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A VARIANT OF JESSEN'S INEQUALITY AND GENERALIZED MEANS

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Abstract

In this paper we give a variant of Jessen's inequality for isotonic linear functionals. Our results generalize some recent results of Gavrea. We also give comparison theorems for generalized means.

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Contents

1. Introduction

Let E be a nonempty set and L be a linear class of real valued functions $f : E \to \mathbb{R}$ having the properties:

$L1$: $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

$L2$: $1 \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$.

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having properties:

$A1$: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L$, $\alpha, \beta \in \mathbb{R}$ (A is linear);

$A2$: $f \in L$, $f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$ (A is isotonic).

The following result is Jessen's generalization of the well known Jensen's inequality for convex functions [3] (see also [5, p. 47]):

Theorem 1.1. *Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let φ be a continuous convex function on an interval $I \subset \mathbb{R}$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in L$ we have $A(g) \in I$ and*

$$\varphi(A(g)) \leq A(\varphi(g)).$$

Similar to Jensen's inequality, Jessen's inequality has a converse [1] (see also [5, p. 98]):

Theorem 1.2. *Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let φ be a convex function on an interval $I = [m, M]$ $(-\infty < m < M < \infty)$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in L$ (so that $m \leq g(t) \leq M$ for all $t \in E$), we have*

$$A(\varphi(g)) \leq \frac{M - A(g)}{M - m} \cdot \varphi(m) + \frac{A(g) - m}{M - m} \cdot \varphi(M).$$

Recently I. Gavrea [2] has obtained the following result which is in connection with Mercer's variant of Jensen's inequality [4]:

Theorem 1.3. *Let A be an isotonic linear functional defined on $C[a,b]$ such that $A(1)=1$. Then for any convex function φ on $[a,b]$,*

$$\begin{aligned}\varphi(a+b-a_1) &\le A(\psi)\\ &\le \varphi(a)+\varphi(b)-\varphi(a)\frac{b-a_1}{b-a}-\varphi(b)\frac{a_1-a}{b-a}\\ &\le \varphi(a)+\varphi(b)-A(\varphi),\end{aligned}$$

where $\psi(t)=\varphi(a+b-t)$ and $a_1=A(id)$.

Remark 1. *Although it is not explicitly stated above, it is obvious that function φ needs to be continuous on $[a,b]$.*

In Section 2 we give the main result of this paper which is an extension of Theorem 1.3 on a linear class L satisfying properties $L1, L2$. In Section 3 we use that result to prove the monotonicity property of generalized power means. We also consider in the same way generalized means with respect to isotonic functionals.

2. Main Result

Theorem 2.1. *Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let φ be a convex function on an interval $I = [m, M]$ $(-\infty < m < M < \infty)$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g), \varphi(m + M - g) \in L$ (so that $m \leq g(t) \leq M$ for all $t \in E$), we have the following variant of Jessen's inequality*

$$\varphi(m + M - A(g)) \leq \varphi(m) + \varphi(M) - A(\varphi(g)). \tag{2.1}$$

In fact, to be more specific, we have the following series of inequalities

$$\begin{aligned} \varphi(m + M - A(g)) &\leq A(\varphi(m + M - g)) \\ &\leq \frac{M - A(g)}{M - m} \cdot \varphi(M) + \frac{A(g) - m}{M - m} \cdot \varphi(m) \\ &\leq \varphi(m) + \varphi(M) - A(\varphi(g)). \end{aligned} \tag{2.2}$$

If the function φ is concave, inequalities (2.1) and (2.2) are reversed.

