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volume 7, issue 1, article 10,
2006.

*Received 26 September, 2005;
accepted 08 November, 2005.*

Communicated by: I. Gavrea

Abstract

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Abstract

In this paper we give a variant of Jessen's inequality for isotonic linear functionals. Our results generalize some recent results of Gavrea. We also give comparison theorems for generalized means.

2000 Mathematics Subject Classification: 26D15, 39B62.

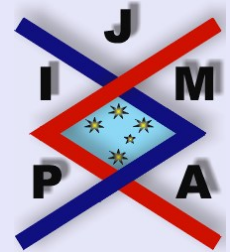
Key words: Isotonic linear functionals, Jessen's inequality, Generalized means.

Research is supported in part by the Research Grants Council of the Hong Kong SAR (Project No. HKU7017/05P).

The authors would like to thank the referee for his invaluable comments and insightful suggestions.

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1. Introduction

Let E be a nonempty set and L be a linear class of real valued functions $f : E \rightarrow \mathbb{R}$ having the properties:

L1: $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

L2: $1 \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$.

An isotonic linear functional is a functional $A : L \rightarrow \mathbb{R}$ having properties:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L, \alpha, \beta \in \mathbb{R}$ (A is linear);

A2: $f \in L, f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$ (A is isotonic).

The following result is Jessen's generalization of the well known Jensen's inequality for convex functions [3] (see also [5, p. 47]):

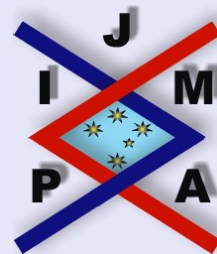
Theorem 1.1. *Let L satisfy properties L1, L2 on a nonempty set E , and let φ be a continuous convex function on an interval $I \subset \mathbb{R}$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in I$ we have $A(g) \in I$ and*

$$\varphi(A(g)) \leq A(\varphi(g)).$$

Similar to Jensen's inequality, Jessen's inequality has a converse [1] (see also [5, p. 98]):

Theorem 1.2. *Let L satisfy properties L1, L2 on a nonempty set E , and let φ be a convex function on an interval $I = [m, M]$ ($-\infty < m < M < \infty$). If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in I$ (so that $m \leq g(t) \leq M$ for all $t \in E$), we have*

$$A(\varphi(g)) \leq \frac{M - A(g)}{M - m} \cdot \varphi(m) + \frac{A(g) - m}{M - m} \cdot \varphi(M).$$



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Recently I. Gavrea [2] has obtained the following result which is in connection with Mercer's variant of Jensen's inequality [4]:

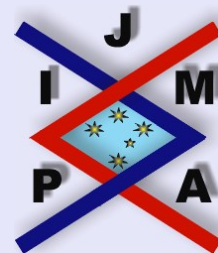
Theorem 1.3. *Let A be an isotonic linear functional defined on $C[a, b]$ such that $A(1) = 1$. Then for any convex function φ on $[a, b]$,*

$$\begin{aligned} \varphi(a + b - a_1) &\leq A(\psi) \\ &\leq \varphi(a) + \varphi(b) - \varphi(a) \frac{b - a_1}{b - a} - \varphi(b) \frac{a_1 - a}{b - a} \\ &\leq \varphi(a) + \varphi(b) - A(\varphi), \end{aligned}$$

where $\psi(t) = \varphi(a + b - t)$ and $a_1 = A(id)$.

Remark 1. *Although it is not explicitly stated above, it is obvious that function φ needs to be continuous on $[a, b]$.*

In Section 2 we give the main result of this paper which is an extension of Theorem 1.3 on a linear class L satisfying properties $L1, L2$. In Section 3 we use that result to prove the monotonicity property of generalized power means. We also consider in the same way generalized means with respect to isotonic functionals.



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2. Main Result

Theorem 2.1. *Let L satisfy properties L1, L2 on a nonempty set E , and let φ be a convex function on an interval $I = [m, M]$ ($-\infty < m < M < \infty$). If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g), \varphi(m + M - g) \in L$ (so that $m \leq g(t) \leq M$ for all $t \in E$), we have the following variant of Jessen's inequality*

$$(2.1) \quad \varphi(m + M - A(g)) \leq \varphi(m) + \varphi(M) - A(\varphi(g)).$$

In fact, to be more specific, we have the following series of inequalities

$$(2.2) \quad \begin{aligned} \varphi(m + M - A(g)) &\leq A(\varphi(m + M - g)) \\ &\leq \frac{M - A(g)}{M - m} \cdot \varphi(M) + \frac{A(g) - m}{M - m} \cdot \varphi(m) \\ &\leq \varphi(m) + \varphi(M) - A(\varphi(g)). \end{aligned}$$

If the function φ is concave, inequalities (2.1) and (2.2) are reversed.

