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ON AN INEQUALITY IN NONLINEAR THERMOELASTICITY

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ABSTRACT. This paper deals with an integral inequality which arises in numerical analysis of the Lax – Friedrichs scheme for the elastodynamics system. It is obtained as a consequence of a more general inequality.

Key words and phrases: Integral inequality, Elastodynamics, Lax – Friedrichs scheme.

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1. INTRODUCTION

Let us consider the following problem:

Theorem 1.1. *Let* $a, b \in \mathbb{R},\ a < 0,\ b > 0$ *and* $f \in C[a,\, b]$*, such that:*

$$0 < f \leq 1 \ \text{ on } \ [a,\, b], \tag{1.1}$$

$$f \ \textit{ is decreasing on } [a,\, 0], \tag{1.2}$$

$$\int_a^0 f\, dx = \int_0^b f\, dx. \tag{1.3}$$

then

$$\int_a^b f^2\, dx \leq 2 \int_a^{\frac{a+b}{2}} f\, dx. \tag{1.4}$$

As we will see later, Theorem 1.1 is a transformed and slightly generalized form of a problem related to the numerical analysis of a nonlinear system of PDEs. This problem is stated below.

Theorem 1.2. *Suppose that* $\sigma \in C^2(\mathbb{R})$ *satisfies*

$$\sigma'(w) > 0 \quad \textit{for all} \quad w \in \mathbb{R} \tag{1.5}$$

$$w\,\sigma''(w) > 0 \quad \textit{for all} \quad w \in \mathbb{R}\backslash\{0\}. \tag{1.6}$$

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Assume further that for $w_1, w_2 \in [-1, \infty),\ w_1 < 0,\ w_2 > 0$ *and* $\alpha > 0$*, the conditions*

$$\int_{w_1}^{0} \sqrt{\sigma'}\, ds = \int_{0}^{w_2} \sqrt{\sigma'}\, ds, \tag{1.7}$$

and

$$\alpha\sqrt{\sigma'(w)} \le 1 \quad \textit{for all } w \in [w_1,\, w_2]. \tag{1.8}$$

hold. Then

$$\int_{w_1}^{\frac{w_1+w_2}{2}} \sqrt{\sigma'}\, ds \ge \frac{\alpha}{2}\big[\sigma(w_2) - \sigma(w_1)\big]. \tag{1.9}$$

The main subject of this paper is the inequality (1.9). In the next section we describe the context in which the inequality arises. We start the third section with the proof of Theorem 1.1, then proceed with the proof of Theorem 1.2 and finally conclude the section with two remarks.

2. Lax – Friedrichs Scheme for the Elastodynamics System

The elastodynamics system governs isentropic processes in thermoelastic nonconductors of heat. The Cauchy problem for the underlying system in the one-dimensional case has the form

$$\partial_t w - \partial_x v = 0, \quad \partial_t v - \partial_x \sigma(w) = 0 \ \text{ in } \mathbb{R} \times (0, T), \tag{2.1}$$

$$w(x, 0) = w_0(x), \qquad v(x, 0) = v_0(x) \text{ in } \mathbb{R}, \tag{2.2}$$

where $w : \mathbb{R} \times [0, T) \to [-1, \infty)$ is the strain and $v : \mathbb{R} \times [0, T) \to \mathbb{R}$ is the velocity. In the theory of nonlinear systems of conservation laws this system plays an important role due to its accessability to a detailed mathematical analysis (see [1]). The special feature that renders these equations amenable to analytical treatment is the existence of the so-called compact invariant regions. Invariant regions are sets $S \subset \mathbb{R}^2$ with the following property: if the initial function $u_0 = (w_0,\, v_0)$ takes its values in S, then so does the solution $u = (w,\, v)$ of (2.1), (2.2). It can be shown (see [1]) that for $N > 0$, the sets given by

$$S_N = \{(w, v) \subset [-1, \infty) \times \mathbb{R} \,:\, |y(w, v)| \le N,\ |z(w, v)| \le N\}, \tag{2.3}$$

are invariant for the Cauchy problem (2.1), (2.2), where

$$y(w, v) = -\int_{w_0}^{w} \sqrt{\sigma'(s)}\, ds + v, \qquad z(w, v) = -\int_{w_0}^{w} \sqrt{\sigma'(s)}\, ds - v$$

are the the so-called Riemann invariants.

The Lax – Friedrichs scheme is frequently used as a discretization procedure for systems of conservation laws. In our particular case, the scheme takes the form

$$u_i^{n+1} = u_i^n - \frac{\alpha}{2}\big(f(u_{i+1}^n) - f(u_{i-1}^n)\big) + \frac{1}{2}\big(u_{i-1}^n - 2u_i^n + u_{i+1}^n\big), \tag{2.4}$$

where $\alpha > 0$ is a parameter and $u_i^n = (w_i^n,\, v_i^n)$ for $i \in \mathbb{Z},\ n \in \mathbb{N}$. Here we used $f(u) = (-v, -\sigma(w))$, with $u = (w,\, v)$. For the numerical stability of the Lax – Friedrichs scheme it is crucial that the sets S_N from (2.3) are also invariant for (2.4). That is, if $u_i^n \in S_N$ for all $i \in \mathbb{Z}$, then $u_i^{n+1} \in S_N$ for all $i \in \mathbb{Z}$, provided $\alpha \cdot \sup_{(w,v)\in S_N} \sqrt{\sigma'(w)} \le 1$, (see [3]). Similarly as in [2], the proof of the invariancy is reduced to some problems associated with certain integral inequalities. The problem stated in Theorem 1.2 is one of them.

