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INTEGRAL MEANS OF MULTIVALENT FUNCTIONS

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ABSTRACT. For analytic functions $f(z)$ and $g(z)$ which satisfy the subordination $f(z) \prec g(z)$, J. E. Littlewood (*Proc. London Math. Soc.*, **23** (1925), 481–519) has shown some interesting results for integral means of $f(z)$ and $g(z)$. The object of the present paper is to derive some applications of integral means by J.E. Littlewood and show interesting examples for our theorems. We also generalize the results of Owa and Sekine (*J. Math. Anal. Appl.*, **304** (2005), 772–782).

Key words and phrases: Integral means, Multivalent function, Subordination, Starlike, Convex.

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1. INTRODUCTION

Let $\mathcal{A}_{p,n}$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are *analytic* and *multivalent* in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ belonging to $\mathcal{A}_{p,n}$ is called to be *multivalently starlike of order* α in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}) \tag{1.2}$$

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for some $\alpha(0 \leqq \alpha < p)$. A function $f(z) \in \mathcal{A}_{p,n}$ is said to be *multivalently convex of order* α in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}) \tag{1.3}$$

for some $\alpha(0 \leqq \alpha < p)$. We denote by $\mathcal{S}^*_{p,n}(\alpha)$ and $\mathcal{K}_{p,n}(\alpha)$ the classes of functions $f(z) \in \mathcal{A}_{p,n}$ which are multivalently starlike of order α in $\mathbb{U}$ and multivalently convex of order α in $\mathbb{U}$, respectively. We note that

$$f(z) \in \mathcal{K}_{p,n}(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*_{p,n}(\alpha).$$

For functions $f(z)$ belonging to the classes $\mathcal{S}^*_{p,n}(\alpha)$ and $\mathcal{K}_{p,n}(\alpha)$, Owa [4] has shown the following coefficient inequalities.

Theorem 1.1. *If a function $f(z) \in \mathcal{A}_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} (k-\alpha)|a_k| \leqq p - \alpha \tag{1.4}$$

for some α $(0 \leqq \alpha < p)$, then $f(z) \in \mathcal{S}^*_{p,n}(\alpha)$.

Theorem 1.2. *If a function $f(z) \in \mathcal{A}_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} k(k-\alpha)|a_k| \leqq p - \alpha \tag{1.5}$$

for some $\alpha(0 \leqq \alpha < p)$, then $f(z) \in \mathcal{K}_{p,n}(\alpha)$.

For analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, $f(z)$ is said to be *subordinate* to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad \text{(cf. Duren [2])}.$$

To discuss our problems for integral means of multivalent functions, we have to recall here the following result due to Littlewood [3].

Theorem 1.3. *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu}\, d\theta. \tag{1.6}$$

Applying Theorem 1.3 by Littlewood [3], Owa and Sekine [5] have considered some integral means inequalities for certain analytic functions. In the present paper, we discuss the integral means inequalities for multivalent functions which are the generalization of the paper by Owa and Sekine [5].

2. Integral Means Inequalities for $f(z)$ and $g(z)$

In this section, we discuss the integral means inequalities for $f(z) \in \mathcal{A}_{p,n}$ and $g(z)$ defined by

$$g(z) = z^p + b_j z^j + b_{2j-p} z^{2j-p} \quad (j \geqq n + p). \tag{2.1}$$

We first derive

Theorem 2.1. *Let $f(z) \in \mathcal{A}_{p,n}$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies*

$$\sum_{k=p+n}^{\infty} |a_k| \leqq |b_{2j-p}| - |b_j| \quad (|b_j| < |b_{2j-p}|) \tag{2.2}$$

and there exists an analytic function $w(z)$ such that

$$b_{2j-p}\,(w(z))^{2(j-p)} + b_j\,(w(z))^{j-p} - \sum_{k=p+n}^{\infty} a_k z^{k-p} = 0,$$