Proof. Since φ is continuous and convex, the same is also true for the function

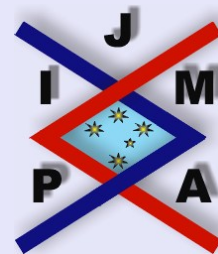
$$\psi : [m, M] \rightarrow \mathbb{R}$$

defined by

$$\psi(t) = \varphi(m + M - t), \quad t \in [m, M].$$

By Theorem 1.1,

$$\psi(A(g)) \leq A(\psi(g)),$$



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i.e.,

$$\varphi(m + M - A(g)) \leq A(\varphi(m + M - g)).$$

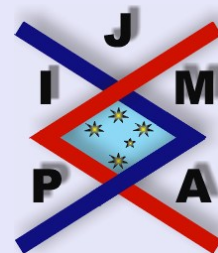
Applying Theorem 1.2 to ψ and then to φ , we have

$$\begin{aligned} & A(\varphi(m + M - g)) \\ & \leq \frac{M - A(g)}{M - m} \cdot \psi(m) + \frac{A(g) - m}{M - m} \cdot \psi(M) \\ & = \frac{M - A(g)}{M - m} \cdot \varphi(M) + \frac{A(g) - m}{M - m} \cdot \varphi(m) \\ & = \varphi(m) + \varphi(M) - \left[\frac{M - A(g)}{M - m} \cdot \varphi(m) + \frac{A(g) - m}{M - m} \cdot \varphi(M) \right] \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(g)). \end{aligned}$$

The last statement follows immediately from the facts that if φ is concave then $-\varphi$ is convex, and that A is linear on L . \square

Remark 2. In Theorem 2.1, taking $L = C[a, b]$ and $g = id$ (so that $m = a$ and $M = b$), we obtain the results of Theorem 1.3. On the other hand, the results of Theorem 1.3 for the functional B defined on L by $B(\varphi) = A(\varphi(g))$, for which $B(1) = 1$ and $B(id) = A(g)$, become the results of Theorem 2.1. Hence, these results are equivalent.

Corollary 2.2. Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space, and let $g : \Omega \rightarrow [m, M]$ ($-\infty < m < M < \infty$) be a measurable function. Then for any contin-



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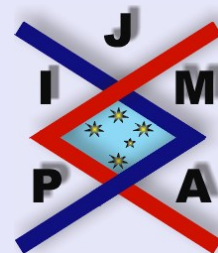
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uous convex function $\varphi : [m, M] \rightarrow \mathbb{R}$,

$$\begin{aligned}\varphi\left(m + M - \int_{\Omega} g d\mu\right) &\leq \int_{\Omega} \varphi(m + M - g) d\mu \\ &\leq \frac{M - \int_{\Omega} g d\mu}{M - m} \cdot \varphi(M) + \frac{\int_{\Omega} g d\mu - m}{M - m} \cdot \varphi(m) \\ &\leq \varphi(m) + \varphi(M) - \int_{\Omega} \varphi(g) d\mu.\end{aligned}$$

Proof. This is a special case of Theorem 2.1 for the functional A defined on class $L^1(\mu)$ as $A(g) = \int_{\Omega} g d\mu$. \square



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3. Some Applications

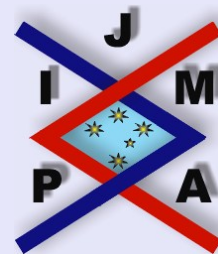
3.1. Generalized Power Means

Throughout this subsection we suppose that:

- (i) L is a linear class having properties $L1, L2$ on a nonempty set E .
- (ii) A is an isotonic linear functional on L such that $A(1) = 1$.
- (iii) $g \in L$ is a function of E to $[m, M]$ ($-\infty < m < M < \infty$) such that all of the following expressions are well defined.

