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ON SOME MAXIMAL INEQUALITIES FOR DEMISUBMARTINGALES AND $N-$DEMISUPER MARTINGALES

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ABSTRACT. We study maximal inequalities for demisubmartingales and N-demisupermartingales and obtain inequalities between dominated demisubmartingales. A sequence of partial sums of zero mean associated random variables is an example of a demimartingale and a sequence of partial sums of zero mean negatively associated random variables is an example of a N-demimartingale.

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{S_n, n \geq 1\}$ be a sequence of random variables defined on it such that $E|S_n| < \infty, n \geq 1$. Suppose that

$$E[(S_{n+1} - S_n)f(S_1, \dots, S_n)] \geq 0 \tag{1.1}$$

for all coordinate-wise nondecreasing functions f whenever the expectation is defined. Then the sequence $\{S_n, n \geq 1\}$ is called a *demimartingale*. If the inequality (1.1) holds for non-negative coordinate-wise nondecreasing functions f, then the sequence $\{S_n, n \geq 1\}$ is called a *demisubmartingale*. If

$$E[(S_{n+1} - S_n)f(S_1, \dots, S_n)] \leq 0 \tag{1.2}$$

for all coordinatewise nondecreasing functions f whenever the expectation is defined, then the sequence $\{S_n, n \geq 1\}$ is called a $N-$*demimartingale*. If the inequality (1.2) holds for non-negative coordinate-wise nondecreasing functions f, then the sequence $\{S_n, n \geq 1\}$ is called a $N-$*demisupermartingale*.

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Remark 1.1. If the function f in (1.1) is not required to be nondecreasing, then the condition defined by the inequality (1.1) is equivalent to the condition that $\{S_n, n \geq 1\}$ is a martingale with respect to the natural choice of σ-algebras. If the inequality defined by (1.1) holds for all nonnegative functions f, then $\{S_n, n \geq 1\}$ is a submartingale with respect to the natural choice of σ-algebras. A martingale with the natural choice of σ-algebras is a demimartingale as well as a $N-$demimartingale since it satisfies (1.1) as well as (1.2). It can be checked that a submartingale is a demisubmartingale and a supermartingale is an N-demisupermartingale. However there are stochastic processes which are demimartingales but not martingales with respect to the natural choice of σ-algebras (cf. [18]).

The concept of demimartingales and demisubmartingales was introduced by Newman and Wright [11] and the notion of $N-$demimartingales (termed earlier as negative demimartingales in [14]) and $N-$demisupermartingales were introduced in [14] and [6].

A set of random variables $X_1, \ldots, X_n$ is said to be *associated* if

$$\operatorname{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0 \tag{1.3}$$

for any two coordinatewise nondecreasing functions f and g whenever the covariance is defined. They are said to be *negatively associated* if

$$\operatorname{Cov}(f(X_i, i \in A), g(X_i, i \in B)) \leq 0 \tag{1.4}$$

for any two disjoint subsets A and B and for any two coordinatewise nondecreasing functions f and g whenever the covariance is defined.

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *associated* (*negatively associated*) if every finite subset of random variables of the sequence is associated (negatively associated).

2. MAXIMAL INEQUALITIES FOR DEMIMARTINGALES AND DEMISUBMARTINGALES

Newman and Wright [11] proved that the partial sums of a sequence of mean zero associated random variables form a demimartingale. We will now discuss some properties of demimartingales and demisubmartingales. The following result is due to Christofides [5].

Theorem 2.1. *Suppose the sequence $\{S_n, n \geq 1\}$ is a demisubmartingale or a demimartingale and $g(\cdot)$ is a nondecreasing convex function. Then the sequence $\{g(S_n), n \geq 1\}$ is a demisubmartingale.*

Let $g(x) = x^+ = \max(0, x)$. Then the function g is nondecreasing and convex. As a special case of the previous result, we get that $\{S_n^+, n \geq 1\}$ is a demisubmartingale. Note that $S_n^+ = \max(0, S_n)$.

Newman and Wright [11] proved the following maximal inequality for demisubmartingales which is an analogue of a maximal inequality for submartingales due to Garsia [8].

Theorem 2.2. *Suppose $\{S_n, n \geq 1\}$ is a demimartingale (demisubmartingale) and $m(\cdot)$ is a nondecreasing (nonnegative and nondecreasing) function with $m(0) = 0$. Let*

$$\begin{aligned} S_{nj} &= j-th \ \ largest\ of \ \ (S_1, \ldots, S_n) \ \ if \ \ j \leq n \\ &= \min(S_1, \ldots, S_n) = S_{n,n} \ \ if \ \ j > n. \end{aligned}$$

Then, for any n and j,

$$E\left(\int_0^{S_{nj}} u dm(u)\right) \leq E\left[S_n m(S_{nj})\right].$$

In particular, for any $\lambda > 0$,

$$\lambda\, P(S_{n1} \geq \lambda) \leq \int_{[S_{n1} \geq \lambda]} S_n dP. \tag{2.1}$$

As an application of the above inequality and an upcrossing inequality for demisubmartingales, the following convergence theorem was proved in [11].