then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu}\, d\theta.$$

Proof. By putting $z = re^{i\theta}$ $(0 < r < 1)$, we see that

$$\begin{aligned}\int_0^{2\pi} |f(z)|^{\mu}\, d\theta &= \int_0^{2\pi} \left| z^p + \sum_{k=p+n}^{\infty} a_k z^k \right|^{\mu} d\theta \\ &= r^{p\mu} \int_0^{2\pi} \left| 1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} \right|^{\mu} d\theta\end{aligned}$$

and

$$\begin{aligned}\int_0^{2\pi} |g(z)|^{\mu}\, d\theta &= \int_0^{2\pi} \left| z^p + b_j z^j + b_{2j-p} z^{2j-p} \right|^{\mu} d\theta \\ &= r^{p\mu} \int_0^{2\pi} \left| 1 + b_j z^{j-p} + b_{2j-p} z^{2j-2p} \right|^{\mu} d\theta.\end{aligned}$$

Applying Theorem 1.3, we have to show that

$$1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} \prec 1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)}.$$

Let us define the function $w(z)$ by

$$1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} = 1 + b_j\,(w(z))^{j-p} + b_{2j-p}\,(w(z))^{2(j-p)}$$

or by

$$b_{2j-p}\,(w(z))^{2(j-p)} + b_j\,(w(z))^{j-p} - \sum_{k=p+n}^{\infty} a_k z^{k-p} = 0. \tag{2.3}$$

Since, for $z = 0$,

$$(w(0))^{j-p} \left\{ b_{2j-p}\,(w(0))^{j-p} + b_j \right\} = 0,$$

there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$.

Next we prove the analytic function $w(z)$ satisfies $|w(z)| < 1$ $(z \in \mathbb{U})$ for

$$\sum_{k=p+n}^{\infty} |a_k| \leqq |b_{2j-p}| - |b_j| \qquad (|b_j| < |b_{2j-p}|)\,.$$

By the inequality (2.3), we know that

$$\left|b_{2j-p}\left(w(z)\right)^{2(j-p)}+b_j\left(w(z)\right)^{j-p}\right|=\left|\sum_{k=p+n}^{\infty}a_k z^{k-p}\right|<\sum_{k=p+n}^{\infty}|a_k|$$

for $z\in\mathbb{U}$, hence

$$|b_{2j-p}|\,|w(z)|^{2(j-p)}-|b_j|\,|w(z)|^{j-p}-\sum_{k=p+n}^{\infty}|a_k|<0. \tag{2.4}$$

Letting $t=|w(z)|^{j-p}$ $(t\geqq 0)$ in (2.4), we define the function $G(t)$ by

$$G(t)=|b_{2j-p}|\,t^2-|b_j|\,t-\sum_{k=p+n}^{\infty}|a_k|\,.$$

If $G(1)\geqq 0$, then we have $t<1$ for $G(t)<0$. Indeed we have

$$G(1)=|b_{2j-p}|-|b_j|-\sum_{k=p+n}^{\infty}|a_k|\geqq 0.$$

that is,

$$\sum_{k=p+n}^{\infty}|a_k|\leqq|b_{2j-p}|-|b_j|\,.$$

Consequently, if the inequality (2.2) holds true, there exists an analytic function $w(z)$ with $w(0)=0$, $|w(z)|<1$ $(z\in\mathbb{U})$ such that $f(z)=g\left(w(z)\right)$. This completes the proof of Theorem 2.1. □

Theorem 2.1 gives us the following corollary.

Corollary 2.2. *Let $f(z)\in\mathcal{A}_{p,n}$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies the conditions of Theorem 2.1, then for $0<\mu\leqq 2$ and $z=re^{i\theta}$ $(0<r<1)$*

$$\begin{aligned}\int_0^{2\pi}|f(z)|^{\mu}\,d\theta &\leqq 2\pi r^{p\mu}\left\{1+|b_j|^2 r^{2(j-p)}+|b_{2j-p}|^2 r^{4(j-p)}\right\}^{\frac{\mu}{2}}\\ &<2\pi\left\{1+|b_j|^2+|b_{2j-p}|^2\right\}^{\frac{\mu}{2}}.\end{aligned}$$

