journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 4, Article 108, 8 pp.

ON THE L^p BOUNDEDNESS OF ROUGH PARAMETRIC MARCINKIEWICZ FUNCTIONS

AHMAD AL-SALMAN AND HUSSAIN AL-QASSEM

DEPARTMENT OF MATHEMATICS AND STATISTICS
SULTAN QABOOS UNIVERSITY
P.O. BOX 36, AL-KHOD 123 MUSCAT
SULTANATE OF OMAN.
alsalman@squ.edu.om

DEPARTMENT OF MATHEMATICS AND PHYSICS
QATAR UNIVERSITY, QATAR
husseink@qu.edu.qa

Received 07 November, 2006; accepted 25 November, 2007
Communicated by L. Debnath

ABSTRACT. In this paper, we study the L^p boundedness of a class of parametric Marcinkiewicz integral operators with rough kernels in $L(\log^+ L)(\mathbf{S}^{n-1})$. Our result in this paper solves an open problem left by the authors of ([6]).

Key words and phrases: Parametric Marcinkiewicz operators, rough kernels, Fourier transforms, Parametric maximal functions.

2000 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B15, 42B25.

1. INTRODUCTION

Let $n \geq 2$ and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Suppose that Ω is a homogeneous function of degree zero on $\mathbb{R}^n$ that satisfies $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x) d\sigma(x) = 0. \tag{1.1}$$

In 1960, Hörmander ([9]) defined the parametric Marcinkiewicz function μ_Ω^ρ of higher dimension by

$$\mu_\Omega^\rho f(x) = \left(\int_{-\infty}^{\infty} \left| 2^{-\rho t} \int_{|y| \leq 2^t} f(x-y) \left|y\right|^{-n+\rho} \Omega(y) dy \right|^2 dt \right)^{\frac{1}{2}}, \tag{1.2}$$

where $\rho > 0$. It is clear that if $\rho = 1$, then μ_Ω^ρ is the classical Marcinkiewicz integral operator introduced by Stein ([11]) which will be denoted by μ_Ω. When $\Omega \in \text{Lip}_\alpha(\mathbf{S}^{n-1})$,

This paper was written during the authors' time in Yarmouk University.

285-06

$(0 < \alpha \leq 1)$, Stein proved that μ_Ω is bounded on L^p for all $1 < p \leq 2$. Subsequently, Benedek-Calderón-Panzone proved the L^p boundedness of μ_Ω for all $1 < p < \infty$ under the condition $\Omega \in C^1(\mathbf{S}^{n-1})$ ([4]). Recently, under various conditions on Ω, the L^p boundedness of μ_Ω and a more general class of operators of Marcinkiewicz type has been investigated (see [1] – [2], [5], among others).

In ([9]), Hörmander proved that μ_Ω^ρ is bounded on L^p for all $1 < p < \infty$, provided that $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$, $(0 < \alpha \leq 1)$ and $\rho > 0$.

A long standing open problem concerning the operator μ_Ω^ρ is whether there are some L^p results on μ_Ω^ρ similar to those on μ_Ω when Ω satisfies only some size conditions. In a recent paper, Ding, Lu, and Yabuta ([6]) studied the operator

$$\mu_{\Omega,h}^\rho f(x) = \left(\int_{-\infty}^{\infty} \left| 2^{-\rho t} \int_{|y| \leq 2^t} f(x-y) \, |y|^{-n+\rho} \, h(|y|) \Omega(y) dy \right|^2 dt \right)^{\frac{1}{2}}, \tag{1.3}$$

where ρ is a complex number, $\mathrm{Re}(\rho) = \alpha > 0$, and h is a radial function on $\mathbb{R}^n$ satisfying $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 \leq q \leq \infty$, where $l^\infty(L^q)(\mathbb{R}^+)$ is defined as follows: For $1 \leq q < \infty$,

$$l^\infty(L^q)(\mathbb{R}^+) = \left\{ h : \|h\|_{l^\infty(L^q)(\mathbb{R}^+)} = \sup_{j \in \mathbf{Z}} \left(\int_{2^{j-1}}^{2^j} |h(r)|^q \, \frac{dr}{r} \right)^{\frac{1}{q}} < C \right\}$$

and for $q = \infty$, $l^\infty(L^\infty)(\mathbb{R}^+) = L^\infty(\mathbb{R}^+)$.