Proof. Since φ is continuous and convex, the same is also true for the function

$$\psi : [m, M] \to \mathbb{R}$$

defined by

$$\psi(t) = \varphi(m + M - t), \quad t \in [m, M].$$

By Theorem 1.1,

$$\psi(A(g)) \leq A(\psi(g)),$$

i.e.,

$$\varphi\left(m+M-A\left(g\right)\right)\le A\left(\varphi(m+M-g)\right).$$

Applying Theorem 1.2 to ψ and then to φ, we have

$$\begin{aligned}
&A\left(\varphi(m+M-g)\right)\\
&\quad\le\frac{M-A\left(g\right)}{M-m}\cdot\psi\left(m\right)+\frac{A\left(g\right)-m}{M-m}\cdot\psi\left(M\right)\\
&\quad=\frac{M-A\left(g\right)}{M-m}\cdot\varphi\left(M\right)+\frac{A\left(g\right)-m}{M-m}\cdot\varphi\left(m\right)\\
&\quad=\varphi\left(m\right)+\varphi\left(M\right)-\left[\frac{M-A\left(g\right)}{M-m}\cdot\varphi\left(m\right)+\frac{A\left(g\right)-m}{M-m}\cdot\varphi\left(M\right)\right]\\
&\quad\le\varphi\left(m\right)+\varphi\left(M\right)-A\left(\varphi\left(g\right)\right).
\end{aligned}$$

The last statement follows immediately from the facts that if φ is concave then $-\varphi$ is convex, and that A is linear on L. □

Remark 2. *In Theorem 2.1, taking $L=C\left[a,b\right]$ and $g=id$ (so that $m=a$ and $M=b$), we obtain the results of Theorem 1.3. On the other hand, the results of Theorem 1.3 for the functional B defined on L by $B(\varphi)=A(\varphi(g))$, for which $B(1)=1$ and $B(id)=A(g)$, become the results of Theorem 2.1. Hence, these results are equivalent.*

Corollary 2.2. *Let $(\Omega,\mathcal{A},\mu)$ be a probability measure space, and let $g:\Omega\to[m,M]$ $(-\infty<m<M<\infty)$ be a measurable function. Then for any contin-*

uous convex function $\varphi : [m, M] \to \mathbb{R}$,

$$\varphi\left(m + M - \int_\Omega g d\mu\right) \leq \int_\Omega \varphi\left(m + M - g\right) d\mu$$
$$\leq \frac{M - \int_\Omega g d\mu}{M - m} \cdot \varphi\left(M\right) + \frac{\int_\Omega g d\mu - m}{M - m} \cdot \varphi\left(m\right)$$
$$\leq \varphi\left(m\right) + \varphi\left(M\right) - \int_\Omega \varphi\left(g\right) d\mu.$$

Proof. This is a special case of Theorem 2.1 for the functional A defined on class $L^1\left(\mu\right)$ as $A(g) = \int_\Omega g d\mu$. □

3. Some Applications

3.1. Generalized Power Means

Throughout this subsection we suppose that:

(i) L is a linear class having properties $L1$, $L2$ on a nonempty set E.

(ii) A is an isotonic linear functional on L such that $A(1) = 1$.

(iii) $g \in L$ is a function of E to $[m, M]$ $(-\infty < m < M < \infty)$ such that all of the following expressions are well defined.

From (iii) it follows especially that $0 < m < M < \infty$, and we define, for any $r, s \in \mathbb{R}$,

$$Q(r,g) := \begin{cases} \left[m^r + M^r - A(g^r)\right]^{\frac{1}{r}}, & r \neq 0 \\ \dfrac{mM}{\exp(A(\log g))}, & r = 0, \end{cases}$$

$$R(r,s,g) := \begin{cases} \left[A\left(\left[m^r + M^r - g^r\right]^{\frac{s}{r}}\right)\right]^{\frac{1}{s}}, & r \neq 0,\ s \neq 0 \\ \exp\left(A\left(\log\left[m^r + M^r - A(g^r)\right]^{\frac{1}{r}}\right)\right), & r \neq 0,\ s = 0 \\ \left[A\left(\left(\frac{mM}{g}\right)^s\right)\right]^{\frac{1}{s}}, & r = 0,\ s \neq 0 \\ \exp\left(A\left(\log\frac{mM}{g}\right)\right), & r = s = 0, \end{cases}$$