From (iii) it follows especially that $0 < m < M < \infty$, and we define, for any $r, s \in \mathbb{R}$,

$$Q(r, g) := \begin{cases} [m^r + M^r - A(g^r)]^{\frac{1}{r}}, & r \neq 0 \\ \frac{mM}{\exp(A(\log g))}, & r = 0, \end{cases}$$
$$R(r, s, g) := \begin{cases} \left[A \left([m^r + M^r - g^r]^{\frac{s}{r}} \right) \right]^{\frac{1}{s}}, & r \neq 0, s \neq 0 \\ \exp \left(A \left(\log [m^r + M^r - A(g^r)]^{\frac{1}{r}} \right) \right), & r \neq 0, s = 0 \\ \left[A \left(\left(\frac{mM}{g} \right)^s \right) \right]^{\frac{1}{s}}, & r = 0, s \neq 0 \\ \exp \left(A \left(\log \frac{mM}{g} \right) \right), & r = s = 0, \end{cases}$$



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and

$$S(r, s, g) := \begin{cases} \left[\frac{M^r - A(g^r)}{M^r - m^r} \cdot M^s + \frac{A(g^r) - m^r}{M^r - m^r} \cdot m^s \right]^{\frac{1}{s}}, & r \neq 0, s \neq 0 \\ \exp \left(\frac{M^r - A(g^r)}{M^r - m^r} \cdot \log M + \frac{A(g^r) - m^r}{M^r - m^r} \cdot \log m \right), & r \neq 0, s = 0 \\ \left[\frac{\log M - A(\log g)}{\log M - \log m} \cdot M^s + \frac{A(\log g) - \log m}{\log M - \log m} \cdot m^s \right]^{\frac{1}{s}}, & r = 0, s \neq 0 \\ \exp \left(\frac{\log M - A(\log g)}{\log M - \log m} \cdot \log M + \frac{A(\log g) - \log m}{\log M - \log m} \cdot \log m \right), & r = s = 0. \end{cases}$$

In [2] Gavrea proved the following result:

“If $r, s \in \mathbb{R}$ such that $r \leq s$, then for every monotone positive function $g \in C[a, b]$,

$$\tilde{Q}(r, g) \leq \tilde{Q}(s, g),$$

where

$$\tilde{Q}(r, g) = \begin{cases} [g^r(a) + g^r(b) - M^r(r, g)]^{\frac{1}{r}} & r \neq 0 \\ \frac{g(a)g(b)}{\exp(A(\log g))} & r = 0 \end{cases},$$

and $M(r, g)$ is power mean of order r .”

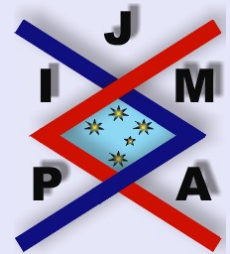
The following is an extension to Gavrea’s result.

Theorem 3.1. *If $r, s \in \mathbb{R}$ and $r \leq s$, then*

$$Q(r, g) \leq Q(s, g).$$

Furthermore,

$$(3.1) \quad Q(r, g) \leq R(r, s, g) \leq S(r, s, g) \leq Q(s, g).$$



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Proof. From above, we know that

$$0 < m \leq g \leq M < \infty .$$

STEP 1: Assume $0 < r \leq s$.

In this case, we have

$$0 < m^r \leq g^r \leq M^r < \infty .$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous convex function

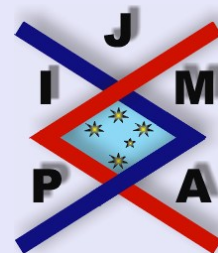
$$\begin{aligned} \varphi : (0, \infty) &\rightarrow \mathbb{R} \\ \varphi(x) &= x^{\frac{s}{r}}, \quad x \in (0, \infty), \end{aligned}$$

we have

$$\begin{aligned} [m^r + M^r - A(g^r)]^{\frac{s}{r}} &\leq A\left((m^r + M^r - g^r)^{\frac{s}{r}}\right) \\ &\leq \frac{M^r - A(g^r)}{M^r - m^r} \cdot M^s + \frac{A(g^r) - m^r}{M^r - m^r} \cdot m^s \\ &\leq m^s + M^s - A(g^s). \end{aligned}$$

Since $s \geq r > 0$, this gives

$$\begin{aligned} [m^r + M^r - A(g^r)]^{\frac{1}{r}} &\leq \left[A\left((m^r + M^r - g^r)^{\frac{s}{r}}\right) \right]^{\frac{1}{s}} \\ &\leq \left[\frac{M^r - A(g^r)}{M^r - m^r} \cdot M^s + \frac{A(g^r) - m^r}{M^r - m^r} \cdot m^s \right]^{\frac{1}{s}} \\ &\leq [m^s + M^s - A(g^s)]^{\frac{1}{s}}, \end{aligned}$$



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or

$$Q(r, g) \leq R(r, s, g) \leq S(r, s, g) \leq Q(s, g).$$

STEP 2: Assume $r \leq s < 0$.