Ding, Lu, and Yabuta ([6]) proved the following:

Theorem 1.1. *Suppose that $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1.1) and $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$ for some $1 < q \leq \infty$. If $\mathrm{Re}(\rho) = \alpha > 0$, then $\left\| \mu_{\Omega,h}^\rho f \right\|_2 \leq C/\sqrt{\alpha} \, \|f\|_2$, where C is independent of ρ and f.*

The L^p boundedness of $\mu_{\Omega,h}^\rho$ for $p \neq 2$ was left open by the authors of ([6]). The main purpose of this paper is to establish the L^p boundedness of $\mu_{\Omega,h}^\rho$ for $p \neq 2$. Our main result of this paper is the following:

Theorem 1.2. *Suppose that $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1.1). If $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 < q \leq \infty$, and $\mathrm{Re}(\rho) = \alpha > 0$, then $\left\| \mu_{\Omega,h}^\rho f \right\|_p \leq C/\alpha \, \|f\|_p$ for all $1 < p < \infty$, where C is independent of ρ and f.*

Also, in this paper, we establish the L^p boundedness of the related parametric maximal function. In fact, we have the following result:

Theorem 1.3. *Suppose that $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$. If $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 < q \leq \infty$, and $\alpha > 0$, then*

$$\|M_\alpha f\|_p \leq \frac{C}{\alpha} \, \|f\|_p$$

for all $1 < p < \infty$ with a constant C independent of α, where M_α is the operator defined by

$$M_\alpha f(x) = \sup_{t \in \mathbb{R}} \left\{ 2^{-\alpha t} \left| \int_{|y| \leq 2^t} \Omega(y) \, |y|^{-n+\rho} \, h(|y|) f(x-y) dy \right| \right\}. \tag{1.4}$$

The method employed in this paper is based in part on ideas from [1], [2] and [3], among others. A variation of this method can be applied to deal with more general integral operators of Marcinkiewicz type. An extensive discussion of more general operators will appear in forthcoming papers.

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one in each occurrence.

2. PREPARATION

Suppose $a \geq 1$. For a suitable family of measures $\tau = \{\tau_t : t \in \mathbb{R}\}$ on $\mathbb{R}^n$ and a suitable family of $\mathcal{C}^\infty$ functions $\Phi_a = \{\varphi_t : t \in \mathbb{R}\}$ on $\mathbb{R}^n$, define the family of operators $\{\Lambda_{\tau,\Phi_a,s} : t, s \in \mathbb{R}\}$ by

$$\Lambda_{\tau,\Phi,s,a}(f)(x) = \left(\int_{-\infty}^{\infty} |\tau_{at} * \varphi_{t+s} * f(x)|^2 \, dt\right)^{\frac{1}{2}}. \tag{2.1}$$

Also, define the operator τ^* by

$$\tau^*(f)(x) = \sup_{t\in\mathbb{R}}(|\tau_t| * |f|)(x). \tag{2.2}$$

The proof of our result will be based on the following lemma:

Lemma 2.1. *Suppose that for some $B > 0$, $\varepsilon > 0$, and $\beta > 0$, we have*

(i) $\|\tau_t\| \leq \beta$ *for* $t \in \mathbb{R}$;
(ii) $|\hat{\tau}_t(\xi)| \leq \beta(2^t|\xi|)^{\pm\frac{\varepsilon}{a}}$ *for* $\xi \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$;
(iii) $\|\tau^*(f)\|_q \leq B\|f\|_q$ *for some* $q > 1$;
(iv) *The functions φ_t, $t \in \mathbb{R}$ satisfy the properties that $\hat{\varphi}_t$ is supported in $\{\xi \in \mathbb{R}^n : 2^{-(t+1)a} \leq |\xi| \leq 2^{-(t-1)a}\}$ and $\left|\frac{d^\gamma\hat{\varphi}_t}{d\xi^\gamma}(\xi)\right| \leq C_\gamma |\xi|^{-|\gamma|}$ for any multi-index $\gamma \in (\mathbb{N}\cup(\mathbf{0}))^n$ with constants C_γ depend only on γ and the dimension of the underlying space $\mathbb{R}^n$.*

Then for $\frac{2q}{q+1} < p < \frac{2q}{q-1}$, there exists a constant C_p independent of a, β, B, s, and ε such that

$$\|\Lambda_{\tau,\Phi,s,a}(f)\|_p \leq C_p(\beta B)^{\frac{1}{2}}(\beta B^{-1})^{\frac{\theta(p)}{2}} 2^{(\varepsilon+1)\theta(p)} 2^{-\varepsilon\theta(p)|s|} \|f\|_p \tag{2.3}$$

for all $f \in L^p(\mathbb{R}^n)$, where $\theta(p) = \frac{2q-pq+p}{p}$ if $p \in \left(2, \frac{2q}{q-1}\right)$ and $\theta(p) = \frac{pq+p-2q}{p}$ if $p \in \left(\frac{2q}{q+1}, 2\right)$.