and

$$S(r,s,g) := \begin{cases} \left[\frac{M^r - A(g^r)}{M^r - m^r} \cdot M^s + \frac{A(g^r) - m^r}{M^r - m^r} \cdot m^s\right]^{\frac{1}{s}}, & r \neq 0,\ s \neq 0 \\ \exp\left(\frac{M^r - A(g^r)}{M^r - m^r} \cdot \log M + \frac{A(g^r) - m^r}{M^r - m^r} \cdot \log m\right), & r \neq 0,\ s = 0 \\ \left[\frac{\log M - A(\log g)}{\log M - \log m} \cdot M^s + \frac{A(\log g) - \log m}{\log M - \log m} \cdot m^s\right]^{\frac{1}{s}}, & r = 0,\ s \neq 0 \\ \exp\left(\frac{\log M - A(\log g)}{\log M - \log m} \cdot \log M + \frac{A(\log g) - \log m}{\log M - \log m} \cdot \log m\right), & r = s = 0. \end{cases}$$

In [2] Gavrea proved the following result:

"If $r, s \in \mathbb{R}$ such that $r \leq s$, then for every monotone positive function $g \in C[a,b]$,

$$\widetilde{Q}(r,g) \leq \widetilde{Q}(s,g),$$

where

$$\widetilde{Q}(r,g) = \begin{cases} \left[g^r(a) + g^r(b) - M^r(r,g)\right]^{\frac{1}{r}} & r \neq 0 \\ \frac{g(a)g(b)}{\exp(A(\log g))} & r = 0 \end{cases},$$

and $M(r,g)$ is power mean of order r."

The following is an extension to Gavrea's result.

Theorem 3.1. *If $r, s \in \mathbb{R}$ and $r \leq s$, then*

$$Q(r,g) \leq Q(s,g).$$

Furthermore,

(3.1) $$Q(r,g) \leq R(r,s,g) \leq S(r,s,g) \leq Q(s,g).$$

Proof. From above, we know that

$$0 < m \leq g \leq M < \infty \, .$$

STEP 1: Assume $0 < r \leq s$.
In this case, we have

$$0 < m^r \leq g^r \leq M^r < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous convex function

$$\begin{aligned} \varphi : \quad & (0,\infty) \to \mathbb{R} \\ & \varphi(x) = x^{\frac{s}{r}} \, , \quad x \in (0,\infty) \, , \end{aligned}$$

we have

$$\begin{aligned} \left[m^r + M^r - A\left(g^r\right)\right]^{\frac{s}{r}} &\leq A\left(\left(m^r + M^r - g^r\right)^{\frac{s}{r}}\right) \\ &\leq \frac{M^r - A\left(g^r\right)}{M^r - m^r} \cdot M^s + \frac{A\left(g^r\right) - m^r}{M^r - m^r} \cdot m^s \\ &\leq m^s + M^s - A\left(g^s\right) . \end{aligned}$$

Since $s \geq r > 0$, this gives

$$\begin{aligned} \left[m^r + M^r - A(g^r)\right]^{\frac{1}{r}} &\leq \left[A\left(\left(m^r + M^r - g^r\right)^{\frac{s}{r}}\right)\right]^{\frac{1}{s}} \\ &\leq \left[\frac{M^r - A\left(g^r\right)}{M^r - m^r} \cdot M^s + \frac{A\left(g^r\right) - m^r}{M^r - m^r} \cdot m^s\right]^{\frac{1}{s}} \\ &\leq \left[m^s + M^s - A\left(g^s\right)\right]^{\frac{1}{s}} , \end{aligned}$$

or

$$Q(r,g) \leq R(r,s,g) \leq S(r,s,g) \leq Q(s,g).$$

STEP 2: Assume $r \leq s < 0$.
In this case we have

$$0 < M^r \leq g^r \leq m^r < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous concave function (note that $0 < \frac{s}{r} \leq 1$ here)

$$\begin{aligned} \varphi : \quad & (0,\infty) \to \mathbb{R} \\ & \varphi(x) = x^{\frac{s}{r}}, \quad x \in (0,\infty), \end{aligned}$$

we have

$$\begin{aligned} [M^r + m^r - A(g^r)]^{\frac{s}{r}} &\geq A\left((M^r + m^r - g^r)^{\frac{s}{r}}\right) \\ &\geq \frac{m^r - A(g^r)}{m^r - M^r} \cdot m^s + \frac{A(g^r) - M^r}{m^r - M^r} \cdot M^s \\ &\geq M^s + m^s - A(g^s). \end{aligned}$$