In this case we have

$$0 < M^r \leq g^r \leq m^r < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous concave function (note that $0 < \frac{s}{r} \leq 1$ here)

$$\begin{aligned} \varphi : (0, \infty) &\rightarrow \mathbb{R} \\ \varphi(x) &= x^{\frac{s}{r}}, \quad x \in (0, \infty), \end{aligned}$$

we have

$$\begin{aligned} [M^r + m^r - A(g^r)]^{\frac{s}{r}} &\geq A\left((M^r + m^r - g^r)^{\frac{s}{r}}\right) \\ &\geq \frac{m^r - A(g^r)}{m^r - M^r} \cdot m^s + \frac{A(g^r) - M^r}{m^r - M^r} \cdot M^s \\ &\geq M^s + m^s - A(g^s). \end{aligned}$$

Since $r \leq s < 0$, this gives

$$\begin{aligned} [m^r + M^r - A(g^r)]^{\frac{1}{r}} &\leq \left[A\left((m^r + M^r - g^r)^{\frac{s}{r}}\right) \right]^{\frac{1}{s}} \\ &\leq \left[\frac{M^r - A(g^r)}{M^r - m^r} \cdot M^s + \frac{A(g^r) - m^r}{M^r - m^r} \cdot m^s \right]^{\frac{1}{s}} \\ &\leq [m^s + M^s - A(g^s)]^{\frac{1}{s}}, \end{aligned}$$



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or

$$Q(r, g) \leq R(r, s, g) \leq S(r, s, g) \leq Q(s, g).$$

STEP 3: Assume $r < 0 < s$.

In this case we have

$$0 < M^r \leq g^r \leq m^r < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous convex function (note that $\frac{s}{r} < 0$ here)

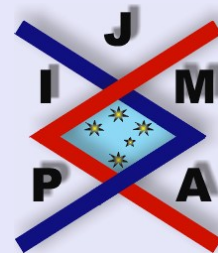
$$\begin{aligned} \varphi : (0, \infty) &\rightarrow \mathbb{R} \\ \varphi(x) &= x^{\frac{s}{r}}, \quad x \in (0, \infty), \end{aligned}$$

we have

$$\begin{aligned} [M^r + m^r - A(g^r)]^{\frac{s}{r}} &\leq A\left((M^r + m^r - g^r)^{\frac{s}{r}}\right) \\ &\leq \frac{m^r - A(g^r)}{m^r - M^r} \cdot m^s + \frac{A(g^r) - M^r}{m^r - M^r} \cdot M^s \\ &\leq M^s + m^s - A(g^s). \end{aligned}$$

Since $r < 0 < s$, this gives

$$\begin{aligned} [m^r + M^r - A(g^r)]^{\frac{1}{r}} &\leq \left[A\left((m^r + M^r - g^r)^{\frac{s}{r}}\right) \right]^{\frac{1}{s}} \\ &\leq \left[\frac{M^r - A(g^r)}{M^r - m^r} \cdot M^s + \frac{A(g^r) - m^r}{M^r - m^r} \cdot m^s \right]^{\frac{1}{s}} \\ &\leq [m^s + M^s - A(g^s)]^{\frac{1}{s}}, \end{aligned}$$



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or

$$Q(r, g) \leq R(r, s, g) \leq S(r, s, g) \leq Q(s, g).$$

STEP 4: Assume $r < 0, s = 0$.