Proof. We start with the case $p = 2$. By Plancherel's formula and the conditions (i)-(ii), we obtain

$$\|\Lambda_{\tau,\Phi,s,a}(f)\|_2 \leq \beta 2^{\varepsilon+1} 2^{-\varepsilon|s|} \|f\|_2 \tag{2.4}$$

for all $f \in L^2(\mathbb{R}^n)$.

Next, set $p_0 = 2q'$ and choose a non-negative function $v \in L^q_+(\mathbb{R}^n)$ with $\|v\|_q = 1$ such that

$$\|\Lambda_{\tau,\Phi,s,a}(f)\|_{p_0}^2 = \int_{\mathbb{R}^n}\int_{-\infty}^{\infty} |\tau_{at} * \varphi_{t+s} * f(x)|^2 \, v(x) dt dx.$$

Now it is easy to see that

$$\|\Lambda_{\tau,\Phi,s,a}(f)\|_{p_0} \leq \sqrt{\beta}\, \|\mathbf{g}_{a,s}(f)\|_{p_0} \|\tau^*(v)\|_q^{\frac{1}{2}} \tag{2.5}$$

where $\mathbf{g}_{a,s}$ is the operator

$$\mathbf{g}_{a,s}(f)(x) = \left(\int_{-\infty}^{\infty} |\varphi_{t+s} * f(x)|^2 \, dt\right)^{\frac{1}{2}}. \tag{2.6}$$

By the condition (iv) and a well-known argument (see [12, p. 26-28]), it is easy to see that

$$\|\mathbf{g}_{a,s}(f)\|_{p_0} \leq C_{p_0}\|f\|_{p_0} \tag{2.7}$$

for all $f \in L^{p_0}(\mathbb{R}^n)$ with constant C_{p_0} independent of a and s. Thus, by (2.5) and (2.7), we have

$$\|\Lambda_{\tau,\Phi,s,a}(f)\|_{p_0} \leq C_{p_0}\sqrt{\beta B}\, \|f\|_{p_0}. \tag{2.8}$$

By duality, we get

$$\|\Lambda_{\tau,\Phi,s,a}(f)\|_{(p_0)'} \le C_{(p_0)'}\sqrt{\beta B}\,\|f\|_{(p_0)'}\,. \tag{2.9}$$

Therefore, by interpolation between (2.4), (2.8), and (2.9), we obtain (2.3). This concludes the proof of the lemma. □

Now we establish the following oscillatory estimates:

Lemma 2.2. *Suppose that* $\Omega \in L^\infty(\mathbf{S}^{n-1})$ *is a homogeneous function of degree zero on* $\mathbb{R}^n$ *satisfying (1.1) and* $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 < q \le 2$. *Then for a complex number* ρ *with* $\mathrm{Re}(\rho) = \alpha > 0$, *we have*

$$\left|2^{-\alpha t}\int_{|y|\le 2^t} e^{-i\xi\cdot y}\Omega(y)\,|y|^{-n+\rho}\,h(|y|)dy\right| \le 2\frac{C}{\alpha}\,\|h\|_{l^\infty(L^q)(\mathbb{R}^+)}\,\|\Omega\|_1^{1-2/q'}\,\|\Omega\|_\infty^{2/q'}\,(2^t\,|\xi|)^{-\varepsilon}, \tag{2.10}$$

and

$$\left|2^{-\alpha t}\int_{|y|\le 2^t} e^{-i\xi\cdot y}\Omega(y)\,|y|^{-n+\rho}\,h(|y|)dy\right| \le 2\frac{C}{\alpha}\,\|h\|_{l^\infty(L^q)(\mathbb{R}^+)}\,\|\Omega\|_1\,(2^t\,|\xi|)^{\varepsilon} \tag{2.11}$$

for all $0 < \varepsilon < \min\{1/2, \alpha\}$. *The constant* C *is independent of* Ω, α, *and* t.