Theorem 2.3. *If $\{S_n, n \geq 1\}$ is a demisubmartingale and $\sup_n E|S_n| < \infty$, then S_n converges almost surely to a finite limit.*

Christofides [5] proved a general version of the inequality (2.1) of Theorem 2.2 which is an analogue of Chow's maximal inequality for martingales [3].

Theorem 2.4. *Let $\{S_n, n \geq 1\}$ be a demisubmartingale with $S_0 = 0$. Let the sequence $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive numbers. Then, for any $\lambda > 0$,*

$$\lambda\, P\left(\max_{1\leq k\leq n} c_k S_k \geq \lambda\right) \leq \sum_{j=1}^{n} c_j E\left(S_j^+ - S_{j-1}^+\right).$$

Wang [16] obtained the following maximal inequality generalizing Theorems 2.2 and 2.4.

Theorem 2.5. *Let $\{S_n, n \geq 1\}$ be a demimartingale and $g(\cdot)$ be a nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$. Suppose that $\{c_i, 1 \leq i \leq n\}$ is a nonincreasing sequence of positive numbers. Let $S_n^* = \max_{1\leq i\leq n} c_i g(S_i)$. Then, for any $\lambda > 0$,*

$$\lambda\, P(S_n^* \geq \lambda) \leq \sum_{i=1}^{n} c_i E\{(g(S_i) - g(S_{i-1}))I[S_n^* \geq \lambda]\}.$$

Suppose $\{S_n, n \geq 1\}$ is a nonnegative demimartingale. As a corollary to the above theorem, it can be proved that

$$E(S_n^{max}) \leq \frac{e}{e-1}[1 + E(S_n \log^+ S_n)].$$

For a proof of this inequality, see Corollary 2.1 in [16].

We now discuss a Whittle type inequality for demisubmartingales due to Prakasa Rao [13]. This result generalizes the Kolmogorov inequality and the Hajek-Renyi inequality for independent random variables [17] and is an extension of the results in [5] for demisubmartingales.

Theorem 2.6. *Let $S_0 = 0$ and $\{S_n, n \geq 1\}$ be a demisubmartingale. Let $\phi(\cdot)$ be a nonnegative nondecreasing convex function such that $\phi(0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$. Further suppose that $0 = u_0 < u_1 \leq \cdots \leq u_n$. Then*

$$P(\phi(S_k) \leq \psi(u_k), 1 \leq k \leq n) \geq 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}.$$

As a corollary of the above theorem, it follows that

$$P\left(\sup_{1\leq j\leq n} \frac{\phi(S_j)}{\psi(u_j)} \geq \epsilon\right) \leq \epsilon^{-1} \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}$$

for any $\epsilon > 0$. In particular, for any fixed $n \geq 1$,

$$P\left(\sup_{k\geq n} \frac{\phi(S_k)}{\psi(u_k)} \geq \epsilon\right) \leq \epsilon^{-1}\left[E\left(\frac{\phi(S_n)}{\psi(u_n)}\right) + \sum_{k=n+1}^{\infty} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}\right]$$

for any $\epsilon > 0$. As a consequence of this inequality, we get the following strong law of large numbers for demisubmartingales [13].

Theorem 2.7. *Let $S_0 = 0$ and $\{S_n, n \geq 1\}$ be a demisubmartingale. Let $\phi(\cdot)$ be a nonnegative nondecreasing convex function such that $\phi(0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$ such that $\psi(u) \to \infty$ as $u \to \infty$. Further suppose that*

$$\sum_{k=1}^{\infty} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)} < \infty$$

for a nondecreasing sequence $u_n \to \infty$ as $n \to \infty$. Then

$$\frac{\phi(S_n)}{\psi(u_n)} \stackrel{a.s}{\to} 0 \ \text{ as } \ n \to \infty.$$

Suppose $\{S_n, n \geq 1\}$ is a demisubmartingale. Let $S_n^{\max} = \max_{1\leq i\leq n} S_i$ and $S_n^{\min} = \min_{1\leq i\leq n} S_i$. As special cases of Theorem 2.2, we get that

$$\lambda \ P[S_n^{\max} \geq \lambda] \leq \int_{[S_n^{\max}\geq\lambda]} S_n dP \tag{2.2}$$

and

$$\lambda \ P[S_n^{\min} \geq \lambda] \leq \int_{[S_n^{\min}\geq\lambda]} S_n dP \tag{2.3}$$

for any $\lambda > 0$.