Further, we have that $f(z)\in\mathcal{H}^q(U)$ for $0<q\leqq 2$, where $\mathcal{H}^q$ denotes the Hardy space (cf. Duren [1]*).*

Proof. Since,

$$\int_0^{2\pi}|g(z)|^{\mu}\,d\theta=\int_0^{2\pi}|z^p|^{\mu}\left|1+b_j z^{j-p}+b_{2j-p}z^{2(j-p)}\right|^{\mu}d\theta,$$

applying Hölder's inequality for $0<\mu<2$, we obtain that

$$\begin{aligned}\int_0^{2\pi}|g(z)|^\mu d\theta &\leqq \left\{\int_0^{2\pi}\left(|z^p|^\mu\right)^{\frac{2}{2-\mu}}d\theta\right\}^{\frac{2-\mu}{2}}\left\{\int_0^{2\pi}\left(\left|1+b_jz^{j-p}+b_{2j-p}z^{2(j-p)}\right|^\mu\right)^{\frac{2}{\mu}}d\theta\right\}^{\frac{\mu}{2}}\\ &=\left\{r^{\frac{2p\mu}{2-\mu}}\int_0^{2\pi}d\theta\right\}^{\frac{2-\mu}{2}}\left\{\int_0^{2\pi}\left|1+b_jz^{j-p}+b_{2j-p}z^{2(j-p)}\right|^2d\theta\right\}^{\frac{\mu}{2}}\\ &=\left\{2\pi r^{\frac{2p\mu}{2-\mu}}\right\}^{\frac{2-\mu}{2}}\left\{2\pi\left(1+|b_j|^2r^{2(j-p)}+|b_{2j-p}|^2r^{4(j-p)}\right)\right\}^{\frac{\mu}{2}}\\ &=2\pi r^{p\mu}\left(1+|b_j|^2r^{2(j-p)}+|b_{2j-p}|^2r^{4(j-p)}\right)^{\frac{\mu}{2}}\\ &<2\pi\left(1+|b_j|^2+|b_{2j-p}|^2\right)^{\frac{\mu}{2}}.\end{aligned}$$

Further, it is easy to see that for $\mu=2$,

$$\begin{aligned}\int_0^{2\pi}|f(z)|^2\,d\theta &\leqq 2\pi r^{p\mu}\left\{1+|b_j|^2r^{2(j-p)}+|b_{2j-p}|^2r^{4(j-p)}\right\}\\ &<2\pi\left\{1+|b_j|^2+|b_{2j-p}|^2\right\}.\end{aligned}$$

From the above, we also have that, for $0<\mu\leqq 2$,

$$\sup_{z\in U}\frac{1}{2\pi}\int_0^{2\pi}|f(z)|^\mu d\theta\leqq 2\pi r^{p\mu}\left\{1+|b_j|^2+|b_{2j-p}|^2\right\}^{\frac{\mu}{2}}<\infty$$

which observe that $f(z)\in\mathcal{H}^2(\mathbb{U})$. Noting that $\mathcal{H}^q\subset\mathcal{H}^r(0<r<q<\infty)$, we complete the proof of the corollary. $\square$

Example 2.1. Let $f(z)\in\mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.4) and

$$g(z)=z^p+\frac{n}{n+p-\alpha}\varepsilon z^j+\delta z^{2j-p}\qquad(|\varepsilon|=|\delta|=1)$$

with $0\leqq\alpha<p$. Note that $b_j=\dfrac{n\varepsilon}{n+p-\alpha}$ and $b_{2j-p}=\delta$.

By virtue of (1.4), we observe that

$$\sum_{k=p+n}^{\infty}|a_k|\le\frac{p-\alpha}{p+n-\alpha}=1-\frac{n}{p+n-\alpha}=|b_{2j-p}|-|b_j|.$$