Proof. For $\xi \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$, let $G(\xi, r) = \int_{\mathbf{S}^{n-1}} e^{-ir\xi\cdot y'}\Omega(y')d\sigma(y')$. Then it is easy to see that

$$\left|2^{-\alpha t}\int_{|y|\le 2^t} e^{-i\xi\cdot y}\Omega(y)\,|y|^{-n+\rho}\,h(|y|)dy\right| \le \sum_{j=0}^{\infty} 2^{-\alpha j}\int_{2^{t-j-1}}^{2^{t-j}} |h(r)|\,|G(\xi,r)|\,r^{-1}dr. \tag{2.12}$$

Using the assumption that $1 < q \le 2$, it is straightforward to show that the right hand side of (2.12) is dominated by

$$2\,\|h\|_{l^\infty(L^q)(\mathbb{R}^+)}\,\|\Omega\|_1^{1-2/q'}\sum_{j=0}^{\infty} 2^{-\alpha j}\left(\int_{2^{t-j-1}}^{2^{t-j}} |G(\xi,r)|^2\,r^{-1}dr\right)^{\frac{1}{q'}}. \tag{2.13}$$

Now, for $\xi \in \mathbb{R}^n$, $y', z' \in \mathbf{S}^{n-1}$, $j \ge 0$, and $t \in \mathbb{R}$, set

$$I_{j,t}(\xi, y', z') = \int_{2^{t-j-1}}^{2^{t-j}} e^{-ir\xi\cdot(y'-z')}r^{-1}dr.$$

Then, we have

$$\left(\int_{2^{t-j-1}}^{2^{t-j}} |G(\xi,r)|^2\,r^{-1}dr\right)^{\frac{1}{q'}} \le \|\Omega\|_\infty^{2/q'}\left[\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} |I_{j,t}(\xi,y',z')|\,d\sigma(y')d\sigma(z')\right]^{\frac{1}{q'}}. \tag{2.14}$$

By integration by parts, we have

$$|I_{j,t}(\xi,y',z')| \le (2^{t-j-1}\,|\xi|\,|\xi'\cdot(y'-z')|)^{-1}. \tag{2.15}$$

On the other hand, we have

$$|I_{j,t}(\xi,y',z')| \le \ln 2. \tag{2.16}$$

Thus, by combining (2.15) and (2.16), we get

$$|I_{j,t}(\xi,y',z')| \le (2^{t-j-1}\,|\xi|\,|\xi'\cdot(y'-z')|)^{-\varepsilon} \tag{2.17}$$

for $0 < \varepsilon < \min\{1/2, \alpha\}$. Therefore, by (2.14) and (2.17), we obtain that

$$\left(\int_{2^{t-j-1}}^{2^{t-j}} |G(\xi,r)|^2 r^{-1}dr\right)^{\frac{1}{q'}} \leq \|\Omega\|_\infty^{2/q'} C(2^{t-j-1}|\xi|)^{-\varepsilon}, \tag{2.18}$$

where the constant C is independent of Ω, j, and t. Moreover, since $\varepsilon \leq 1/2$, it can be shown that C is also independent of α. Hence by (2.12), (2.13), and (2.18), we get (2.10).

Now we prove (2.11). Using the cancellation property (1.1), it is clear that

$$\left|2^{-\alpha t}\int_{|y|\leq 2^t} e^{-i\xi\cdot y}\Omega(y)|y|^{-n+\rho} h(|y|)dy\right| \leq \frac{2(\ln 2)^{\frac{1}{q'}}}{\alpha} \|h\|_{l^\infty(L^q)(\mathbb{R}^+)} \|\Omega\|_1 2^t |\xi|. \tag{2.19}$$

On the other hand, we have

$$\left|2^{-\alpha t}\int_{|y|\leq 2^t} e^{-i\xi\cdot y}\Omega(y)|y|^{-n+\rho} h(|y|)dy\right| \leq \frac{2(\ln 2)^{\frac{1}{q'}}}{\alpha} \|h\|_{l^\infty(L^q)(\mathbb{R}^+)} \|\Omega\|_1. \tag{2.20}$$

Thus, by interpolation between (2.19) and (2.20), we get (2.11). This completes the proof of Lemma 2.2. □

3. Rough Parametric Maximal Functions

In this section we shall establish the boundedness of certain maximal functions which will be needed to prove our main result.