The inequality (2.2) can also be obtained directly without using Theorem 2.2 by the standard methods used to prove Kolomogorov's inequality. We now prove a variant of the inequality given by (2.3).

Suppose $\{S_n, n \geq 1\}$ is a demisubmartingale. Let $\lambda > 0$. Let

$$N = \left[\min_{1\leq k\leq n} S_k < \lambda\right], \quad N_1 = [S_1 < \lambda]$$

and

$$N_k = [S_k < \lambda, \ S_j \geq \lambda, \ 1 \leq j \leq k-1], \quad k > 1.$$

Observe that

$$N = \bigcup_{k=1}^{n} N_k$$

and $N_k \in \mathcal{F}_k = \sigma\{S_1, \ldots, S_k\}$. Furthermore N_k, $1 \leq k \leq n$ are disjoint and

$$N_k \subset \left(\bigcup_{i=1}^{k-1} N_i\right)^c,$$

where A^c denotes the complement of the set A in Ω. Note that

$$\begin{aligned} E(S_1) &= \int_{N_1} S_1 dP + \int_{N_1^c} S_1 dP \\ &\leq \lambda \int_{N_1} dP + \int_{N_1^c} S_2 dP. \end{aligned}$$

The last inequality follows by observing that

$$\begin{aligned} \int_{N_1^c} S_1 dP - \int_{N_1^c} S_2 dP &= \int_{N_1^c} (S_1 - S_2) dP \\ &= E((S_1 - S_2) I[N_1^c]). \end{aligned}$$

Since the indicator function of the set $N_1^c = [S_1 \geq \lambda]$ is a nonnegative nondecreasing function of S_1 and $\{S_k, 1 \leq k \leq n\}$ is a demisubmartingale, it follows that

$$E((S_2 - S_1)I[N_1^c]) \geq 0.$$

Therefore

$$E((S_1 - S_2)I[N_1^c]) \leq 0,$$

which implies that

$$\int_{N_1^c} S_1 dP \leq \int_{N_1^c} S_2 dP.$$

This proves the inequality

$$\begin{aligned} E(S_1) &\leq \lambda \int_{N_1} dP + \int_{N_1^c} S_2 dP \\ &= \lambda P(N_1) + \int_{N_1^c} S_2 dP. \end{aligned}$$

Observe that $N_2 \subset N_1^c$. Hence

$$\begin{aligned} \int_{N_1^c} S_2 dP &= \int_{N_2} S_2 dP + \int_{N_2^c \cap N_1^c} S_2 dP \\ &\leq \int_{N_2} S_2 dP + \int_{N_2^c \cap N_1^c} S_3 dP \\ &\leq \lambda\ P(N_2) + \int_{N_2^c \cap N_1^c} S_3 dP. \end{aligned}$$

The second inequality in the above chain follows from the observation that the indicator function of the set $N_2^c \cap N_1^c = I[S_1 \geq \lambda, S_2 \geq \lambda]$ is a nonnegative nondecreasing function of S_1, S_2 and the fact that $\{S_k, 1 \leq k \leq n\}$ is a demisubmartingale. By repeated application of these arguments, we get that

$$\begin{aligned} E(S_1) &\leq \lambda \sum_{i=1}^{n} P(N_i) + \int_{\cap_{i=1}^n N_i^c} S_n dP \\ &= \lambda\ P(N) + \int_{\Omega} S_n dP - \int_N S_n dP. \end{aligned}$$

Hence

$$\lambda\ P(N) \geq \int_N S_n dP - \int_{\Omega} (S_n - S_1) dP$$

and we have the following result.

Theorem 2.8. *Suppose that $\{S_n, n \geq 1\}$ is a demisubmartingale . Let*

$$N = \left[\min_{1 \leq k \leq n} S_k < \lambda\right]$$

for any $\lambda > 0$. Then

$$\lambda\ P(N) \geq \int_N S_n dP - \int_{\Omega} (S_n - S_1) dP. \tag{2.4}$$

In particular, if $\{S_n, n \geq 1\}$ is a demimartingale, then it is easy to check that $E(S_n) = E(S_1)$ for all $n \geq 1$ and hence we have the following result as a corollary to Theorem 2.8.

Theorem 2.9. *Suppose that* $\{S_n, n \geq 1\}$ *is a demimartingale . Let* $N = [\min_{1\leq k\leq n} S_k < \lambda]$ *for any* $\lambda > 0$. *Then*

$$\lambda\ P(N) \geq \int_N S_n dP. \tag{2.5}$$

We now prove some new maximal inequalities for nonnegative demisubmartingales.