In this case we have

$$0 < M^r \leq g^r \leq m^r < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous convex function

$$\begin{aligned} \varphi : (0, \infty) &\rightarrow \mathbb{R} \\ \varphi(x) &= \frac{1}{r} \log x, \quad x \in (0, \infty), \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{r} \log (M^r + m^r - A(g^r)) &\leq A\left(\frac{1}{r} \log (M^r + m^r - g^r)\right) \\ &\leq \frac{m^r - A(g^r)}{m^r - M^r} \cdot \frac{1}{r} \log m^r + \frac{A(g^r) - M^r}{m^r - M^r} \cdot \frac{1}{r} \log M^r \\ &\leq \frac{1}{r} \log M^r + \frac{1}{r} \log m^r - A\left(\frac{1}{r} \log g^r\right), \end{aligned}$$

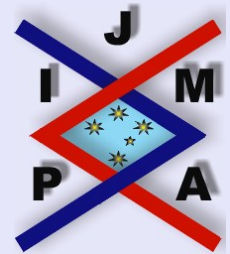
or

$$\log Q(r, g) \leq \log R(r, 0, g) \leq \log S(r, 0, g) \leq \log Q(0, g).$$

Hence

$$Q(r, g) \leq R(r, 0, g) \leq S(r, 0, g) \leq Q(0, g).$$

STEP 5: Assume $r = 0, s > 0$.



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In this case we have

$$-\infty < \log m \leq \log g \leq \log M < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous convex function

$$\begin{aligned} \varphi : \quad \mathbb{R} &\rightarrow (0, \infty) \\ \varphi(x) &= \exp(sx), \quad x \in \mathbb{R}, \end{aligned}$$

we have

$$\begin{aligned} &\exp(s(\log m + \log M - A(\log g))) \\ &\leq A(\exp(s(\log m + \log M - \log g))) \\ &\leq \frac{\log M - A(\log g)}{\log M - \log m} \cdot \exp(s \log M) + \frac{A(\log g) - \log m}{\log M - \log m} \cdot \exp(s \log m) \\ &\leq \exp(s \log m) + \exp(s \log M) - A(\exp(s \log g)), \end{aligned}$$

or

$$Q(0, g)^s \leq R(0, s, g)^s \leq S(0, s, g)^s \leq Q(s, g)^s.$$

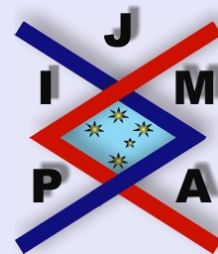
Since $s > 0$, we have

$$Q(0, g) \leq R(0, s, g) \leq S(0, s, g) \leq Q(s, g).$$

This completes the proof of the theorem, since when $r = s = 0$ we have

$$Q(0, g) = R(0, 0, g) = S(0, 0, g).$$

□



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Corollary 3.2. Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space, and let $g : \Omega \rightarrow [m, M]$ ($0 < m < M < \infty$) be a measurable function. Let A be defined as $A(g) = \int_{\Omega} g d\mu$. Then for any continuous convex function $\varphi : [m, M] \rightarrow \mathbb{R}$, and any $r, s \in \mathbb{R}$ with $r \leq s$, (3.1) holds.

3.2. Generalized Means

Let L satisfy properties $L1, L2$ on a nonempty set E , and let A be an isotonic linear functional on L with $A(1) = 1$. Let ψ, χ be continuous and strictly monotonic functions on an interval $I = [m, M]$ ($-\infty < m < M < \infty$). Then for any $g \in L$ such that $\psi(g), \chi(g), \chi(\psi^{-1}(\psi(m) + \psi(M) - \psi(g))) \in L$ (so that $m \leq g(t) \leq M$ for all $t \in E$), we define the *generalized mean of g* with respect to the functional A and the function ψ by (see for example [5, p. 107])

$$M_{\psi}(g, A) = \psi^{-1}(A(\psi(g))).$$

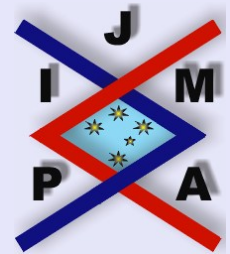
Observe that if $\psi(m) \leq \psi(g) \leq \psi(M)$ for $t \in E$, then by the isotonic character of A , we have $\psi(m) \leq A(\psi(g)) \leq \psi(M)$, so that M_{ψ} is well defined. We further define

$$\widetilde{M}_{\psi}(g, A) = \psi^{-1}(\psi(m) + \psi(M) - A(\psi(g))).$$

From the above observation we know that

$$\psi(m) \leq \psi(m) + \psi(M) - A(\psi(g)) \leq \psi(M)$$

so that \widetilde{M}_{ψ} is also well defined.