Theorem 3.1. *Suppose that $\Omega \in L^\infty(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ with $\|\Omega\|_1 \leq 1$ and $\|\Omega\|_\infty \leq 2^a$ for some $a > 1$. Suppose also that $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 < q \leq \infty$ and let M_α be the operator defined as in (1.4). Then*

$$\|M_\alpha f\|_p \leq \frac{aC}{\alpha}\|f\|_p \tag{3.1}$$

for all $1 < p < \infty$ with constant C independent of a, f, and α.

Proof. Since $l^\infty(L^{q_1})(\mathbb{R}^+) \subset l^\infty(L^{q_2})(\mathbb{R}^+)$ whenever $q_2 \leq q_1$, it suffices to assume that $1 < q \leq 2$. By a similar argument as in ([2]), choose a collection of $\mathcal{C}^\infty$ functions $\Phi_a = \{\varphi_t : t \in \mathbb{R}\}$ on $\mathbb{R}^n$ that satisfies the following properties: $\hat{\varphi}_t$ is supported in $\{\xi \in \mathbb{R}^n : 2^{-(t+1)a} \leq |\xi| \leq 2^{-(t-1)a}\}$, $\left|\frac{d^\gamma \hat{\varphi}_t}{d\xi^\gamma}(\xi)\right| \leq C_\gamma |\xi|^{-|\gamma|}$ for any multi-index $\gamma \in (\mathbb{N}\cup\{\mathbf{0}\})^n$ with constants C_γ depending only on the underlying dimension and γ, and

$$\sum_{j\in\mathbf{Z}} \hat{\varphi}_{t+j}(\xi) = 1. \tag{3.2}$$

For $t \in \mathbb{R}$, let $\{\sigma_t : t \in \mathbb{R}\}$ be the family of measures on $\mathbb{R}^n$ defined via the Fourier transform by

$$\hat{\sigma}_t(\xi) = 2^{-\alpha t}\int_{|y|\leq 2^t} e^{-i\xi\cdot y}|\Omega(y)||y|^{-n+\rho}|h(|y|)|dy \tag{3.3}$$

Then it is easy to see that

$$M_\alpha f(x) = \sup_{t\in\mathbb{R}}\{|\sigma_t| * |f(x)|\}. \tag{3.4}$$

Now choose $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\phi}(\eta) = 1$ for $|\eta| \leq \frac{1}{2}$, and $\hat{\phi}(\eta) = 0$ for $|\eta| \geq 1$. Let $\{\tau_t : t \in \mathbb{R}\}$ be the family of measures on $\mathbb{R}^n$ defined via the Fourier transform by

$$\hat{\tau}_t(\xi) = \hat{\sigma}_t(\xi) - \hat{\phi}(2^t\xi)\hat{\sigma}_t(0). \tag{3.5}$$

Then by Lemma 2.2, the choice of ϕ, the definitions of σ_t, τ_t, and the assumptions on Ω, we have

$$|\hat{\tau}_t(\xi)| \leq \frac{C2^{la}}{\alpha}(2^t|\xi|)^{-\varepsilon} \tag{3.6}$$

for some $l, \varepsilon > 0$. Moreover, it is easy to see that

$$\|\tau_t\| \leq \frac{C}{\alpha} \tag{3.7}$$

Therefore by interpolation between (3.6) and (3.7), we get

$$|\hat{\tau}_t(\xi)| \leq \frac{C}{\alpha}(2^t|\xi|)^{-\frac{\varepsilon}{a}}. \tag{3.8}$$

Now by (3.2), and the definitions of σ_t, and τ_t, it is easy to see that

$$M_\alpha f(x) \leq 2\sqrt{a}\sum_{j\in\mathbf{Z}}\Lambda_{\tau,\Phi,j,a}(f)(x) + C\alpha^{-1}MH(f)(x), \tag{3.9}$$

$$\tau^*(f)(x) \leq 2\sqrt{a}\sum_{j\in\mathbf{Z}}\Lambda_{\tau,\Phi,j,a}(f)(x) + C\alpha^{-1}MH(f)(x), \tag{3.10}$$

where MH stands for the Hardy-Littlewood maximal function on $\mathbb{R}^n$, τ^* the maximal function that corresponds to $\{\tau_t : t \in \mathbb{R}\}$, and $\Lambda_{\tau,\Phi,s,a}$ is the operator defined by (2.1).