Therefore, if there exists the function $w(z)$ satisfying the condition in Theorem 2.1, then $f(z)$ and $g(z)$ satisfy the conditions in Theorem 2.1. Thus we have for $0<\mu\leqq 2$ and $z=re^{i\theta}$ $(0<r<1)$,

$$\begin{aligned}\int_0^{2\pi}|f(z)|^\mu d\theta &\leqq 2\pi r^{p\mu}\left\{1+\left(\frac{n}{p+n-\alpha}\right)^2r^{2(j-p)}+r^{4(j-p)}\right\}^{\frac{\mu}{2}}\\ &<2\pi\left\{2+\left(\frac{n}{p+n-\alpha}\right)^2\right\}^{\frac{\mu}{2}}.\end{aligned}$$

Using the same technique as in the proof of Theorem 2.1, we also derive

Theorem 2.3. *Let $f(z)\in\mathcal{A}_{p,n}$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies*

$$\sum_{k=p+n}^{\infty}k|a_k|\leqq(2j-p)|b_{2j-p}|-j|b_j|\qquad(|b_j|<|b_{2j-p}|)\tag{2.5}$$

and there exists an analytic function $w(z)$ *such that*

$$(2j-p)b_{2j-p}\left(w(z)\right)^{2(j-p)} + jb_j\left(w(z)\right)^{j-p} - \sum_{k=p+n}^{\infty} ka_k z^{k-p} = 0,$$

then for $\mu > 0$ *and* $z = re^{i\theta}$ $(0 < r < 1)$

$$\int_0^{2\pi} |f'(z)|^{\mu} d\theta \leqq \int_0^{2\pi} |g'(z)|^{\mu} d\theta.$$

Further, with the help of Hölder's inequality, we have

Corollary 2.4. *Let* $f(z) \in \mathcal{A}_{p,n}$ *and* $g(z)$ *be given by (2.1). If.* $f(z)$ *satisfies conditions of Theorem 2.3, then for* $0 < \mu \leqq 2$ *and* $z = re^{i\theta}$ $(0 < r < 1)$

$$\int_0^{2\pi} |f'(z)|^{\mu} d\theta \leqq 2\pi r^{(p-1)\mu}\left\{p^2 + j^2|b_j|^2 r^{2(j-p)} + (2j-p)^2|b_{2j-p}|^2 r^{4(j-p)}\right\}^{\frac{\mu}{2}}$$
$$< 2\pi\left\{p^2 + j^2|b_j|^2 + (2j-p)^2|b_{2j-p}|^2\right\}^{\frac{\mu}{2}}.$$

Example 2.2. Let $f(z) \in \mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.5) and

$$g(z) = z^p + \frac{n}{j(n+p-\alpha)}\varepsilon z^j + \frac{\delta}{2j-p} z^{2j-p} \qquad (|\varepsilon| = |\delta| = 1)$$

with $0 \leqq \alpha < p$. Then

$$b_j = \frac{n\varepsilon}{j(n+p-\alpha)} \quad \text{and} \quad b_{2j-p} = \frac{\delta}{2j-p}.$$

Since

$$\sum_{k=p+n}^{\infty} k|a_k| \leqq \frac{p-\alpha}{p+n-\alpha} = 1 - \frac{n}{p+n-\alpha} = (2j-p)|b_{2j-p}| - j|b_j|,$$

if there exists the function $w(z)$ satisfying the condition in Theorem 2.3, then $f(z)$ and $g(z)$ satisfy the conditions in Theorem 2.3. Thus by Corollary 2.4, we have for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^{\mu} d\theta \leqq 2\pi r^{(p-1)\mu}\left\{p^2 + \left(\frac{n}{p+n-\alpha}\right)^2 r^{2(j-p)} + r^{4(j-p)}\right\}^{\frac{\mu}{2}}$$
$$< 2\pi\left\{p^2 + 1 + \left(\frac{n}{p+n-\alpha}\right)^2\right\}^{\frac{\mu}{2}}.$$

3. Integral Means Inequalities for $f(z)$ and $h(z)$

In this section, we introduce an analytic and multivalent function $h(z)$ defined by

(3.1) $$h(z) = z^p + b_j z^j + b_{2j-p} z^{2j-p} + b_{3j-2p} z^{3j-2p} \qquad (j \geqq n+p).$$

For the above function $h(z)$, we show

Theorem 3.1. *Let* $f(z) \in \mathcal{A}_{p,n}$ *and* $h(z)$ *be given by (3.1). If* $f(z)$ *satisfies*