Since $r \leq s < 0$, this gives

$$\begin{aligned} [m^r + M^r - A(g^r)]^{\frac{1}{r}} &\leq \left[A\left((m^r + M^r - g^r)^{\frac{s}{r}}\right)\right]^{\frac{1}{s}} \\ &\leq \left[\frac{M^r - A(g^r)}{M^r - m^r} \cdot M^s + \frac{A(g^r) - m^r}{M^r - m^r} \cdot m^s\right]^{\frac{1}{s}} \\ &\leq [m^s + M^s - A(g^s)]^{\frac{1}{s}}, \end{aligned}$$

or

$$Q(r,g) \leq R(r,s,g) \leq S(r,s,g) \leq Q(s,g).$$

STEP 3: Assume $r < 0 < s$.
In this case we have

$$0 < M^r \leq g^r \leq m^r < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous convex function (note that $\frac{s}{r} < 0$ here)

$$\begin{aligned} \varphi : \quad & (0,\infty) \to \mathbb{R} \\ & \varphi(x) = x^{\frac{s}{r}}, \quad x \in (0,\infty), \end{aligned}$$

we have

$$\begin{aligned} [M^r + m^r - A(g^r)]^{\frac{s}{r}} &\leq A\left((M^r + m^r - g^r)^{\frac{s}{r}}\right) \\ &\leq \frac{m^r - A(g^r)}{m^r - M^r} \cdot m^s + \frac{A(g^r) - M^r}{m^r - M^r} \cdot M^s \\ &\leq M^s + m^s - A(g^s). \end{aligned}$$

Since $r < 0 < s$, this gives

$$\begin{aligned} [m^r + M^r - A(g^r)]^{\frac{1}{r}} &\leq \left[A\left((m^r + M^r - g^r)^{\frac{s}{r}}\right)\right]^{\frac{1}{s}} \\ &\leq \left[\frac{M^r - A(g^r)}{M^r - m^r} \cdot M^s + \frac{A(g^r) - m^r}{M^r - m^r} \cdot m^s\right]^{\frac{1}{s}} \\ &\leq [m^s + M^s - A(g^s)]^{\frac{1}{s}}, \end{aligned}$$

or

$$Q(r,g) \le R(r,s,g) \le S(r,s,g) \le Q(s,g).$$

STEP 4: Assume $r<0$, $s=0$.

In this case we have

$$0 < M^r \le g^r \le m^r < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous convex function

$$\varphi : \quad (0,\infty) \to \mathbb{R}$$
$$\varphi(x) = \tfrac{1}{r}\log x\,, \quad x \in (0,\infty)\,,$$

we have

$$\begin{aligned}
\frac{1}{r}\log\left(M^r+m^r-A(g^r)\right) &\le A\left(\frac{1}{r}\log\left(M^r+m^r-g^r\right)\right)\\
&\le \frac{m^r-A(g^r)}{m^r-M^r}\cdot\frac{1}{r}\log m^r + \frac{A(g^r)-M^r}{m^r-M^r}\cdot\frac{1}{r}\log M^r\\
&\le \frac{1}{r}\log M^r + \frac{1}{r}\log m^r - A\left(\frac{1}{r}\log g^r\right),
\end{aligned}$$

or

$$\log Q(r,g) \le \log R(r,0,g) \le \log S(r,0,g) \le \log Q(0,g).$$

Hence

$$Q(r,g) \le R(r,0,g) \le S(r,0,g) \le Q(0,g).$$

STEP 5: Assume $r=0$, $s>0$.