By (3.8), it is easy to see that

$$\|\Lambda_{\tau,\Phi,j,a}(f)\|_2 \leq C2^{-\varepsilon|j|}\alpha^{-1}\|f\|_2 \tag{3.11}$$

for all $f \in L^2(\mathbb{R}^n)$. Therefore, by (3.10) and (3.11) we have

$$\|\tau^*(f)\|_2 \leq C\alpha^{-1}a\|f\|_2. \tag{3.12}$$

Thus by (3.7), (3.8), (3.11), (3.12), and Lemma 2.1 with $q = 2$, we get

$$\|\Lambda_{\tau,\Phi,j,a}(f)\|_p \leq C\alpha^{-1}\sqrt{a}\|f\|_p \tag{3.13}$$

for $p \in (\frac{4}{3}, 4)$. Hence, by interpolation between (3.11) and (3.13), we obtain

$$\|\Lambda_{\tau,\Phi,j,a}(f)\|_p \leq C\alpha^{-1}\sqrt{a}2^{-\varepsilon'|j|}\|f\|_p \tag{3.14}$$

for $p \in (\frac{4}{3}, 4)$. Hence by (3.10) and (3.14), we get

$$\|\tau^*(f)\|_p \leq C\alpha^{-1}a\|f\|_p \tag{3.15}$$

for $p \in (\frac{4}{3}, 4)$. Next, by repeating the above argument with $q = \frac{4}{3} + \varepsilon$ $(\varepsilon \to 0^+)$, we get that

$$\|\Lambda_{\tau,\Phi,j,a}(f)\|_p \leq C\alpha^{-1}\sqrt{a}2^{-\varepsilon'|j|}\|f\|_p \tag{3.16}$$

$$\|\tau^*(f)\|_p \leq C\alpha^{-1}a\|f\|_p \tag{3.17}$$

for $p \in (\frac{7}{8}, 8)$. Now the result follows by successive applications of the above argument. This completes the proof. □

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.2. Suppose that $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$ and $h(|x|) \in l^\infty(L^q)(\mathbb{R}^+)$, $1 < q \leq \infty$. A key element in proving our results is decomposing the function Ω as follows (for more information see [3]): For a natural number w, let $\mathbf{E}_w$ be the set of points $x' \in \mathbf{S}^{n-1}$ which satisfy $2^{w+1} \leq |\Omega(x')| < 2^{w+2}$. Also, we let $\mathbf{E}_0$ be the set of points $x' \in \mathbf{S}^{n-1}$ which satisfy $|\Omega(x')| < 2^2$. Set $b_w = \Omega\chi_{\mathbf{E}_w}$. Set $\mathbf{D} = \{w : \|b_w\|_1 \geq 2^{-3w}\}$ and define the sequence of functions $\{\Omega_w\}_{w\in\mathbf{D}\cup\{0\}}$ by

$$\Omega_0(x) = b_0(x) + \sum_{w\notin\mathbf{D}} b_w(x) - \int_{\mathbf{S}^{n-1}} b_0(x)d\sigma(x) - \sum_{w\notin\mathbf{D}} \left(\int_{\mathbf{S}^{n-1}} b_w(x)d\sigma(x)v\right) \tag{4.1}$$

and for $w \in \mathbf{D}$,

$$\Omega_w(x) = (\|b_w\|_1)^{-1} \left(b_w(x) - \int_{\mathbf{S}^{n-1}} b_w(x)d\sigma(x)\right). \tag{4.2}$$

Then, it is easy to see that for $w \in \mathbf{D}\cup\{0\}$, Ω_w satisfies (1.1),

$$\|\Omega_w\|_1 \leq C, \quad \|\Omega_w\|_\infty \leq C2^{4(w+2)}, \tag{4.3}$$

$$\Omega(x) = \sum_{w\in\mathbf{D}\cup\{0\}} \theta_w \Omega_w(x), \tag{4.4}$$

where $\theta_0 = 1$, and $\theta_w = \|b_w\|_1$ if $w \in \mathbf{D}$.

