, it suffices to prove that

$$a^a + c^b \ge a^b + c^a.$$

For $a = b$, this inequality is an equality. Otherwise, for $a > b$, we substitute $x = a^b$, $y = c^b$ and $s = \frac{a}{b}$ (where $x \ge 1$, $x \ge y$ and $s > 1$) to rewrite the inequality as $f(x) \ge 0$, where

$$f(x) = x^s - x - y^s + y.$$

Since

$$f'(x) = sx^{s-1} - 1 \ge s - 1 > 0,$$

$f(x)$ is strictly increasing for $x \ge y$, and therefore $f(x) \ge f(y) = 0$.

Case $c \le b \le a < 1$. By Theorem 2.3, we have $a^a + b^b \ge a^b + b^a$. Thus, it suffices to show that

$$b^a + c^c \ge b^c + c^a,$$

which is equivalent to $f(b) \ge f(c)$, where $f(x) = x^a - x^c$. This inequality is true if $f'(x) \ge 0$ for $c \le x \le b$. From

$$\begin{aligned} f'(x) &= ax^{a-1} - cx^{c-1} \\ &= x^{c-1}(ax^{a-c} - c) \\ &\ge x^{c-1}(ac^{a-c} - c) = x^{c-1}c^{a-c}(a - c^{1-a+c}), \end{aligned}$$

we need to show that $a - c^{1-a+c} \ge 0$. Since $0 < 1 - a + c \le 1$, by Bernoulli's inequality we have

$$\begin{aligned} c^{1-a+c} &= (1 + (c-1))^{1-a+c} \\ &\le 1 + (1-a+c)(c-1) = a - c(a-c) \le a. \end{aligned}$$

Case $b \le c \le a < 1$. The proof of this case is similar to the previous case. So the proof is completed.

Equality holds if and only if $a = b = c$. □

Conjecture 4.3. *If a, b, c are positive real numbers, then*

$$a^{2a} + b^{2b} + c^{2c} \geq a^{2b} + b^{2c} + c^{2a}. \tag{4.3}$$

Conjecture 4.4. *Let r be a positive real number. The inequality*

$$a^{ra} + b^{rb} + c^{rc} \geq a^{rb} + b^{rc} + c^{ra} \tag{4.4}$$

holds for all positive real numbers a, b, c with $a \leq b \leq c$ if and only if $r \leq e$.

We can prove that the condition $r \leq e$ in Conjecture 4.4 is necessary by setting $c = b$ and applying Theorem 2.5.

Proposition 4.5. *If a and b are nonnegative real numbers such that $a + b = 2$, then*

$$a^{2b} + b^{2a} \leq 2. \tag{4.5}$$

Proof. We will show the stronger inequality

$$a^{2b} + b^{2a} + \left(\frac{a-b}{2}\right)^2 \leq 2.$$

Without loss of generality, assume that $a \geq b$. Since $0 \leq a - 1 < 1$ and $0 < b \leq 1$, by Bernoulli's inequality we have

$$a^b \leq 1 + b(a-1) = 1 + b - b^2$$

and

$$b^a = b \cdot b^{a-1} \leq b[1 + (a-1)(b-1)] = b^2(2-b).$$

Therefore,

$$\begin{aligned} a^{2b} + b^{2a} + \left(\frac{a-b}{2}\right)^2 - 2 &\leq (1 + b - b^2)^2 + b^4(2-b)^2 + (1-b)^2 - 2 \\ &= b^3(b-1)^2(b-2) \leq 0. \end{aligned}$$

□

Conjecture 4.6. *Let r be a positive real number. The inequality*

$$a^{rb} + b^{ra} \leq 2 \tag{4.6}$$

holds for all nonnegative real numbers a and b with $a + b = 2$ if and only if $r \leq 3$.

Conjecture 4.7. *If a and b are nonnegative real numbers such that $a + b = 2$, then*

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \leq 2. \tag{4.7}$$

Conjecture 4.8. *If a and b are nonnegative real numbers such that $a + b = 1$, then*

$$a^{2b} + b^{2a} \leq 1. \tag{4.8}$$

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[1] A. ZEIKII, V. CÎRTOAJE AND W. BERNDT, *Mathlinks Forum*, Nov. 2006, [ONLINE: `http://www.mathlinks.ro/Forum/viewtopic.php?t=118722`].