In this case we have

$$-\infty < \log m \leq \log g \leq \log M < \infty.$$

Applying Theorem 2.1 or more precisely inequality (2.2) to the continuous convex function

$$\begin{aligned} \varphi : \quad & \mathbb{R} \to (0, \infty) \\ \varphi(x) &= \exp(sx) \ , \quad x \in \mathbb{R} \ , \end{aligned}$$

we have

$$\begin{aligned} &\exp\left(s\left(\log m + \log M - A\left(\log g\right)\right)\right) \\ &\quad \leq A\left(\exp\left(s\left(\log m + \log M - \log g\right)\right)\right) \\ &\quad \leq \frac{\log M - A\left(\log g\right)}{\log M - \log m} \cdot \exp\left(s \log M\right) + \frac{A\left(\log g\right) - \log m}{\log M - \log m} \cdot \exp\left(s \log m\right) \\ &\quad \leq \exp\left(s \log m\right) + \exp s\left(\log M\right) - A\left(\exp\left(s \log g\right)\right), \end{aligned}$$

or

$$Q(0,g)^s \leq R\left(0,s,g\right)^s \leq S\left(0,s,g\right)^s \leq Q(s,g)^s.$$

Since $s > 0$, we have

$$Q(0,g) \leq R\left(0,s,g\right) \leq S\left(0,s,g\right) \leq Q(s,g).$$

This completes the proof of the theorem, since when $r = s = 0$ we have

$$Q(0,g) = R\left(0,0,g\right) = S\left(0,0,g\right).$$

□

Corollary 3.2. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a probability measure space, and let* $g : \Omega \to [m, M]$ $(0 < m < M < \infty)$ *be a measurable function. Let* A *be defined as* $A(g) = \int_\Omega g d\mu$. *Then for any continuous convex function* $\varphi : [m, M] \to \mathbb{R}$, *and any* $r, s \in \mathbb{R}$ *with* $r \leq s$, (3.1) *holds.*

3.2. Generalized Means

Let L satisfy properties $L1$, $L2$ on a nonempty set E, and let A be an isotonic linear functional on L with $A(1) = 1$. Let ψ, χ be continuous and strictly monotonic functions on an interval $I = [m, M]$ $(-\infty < m < M < \infty)$. Then for any $g \in L$ such that $\psi(g), \chi(g), \chi(\psi^{-1}(\psi(m) + \psi(M) - \psi(g))) \in L$ (so that $m \leq g(t) \leq M$ for all $t \in E$), we define the *generalized mean of* g with respect to the functional A and the function ψ by (see for example [5, p. 107])

$$M_\psi(g, A) = \psi^{-1}(A(\psi(g))).$$

Observe that if $\psi(m) \leq \psi(g) \leq \psi(M)$ for $t \in E$, then by the isotonic character of A, we have $\psi(m) \leq A(\psi(g)) \leq \psi(M)$, so that M_ψ is well defined. We further define

$$\widetilde{M}_\psi(g, A) = \psi^{-1}(\psi(m) + \psi(M) - A(\psi(g))).$$

From the above observation we know that

$$\psi(m) \leq \psi(m) + \psi(M) - A(\psi(g)) \leq \psi(M)$$

so that $\widetilde{M}_\psi$ is also well defined.

Theorem 3.3. *Under the above hypotheses, we have*

(i) *if either $\chi \circ \psi^{-1}$ is convex and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is concave and χ is strictly decreasing, then*

$$\widetilde{M}_{\psi}(g, A) \leq \widetilde{M}_{\chi}(g, A). \tag{3.2}$$

In fact, to be more specific we have the following series of inequalities

$$\begin{aligned} \widetilde{M}_{\psi}(g, A) &\leq \chi^{-1}\left(A\left(\chi\left(\psi^{-1}\left(\psi(m)+\psi(M)-\psi(g)\right)\right)\right)\right) \\ &\leq \chi^{-1}\left(\frac{\psi(M)-A(\psi(g))}{\psi(M)-\psi(m)} \cdot \chi(M)\right. \\ &\qquad \left. + \frac{A(\psi(g))-\psi(m)}{\psi(M)-\psi(m)} \cdot \chi(m)\right) \\ &\leq \widetilde{M}_{\chi}(g, A); \end{aligned} \tag{3.3}$$