For $w \in \mathbf{D}\cup\{0\}$, let $\mu^\rho_{\Omega_w,h}$ be the operator defined as in (1.3) with Ω replaced by Ω_w. Then by (4.4), we have

$$\mu^\rho_{\Omega,h} f(x) \leq \sum_{w\in\mathbf{D}\cup\{0\}} \theta_w \mu^\rho_{\Omega_w,h} f(x). \tag{4.5}$$

Now, for $w \in \mathbf{D}\cup\{0\}$, let $\tau_w = \{\tau_{t,w} : t \in \mathbb{R}\}$ be the family of measures on $\mathbb{R}^n$ defined via the Fourier transform by

$$\hat{\tau}_{t,w}(\xi) = 2^{-\alpha t} \int_{|y|\leq 2^t} e^{-i\xi\cdot y}\Omega_w(y)\,|y|^{-n+\rho}\,h(|y|)dy \tag{4.6}$$

and let $\Phi_{w+2} = \{\varphi_t : t \in \mathbb{R}\}$ be a collection of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ defined as in the proof of Theorem 3.1. Let $\Lambda_{\tau_w,\Phi_{w+2},j,w+2}$, $j \in \mathbf{Z}$ be the operators given by (2.1). Then by a simple change of variable we obtain

$$\mu^\rho_{\Omega_w,h} f(x) \leq \sqrt{w+2} \sum_{j\in\mathbf{Z}} \Lambda_{\tau_w,\Phi_{w+2},j,w+2}(f)(x). \tag{4.7}$$

Thus by Lemma 2.2, the properties of Ω_w, Theorem 3.1, and Lemma 2.1, we get

$$\left\|\mu^\rho_{\Omega_w,h} f\right\|_p \leq \frac{(w+2)C}{\alpha}\,\|f\|_p \tag{4.8}$$

for all $1 < p < \infty$.

Therefore, for $1 < p < \infty$, by (4.7) and (4.8), we get

$$\begin{aligned}\left\|\mu^\rho_{\Omega,h} f\right\|_p &\leq \frac{C}{\alpha}\left\{\sum_{w\in\mathbf{D}\cup\{0\}} (w+2)\theta_w\right\}\|f\|_p \\ &\leq \frac{C}{\alpha}\,\|\Omega\|_{L(\log L)(\mathbf{S}^{n-1})}\,\|f\|_p\,.\end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 1.3. A proof of Theorem 1.3 can be obtained using the decomposition (4.4) and Theorem 3.1. We omit the details □

REFERENCES

[1] A. AL-SALMAN AND H. AL-QASSEM, Flat Marcinkiewicz integral operators, *Turkish J. Math.*, **26**(3) (2002), 329–338.

[2] A. AL-SALMAN AND H. AL-QASSEM, Integral operators of Marcinkiewicz type, *J. Integral Equations Appl.*, **14**(4) (2002), 343–354.

[3] A. AL-SALMAN AND Y. PAN, Singular Integrals with Rough Kernels in $Llog^+L(\mathbf{S}^{n-1})$, *J. London Math. Soc.*, (2) **66** (2002) 153–174.

[4] A. BENEDEK, A. CALDERÓN AND R. PANZONE, convolution operators on Banach space valued functions, *Proc. Nat. Acad. Sci. USA*, **48** (1962), 356–365.

[5] Y. DING, D. FAN AND Y. PAN, L^p boundedness of Marcinkiewicz integrals with Hardy space function kernel, *Acta. Math. Sinica* (English Series), **16** (2000), 593–600.

[6] Y. DING, S. LU AND K. YABUTA, A problem on rough parametric Marcinkiewicz functions, *J. Austral. Math. Soc.*, **72** (2002), 13–21.

[7] J. DUOANDIKOETXEA AND J.L. RUBIO DE FRANCIA, Maximal and singular integral operators via Fourier transform estimates, *Invent. Math.*, **84** (1986), 541–561.

[8] D. FAN AND Y. PAN, Singular integrals with rough kernels supported by subvarieties, *Amer. J. Math.*, **119** (1997), 799–839.

[9] HÖRMANDER, 'Translation invariant operators', *Acta Math.*, **104** (1960), 93–139.

[10] M. SAKAMOTO AND K. YABUTA, Boundedness of Marcinkiewicz functions, *Studia Math.*, **135** (1999), 103–142.

[11] E.M. STEIN, On the function of Littlewood-Paley, Lusin and Marcinkiewicz, *Trans. Amer. Math. Soc.*, **88** (1958), 430–466.

[12] E.M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.