3. PROOF OF THE INEQUALITIES

Proof of Theorem 1.1. We will consider two cases.

1. Case: $\mathbf{a+b \geq 0}$.

By (1.1) and (1.3), we have

$$\int_a^b f^2\,dx \leq \int_a^b f\,dx = 2\int_a^0 f\,dx \leq 2\int_a^{\frac{a+b}{2}} f\,dx.$$

2. Case: $\mathbf{a+b < 0}$.

First, note that due to (1.1) and (1.3), for every $a' \in [a,\,0]$ there exists a unique $b' \in [0,\,b]$, such that $\int_{a'}^0 f\,dx = \int_0^{b'} f\,dx$. Therefore, one can introduce a function $\varphi : [a,\,0] \to [0,\,b]$ with the property $\int_x^0 f\,ds = \int_0^{\varphi(x)} f\,dx$. Obviously, $\varphi(a) = b$ and $\varphi(0) = 0$. It is a simple matter to prove that φ is differentiable and that for all $x \in [a,\,0]$,

$$f(\varphi(x))\,\varphi'(x) = -f(x). \tag{3.1}$$

We will show that the inequality

$$\int_x^{\varphi(x)} f^2\,ds \leq 2\int_x^{\frac{x+\varphi(x)}{2}} f\,ds, \tag{3.2}$$

holds for all $x \in [a,\,0]$. Then (1.4) will be a consequence of (3.2), when $x = a$.

Let $x \in [a,\,0]$ be arbitrary. If $x + \varphi(x) \geq 0$, then we proceed as in Case 1. Therefore, suppose that

$$x + \varphi(x) < 0. \tag{3.3}$$

Define a function $\psi : [a,\,0] \to \mathbb{R}$ with

$$\psi(x) = 2\int_x^{\frac{x+\varphi(x)}{2}} f\,ds - \int_x^{\varphi(x)} f^2\,ds.$$

From

$$\psi'(x) = (1+\varphi'(x))\,f\left(\frac{x+\varphi(x)}{2}\right) - 2f(x) - f^2(\varphi(x))\,\varphi'^2(x),$$

using (3.1) follows

$$\begin{aligned} f(\varphi(x))\,\psi'(x) = \big[f(\varphi(x)) - f(x)\big]\,f\left(\frac{x+\varphi(x)}{2}\right)& \\ - 2f(x)\,f(\varphi(x)) + f^2(\varphi(x))\,f(x) + f^2(x)\,f(\varphi(x)).& \end{aligned}$$

If $f(\varphi(x)) - f(x) \leq 0$, then obviously $\psi'(x) \leq 0$. Assume now $f(\varphi(x)) - f(x) > 0$. Using the fact that $x \leq 0$, $\varphi(x) \geq 0$ and (3.3), we obtain $0 > \frac{x+\varphi(x)}{2} \geq x$, which together with (1.2) yields, $f\left(\frac{x+\varphi(x)}{2}\right) \leq f(x)$. Hence,

$$\begin{aligned} f(\varphi(x))\,\psi'(x) &\leq \big[f(\varphi(x)) - f(x)\big]\,f(x) - 2f(x)\,f(\varphi(x)) + f^2(\varphi(x))\,f(x) + f^2(x)\,f(\varphi(x)) \\ &= f(x)\,\big[f(x) + f(\varphi(x))\big]\,\big[f(\varphi(x)) - 1\big] \leq 0. \end{aligned}$$

Hence we have shown that $\psi'(x) \leq 0$ for all $x \in [a,\,0]$. Since $\psi(0) = 0$, one concludes that $\psi \geq 0$ on $[a,\,0]$, that is, (3.2) holds. $\square$

Proof of Theorem 1.2. Since $\sigma(w_2) - \sigma(w_1) = \int_{w_1}^{w_2} \sigma' \, ds$, then by multiplying (1.9) by 2α and introducing $f = \alpha\sqrt{\sigma'}$, the inequality (1.9) is transformed into (1.4), with $a = w_1$, $b = w_2$. Due to (1.6), σ' decreases on $[w_1, \, 0]$, so f does on $[a, \, 0]$, as well. The relations (1.5) and (1.8) yield (1.1). The equality (1.7) implies (1.3). Therefore, Theorem 1.1 applies. □

Remark 3.1. Assume that (1.1), (1.2) and (1.3) hold for $f \in C[a, b]$.

(a) The constant $A = 2$ in

$$\int_a^b f^2 \, dx \le A \int_a^{\frac{a+b}{2}} f \, dx \tag{3.4}$$

is optimal in the case $a + b = 0$; indeed, taking $f = 1$ in (3.4), one obtains $A \ge 2$.

(b) It is easy to see that if $p \ge 2$, then the inequality

$$\int_a^b f^p \, dx \le A_p \int_a^{\frac{a+b}{2}} f \, dx \tag{3.5}$$

holds for all $A_p \ge 2$. However, if $1 \le p < 2$, then proceeding similarly as in the proof of Theorem 1.1, one can deduce that (3.5) is satisfied for all $A_p \ge 4$.

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