(ii) *if either $\chi \circ \psi^{-1}$ is concave and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is convex and χ is strictly decreasing, then the reverse inequalities hold.*

Proof. Since ψ is strictly monotonic and $-\infty < m \leq g(t) \leq M < \infty$, we have $-\infty < \psi(m) \leq \psi(g) \leq \psi(M) < \infty$, or $-\infty < \psi(M) \leq \psi(g) \leq \psi(m) < \infty$.

Suppose that $\chi \circ \psi^{-1}$ is convex. Letting $\varphi = \chi \circ \psi^{-1}$ in Theorem 2.1 we obtain

$$
\begin{aligned}
\left(\chi\circ\psi^{-1}\right)\left(\psi(m)+\psi(M)-A\left(\psi(g)\right)\right)&\\
&\le A\left(\left(\chi\circ\psi^{-1}\right)\left(\psi\left(m\right)+\psi\left(M\right)-\psi\left(g\right)\right)\right)\\
&\le \frac{\psi\left(M\right)-A\left(\psi\left(g\right)\right)}{\psi\left(M\right)-\psi\left(m\right)}\cdot\left(\chi\circ\psi^{-1}\right)\left(\psi\left(M\right)\right)\\
&\qquad+\frac{A\left(\psi\left(g\right)\right)-\psi\left(m\right)}{\psi\left(M\right)-\psi\left(m\right)}\cdot\left(\chi\circ\psi^{-1}\right)\left(\psi\left(m\right)\right)\\
&\le \left(\chi\circ\psi^{-1}\right)\left(\psi\left(m\right)\right)+\left(\chi\circ\psi^{-1}\right)\left(\psi\left(M\right)\right)-A\left(\left(\chi\circ\psi^{-1}\right)\left(\psi\left(g\right)\right)\right),
\end{aligned}
$$

or

$$
\begin{aligned}
\chi\left(\psi^{-1}\left(\psi(m)+\psi(M)-A\left(\psi(g)\right)\right)\right)&\\
&\le A\left(\chi\left(\psi^{-1}\left(\psi\left(m\right)+\psi\left(M\right)-\psi\left(g\right)\right)\right)\right)\\
&\le \frac{\psi\left(M\right)-A\left(\psi\left(g\right)\right)}{\psi\left(M\right)-\psi\left(m\right)}\cdot\chi\left(M\right)+\frac{A\left(\psi\left(g\right)\right)-\psi\left(m\right)}{\psi\left(M\right)-\psi\left(m\right)}\cdot\chi\left(m\right)\\
&\le \chi\left(m\right)+\chi\left(M\right)-A\left(\chi\left(g\right)\right).
\end{aligned}
\tag{3.4}
$$

If $\chi\circ\psi^{-1}$ is concave we have the reverse of inequalities (3.4).

If χ is strictly increasing, then the inverse function χ^{-1} is also strictly increasing, so that (3.4) implies (3.3). If χ is strictly decreasing, then the inverse function χ^{-1} is also strictly decreasing, so in that case the reverse of (3.4) implies (3.3). Analogously, we get the reverse of (3.3) in the cases when $\chi\circ\psi^{-1}$ is convex and χ is strictly decreasing, or $\chi\circ\psi^{-1}$ is concave and χ is strictly increasing. □

Remark 3. *If we let*

$$\psi(g) = \begin{cases} g^r, & r \neq 0 \\ \log g, & r = 0 \end{cases} \quad \textit{and} \quad \chi(g) = \begin{cases} g^s, & r \neq 0 \\ \log g, & r = 0 \end{cases},$$

then Theorem 3.3 reduces to Theorem 3.1.

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