(3.2) $$\sum_{k=p+n}^{\infty} |a_k| \leqq |b_{3j-2p}| - |b_{2j-p}| - |b_j| \qquad (|b_j| + |b_{2j-p}| < |b_{3j-2p}|)$$

and there exists an analytic function $w(z)$ such that

$$b_{3j-2p}\left(w(z)\right)^{3(j-p)} + b_{2j-p}\left(w(z)\right)^{2(j-p)} + b_j\left(w(z)\right)^{j-p} - \sum_{k=p+n}^{\infty} a_k z^{k-p} = 0, \tag{3.3}$$

then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leqq \int_0^{2\pi} |h(z)|^{\mu} d\theta.$$

Proof. In the same way as in the proof of Theorem 2.1, we have to show that there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = h(w(z))$. Note that this function $w(z)$ is defined by (3.3).

Since, for $z = 0$,

$$(w(0))^{j-p}\left\{b_{3j-2p}\left(w(0)\right)^{2(j-p)} + b_{2j-p}\left(w(0)\right)^{j-p} + b_j\right\} = 0,$$

we consider $w(z)$ satisfies $w(0) = 0$.

On the other hand, we have that

$$|b_{3j-2p}||w(z)|^{3(j-p)} - |b_{2j-p}||w(z)|^{2(j-p)} - |b_j||w(z)|^{j-p} - \sum_{k=n+p}^{\infty} |a_k| < 0.$$

Putting $t = |w(z)|^{j-p}$ $(t \geqq 0)$, we define the function $H(t)$ by

$$H(t) = |b_{3j-2p}|t^3 - |b_{2j-p}|t^2 - |b_j|t - \sum_{k=n+p}^{\infty} |a_k|.$$

It follows that $H(0) \leqq 0$ and

$$H'(t) = 3|b_{3j-2p}|t^2 - 2|b_{2j-p}|t - |b_j|.$$

Since the discriminant of $H'(t) = 0$ is greater than 0, if $H'(1) \geqq 0$, then $t < 1$ for $H(t) < 0$. Therefore, we need the following inequality:

$$H(1) = |b_{3j-2p}| - |b_{2j-p}| - |b_j| - \sum_{k=p+n}^{\infty} |a_k| \geqq 0$$

or

$$\sum_{k=p+n}^{\infty} |a_k| \leqq |b_{3j-2p}| - |b_{2j-p}| - |b_j|.$$

This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let $f(z) \in \mathcal{A}_{p,n}$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies conditions of Theorem 3.1, then for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\begin{aligned}\int_0^{2\pi} |f(z)|^{\mu} d\theta &\leqq 2\pi r^{p\mu}\left\{1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} + |b_{3j-2p}|^2 r^{6(j-p)}\right\}^{\frac{\mu}{2}} \\ &< 2\pi\left\{1 + |b_j|^2 + |b_{2j-p}|^2 + |b_{3j-2p}|^2\right\}^{\frac{\mu}{2}}.\end{aligned}$$

Proof. Since

$$\int_0^{2\pi} |h(z)|^{\mu} d\theta = \int_0^{2\pi} |z^p|^{\mu} |1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)} + b_{3j-2p} z^{3(j-p)}|^{\mu} d\theta,$$