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Theorem 3.3. *Under the above hypotheses, we have*

(i) *if either $\chi \circ \psi^{-1}$ is convex and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is concave and χ is strictly decreasing, then*

$$(3.2) \quad \widetilde{M}_\psi(g, A) \leq \widetilde{M}_\chi(g, A).$$

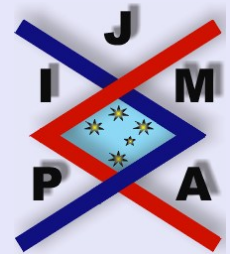
In fact, to be more specific we have the following series of inequalities

$$(3.3) \quad \begin{aligned} \widetilde{M}_\psi(g, A) &\leq \chi^{-1} \left(A \left(\chi \left(\psi^{-1} \left(\psi(m) + \psi(M) - \psi(g) \right) \right) \right) \right) \\ &\leq \chi^{-1} \left(\frac{\psi(M) - A(\psi(g))}{\psi(M) - \psi(m)} \cdot \chi(M) \right. \\ &\quad \left. + \frac{A(\psi(g)) - \psi(m)}{\psi(M) - \psi(m)} \cdot \chi(m) \right) \\ &\leq \widetilde{M}_\chi(g, A); \end{aligned}$$

(ii) *if either $\chi \circ \psi^{-1}$ is concave and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is convex and χ is strictly decreasing, then the reverse inequalities hold.*

Proof. Since ψ is strictly monotonic and $-\infty < m \leq g(t) \leq M < \infty$, we have $-\infty < \psi(m) \leq \psi(g) \leq \psi(M) < \infty$, or $-\infty < \psi(M) \leq \psi(g) \leq \psi(m) < \infty$.

Suppose that $\chi \circ \psi^{-1}$ is convex. Letting $\varphi = \chi \circ \psi^{-1}$ in Theorem 2.1 we



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obtain

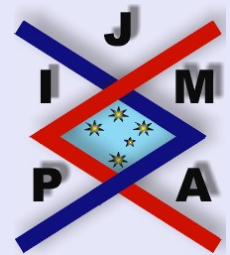
$$\begin{aligned}
 & (\chi \circ \psi^{-1}) (\psi(m) + \psi(M) - A(\psi(g))) \\
 & \leq A((\chi \circ \psi^{-1}) (\psi(m) + \psi(M) - \psi(g))) \\
 & \leq \frac{\psi(M) - A(\psi(g))}{\psi(M) - \psi(m)} \cdot (\chi \circ \psi^{-1}) (\psi(M)) \\
 & \quad + \frac{A(\psi(g)) - \psi(m)}{\psi(M) - \psi(m)} \cdot (\chi \circ \psi^{-1}) (\psi(m)) \\
 & \leq (\chi \circ \psi^{-1}) (\psi(m)) + (\chi \circ \psi^{-1}) (\psi(M)) - A((\chi \circ \psi^{-1}) (\psi(g))),
 \end{aligned}$$

or

$$\begin{aligned}
 & \chi(\psi^{-1}(\psi(m) + \psi(M) - A(\psi(g)))) \\
 (3.4) \quad & \leq A(\chi(\psi^{-1}(\psi(m) + \psi(M) - \psi(g)))) \\
 & \leq \frac{\psi(M) - A(\psi(g))}{\psi(M) - \psi(m)} \cdot \chi(M) + \frac{A(\psi(g)) - \psi(m)}{\psi(M) - \psi(m)} \cdot \chi(m) \\
 & \leq \chi(m) + \chi(M) - A(\chi(g)).
 \end{aligned}$$

If $\chi \circ \psi^{-1}$ is concave we have the reverse of inequalities (3.4).

If χ is strictly increasing, then the inverse function χ^{-1} is also strictly increasing, so that (3.4) implies (3.3). If χ is strictly decreasing, then the inverse function χ^{-1} is also strictly decreasing, so in that case the reverse of (3.4) implies (3.3). Analogously, we get the reverse of (3.3) in the cases when $\chi \circ \psi^{-1}$ is convex and χ is strictly decreasing, or $\chi \circ \psi^{-1}$ is concave and χ is strictly increasing. \square



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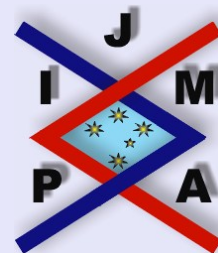
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Remark 3. *If we let*

$$\psi(g) = \begin{cases} g^r, & r \neq 0 \\ \log g, & r = 0 \end{cases} \quad \text{and} \quad \chi(g) = \begin{cases} g^s, & r \neq 0 \\ \log g, & r = 0 \end{cases},$$

then Theorem 3.3 reduces to Theorem 3.1.



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