applying Hölder's inequality for $0 < \mu < 2$, we obtain that

$$\begin{aligned}
&\int_0^{2\pi} |h(z)|^\mu d\theta \\
&\leqq \left\{ \int_0^{2\pi} (|z^p|^\mu)^{\frac{2}{2-\mu}} d\theta \right\}^{\frac{2-\mu}{2}} \left\{ \int_0^{2\pi} \left(|1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)} + b_{3j-2p} z^{3(j-p)}|^\mu \right)^{\frac{2}{\mu}} d\theta \right\}^{\frac{\mu}{2}} \\
&= \left\{ r^{\frac{2p\mu}{2-\mu}} \int_0^{2\pi} d\theta \right\}^{\frac{2-\mu}{2}} \left\{ \int_0^{2\pi} |1 < + b_j z^{j-p} + b_{2j-p} z^{2(j-p)} + b_{2j-2p} z^{3(j-p)}|^2 d\theta \right\}^{\frac{\mu}{2}} \\
&= \left\{ 2\pi r^{\frac{2p\mu}{2-\mu}} \right\}^{\frac{2-\mu}{2}} \left\{ 2\pi (1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} + |b_{3j-2p}|^2 r^{6(j-p)}) \right\}^{\frac{\mu}{2}} \\
&= 2\pi r^{p\mu} \left(1 < + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} + |b_{3j-2p}|^2 r^{6(j-p)} \right)^{\frac{\mu}{2}} \\
&< 2\pi \left(1 + |b_j|^2 + |b_{2j-p}|^2 + |b_{3j-2p}|^2 \right)^{\frac{\mu}{2}}.
\end{aligned}$$

Further, we have that $f(z) \in \mathcal{H}^q(\mathbb{U})$ for $0 < q < 2$. □

We consider the example for Theorem 3.1.

Example 3.1. Let $f(z) \in \mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.4) and

$$h(z) = z^p + \frac{nt}{p+n-\alpha} \varepsilon z^j + \frac{n(1-t)}{p+n-\alpha} \delta z^{2j-p} + \sigma z^{3j-2p}$$
$$(|\varepsilon| = |\delta| = \sigma| = 1; 0 \leqq t \leqq 1)$$

with $0 \leqq \alpha < p$. Then

$$b_j = \frac{nt}{p+n-\alpha} \varepsilon, b_{2j-p} = \frac{n(1-t)}{p+n-\alpha} \delta \quad \text{and} \quad b_{3j-2p} = \sigma.$$

In view of (1.4), we see that

$$\begin{aligned}
\sum_{k=p+n}^{\infty} |a_k| &\leqq \frac{p-\alpha}{p+n-\alpha} \\
&= 1 - \frac{n(1-t)}{p+n-\alpha} - \frac{nt}{p+n-\alpha} \\
&= |b_{3j-2p}| - |b_{2j-p}| - |b_j|.
\end{aligned}$$

Therefore, if there exists the function $w(z)$ satisfying the condition in Theorem 3.1, then $f(z)$ and $g(z)$ satisfy the conditions in Theorem 3.1. Thus applying Corollary 3.2, we have for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\begin{aligned}
&\int_0^{2\pi} |f(z)|^\mu d\theta \\
&\leqq 2\pi r^{p\mu} \left\{ 1 + \left(\frac{nt}{p+n-\alpha} \right)^2 r^{2(j-p)} + \left(\frac{n(1-t)}{p+n-\alpha} \right)^2 r^{4(j-p)} + r^{6(j-p)} \right\}^{\frac{\mu}{2}} \\
&< 2\pi \left\{ 2 + (2t^2 - 2t + 1) \left(\frac{n}{p+n-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}.
\end{aligned}$$

Next, we derive

Theorem 3.3. *Let $f(z) \in \mathcal{A}_{p,n}$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies*

$$\sum_{k=p+n}^{\infty} k|a_k| \leqq (3j-2p)|b_{3j-2p}| - (2j-p)|b_{2j-p}| - j|b_j| \tag{3.4}$$
$$(j|b_j| + (2j-p)|b_{2j-p}| < (3j-2p)|b_{3j-2p}|)$$

and there exists an analytic function $w(z)$ such that

$$(3j-2p)b_{3j-2p}\,(w(z))^{3(j-p)} + (2j-p)b_{2j-p}\,(w(z))^{2(j-p)} + jb_j\,(w(z))^{j-p} - \sum_{k=p+n}^{\infty} ka_k z^{k-p} = 0,$$

then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$

$$\int_0^{2\pi} |f'(z)|^{\mu} d\theta \leqq \int_0^{2\pi} |h'(z)|^{\mu} d\theta.$$

Corollary 3.4. *Let $f(z) \in \mathcal{A}_{p,n}$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies conditions in Theorem 3.3, then for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\begin{aligned}\int_0^{2\pi} |f'(z)|^{\mu} d\theta &\leqq 2\pi r^{(p-1)\mu}\{p^2 + j^2|b_j|^2 r^{2(j-p)} + (2j-p)^2|b_{2j-p}|^2 r^{4(j-p)} \\ &\quad + (3j-2p)^2|b_{3j-2p}|^2 r^{6(j-p)}\}^{\frac{\mu}{2}} \\ &< 2\pi\left\{p^2 + j^2|b_j|^2 + (2j-p)^2|b_{2j-p}|^2 + (3j-2p)^2|b_{3j-2p}|^2\right\}^{\frac{\mu}{2}}.\end{aligned}$$

Finally, we show

Example 3.2. Let $f(z) \in \mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.5) and

$$h(z) = z^p + \frac{nt}{j(p+n-\alpha)}\varepsilon z^j + \frac{n(1-t)}{(2j-p)(p+n-\alpha)}\delta z^{2j-p} + \frac{\sigma}{3j-2p} z^{3j-2p}$$
$$(|\varepsilon| = |\delta| = |\sigma| = 1; 0 \leqq t \leqq 1)$$

with $0 \leqq \alpha < p$. Then

$$b_j = \frac{nt}{p+n-\alpha}\varepsilon, b_{2j-p} = \frac{n(1-t)}{(2j-p)(p+n-\alpha)}\delta \quad \text{and}$$
$$b_{3j-2p} = \frac{\sigma}{3j-2p}.$$

Since

$$\begin{aligned}\sum_{k=p+n}^{\infty} k|a_k| &\leqq \frac{p-\alpha}{p+n-\alpha} \\ &= 1 - \frac{n}{p+n-\alpha} \\ &= (3j-2p)|b_{3j-2p}| - (2j-p)|b_{2j-p}| - j|b_j|,\end{aligned}$$

if there exists the function $w(z)$ satisfying the condition in Theorem 3.3, then $f(z)$ and $g(z)$ satisfy the conditions in Theorem 3.3. Thus by Corollary 3.4, we have for $0 < \mu \leqq 2$ and

$z = re^{i\theta}$ $(0 < r < 1)$,

$$\begin{aligned}
&\int_0^{2\pi} |f'(z)|^{\mu} d\theta \\
&\leqq 2\pi r^{(p-1)\mu} \left\{ p^2 + \left(\frac{nt}{p+n-\alpha} \right)^2 r^{2(j-p)} + \left(\frac{n(1-t)}{p+n-\alpha} \right)^2 r^{4(j-p)} + r^{6(j-p)} \right\}^{\frac{\mu}{2}} \\
&< 2\pi \left\{ p^2 + 1 + (2t^2 - 2t + 1) \left(\frac{n}{p+n-\alpha} \right)^2 \right\}^{\frac{\mu}{2}} .
\end{aligned}$$

Remark 3.5. We have not been able to prove that the analytic function $w(z)$ satisfying each condition of the theorems in this paper exists. However, if we consider some special function $f(z)$ in our theorems, then we know that there is the analytic function $w(z)$ satisfying each condition of our theorems. Thus, if we prove that such a function $w(z)$ exists for any function $f(z) \in \mathcal{A}_{p,n}$, then we do not need to give the condition for $w(z)$ in our theorems.

Remark 3.6. In the above theorems and examples, if we take $p = 1$, we obtain the results by Owa and Sekine [5]. Therefore, the results of our paper are a generalization of the results in [5].

REFERENCES

[1] P.L. DUREN, *Theory of $\mathcal{H}^p$ Space*, Academic Press, New York, 1970.

[2] P.L. DUREN, *Univalent Functions*, Springer-Verlag, New York, 1983.

[3] J.E. LITTLEWOOD, On inequalities in the theory of functions, *Proc. London Math. Soc.*, (2) **23** (1925), 481–519.

[4] S. OWA, On certain classes of p-valent functions with negative coefficients, *Simon Stevin*, **59** (1985), 385–402.

[5] S. OWA AND T. SEKINE, Integral means for analytic functions, *J. Math. Anal. Appl.*, **304** (2005), 772–782.