Theorem 2.10. *Suppose that* $\{S_n, n \geq 1\}$ *is a positive demimartingale with* $S_1 = 1$. *Let* $\gamma(x) = x - 1 - \log x$ *for* $x > 0$. *Then*

$$\gamma(E\left[S_n^{\max}\right]) \leq E\left[S_n \log S_n\right] \tag{2.6}$$

and

$$\gamma(E\left[S_n^{\min}\right]) \leq E\left[S_n \log S_n\right]. \tag{2.7}$$

Proof. Note that the function $\gamma(x)$ is a convex function with minimum $\gamma(1) = 0$. Let $I(A)$ denote the indicator function of the set A. Observe that $S_n^{\max} \geq S_1 = 1$ and hence

$$\begin{aligned}
E(S_n^{\max}) - 1 &= \int_0^\infty P[S_n^{\max} \geq \lambda] d\lambda - 1 \\
&= \int_0^1 P[S_n^{\max} \geq \lambda] d\lambda + \int_1^\infty P[S_n^{\max} \geq \lambda] d\lambda - 1 \\
&= \int_1^\infty P[S_n^{\max} \geq \lambda] d\lambda \ \ (\text{since } \ S_1 = 1) \\
&\leq \int_1^\infty \left\{ \frac{1}{\lambda} \int_{[S_n^{\max} \geq \lambda]} S_n dP \right\} d\lambda \ \ (\text{by (2.2)}) \\
&= E\left(\int_1^\infty \frac{S_n I[S_n^{\max} \geq \lambda]}{\lambda} d\lambda \right) \\
&= E\left(S_n \int_1^{S_n^{\max}} \frac{1}{\lambda} d\lambda \right) \\
&= E(S_n \log(S_n^{\max})).
\end{aligned}$$

Using the fact that $\gamma(x) \geq 0$ for all $x > 0$, we get that

$$\begin{aligned}
E(S_n^{\max}) - 1 &\leq E\left[S_n \left(\log(S_n^{\max}) + \gamma\left(\frac{S_n^{\max}}{S_n E(S_n^{\max})} \right) \right) \right] \\
&= E\left[S_n \left(\log(S_n^{\max}) + \frac{S_n^{\max}}{S_n E(S_n^{\max})} - 1 - \log\left(\frac{S_n^{\max}}{S_n E(S_n^{\max})} \right) \right) \right] \\
&= 1 - E(S_n) + E(S_n \log S_n) + E(S_n) \log E(S_n^{\max}).
\end{aligned}$$

Rearranging the terms in the above inequality, we obtain

$$\begin{aligned}
\gamma(E(S_n^{\max})) &= E(S_n^{\max}) - 1 - \log E(S_n^{\max}) \\
&\leq 1 - E(S_n) + E(S_n \log S_n) + E(S_n) \log E(S_n^{\max}) - \log E(S_n^{\max}) \\
&= E(S_n \log S_n) + (E(S_n) - 1)\left(\log E\left(S_n^{(\max)} \right) - 1 \right) \\
&= E(S_n \log S_n)
\end{aligned} \tag{2.8}$$

since $E(S_n) = E(S_1) = 1$ for all $n \geq 1$. This proves the inequality (2.6).

Observe that $0 \le S_n^{\min} \le S_1 = 1$, which implies that

$$\begin{aligned} E(S_n^{\min}) &= \int_0^1 P[S_n^{\min} \ge \lambda] d\lambda \\ &= 1 - \int_0^1 P[S_n^{\min} < \lambda] d\lambda \\ &\le 1 - \int_0^1 \left\{ \frac{1}{\lambda} \int_{[S_n^{\min} < \lambda]} S_n dP \right\} d\lambda \quad \text{(by Theorem 2.9)} \\ &= 1 - E\left(\int_0^1 \frac{S_n I[S_n^{\min} < \lambda]}{\lambda} d\lambda \right) \\ &= 1 - E\left(S_n \int_{S_n^{\min}}^1 \frac{1}{\lambda} d\lambda \right) \\ &= 1 + E(S_n \log(S_n^{\min})). \end{aligned}$$

Applying arguments similar to those given above to prove the inequality (2.8), we get that

$$\gamma(E(S_n^{\min})) \le E(S_n \log S_n) \tag{2.9}$$

which proves the inequality (2.7). □

The above inequalities for positive demimartingales are analogues of maximal inequalities for nonnegative martingales proved in [9].

3. MAXIMAL ϕ-INEQUALITIES FOR NONNEGATIVE DEMISUBMARTINGALES

Let $\mathcal{C}$ denote the class of *Orlicz functions*, that is, unbounded, nondecreasing convex functions $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$. If the right derivative ϕ' is unbounded, then the function ϕ is called a *Young function* and we denote the subclass of such functions by $\mathcal{C}'$. Since

$$\phi(x) = \int_0^x \phi'(s) ds \le x\phi'(x)$$

by convexity, it follows that

$$p_\phi = \inf_{x>0} \frac{x\phi'(x)}{\phi(x)}$$

and

$$p_\phi^* = \sup_{x>0} \frac{x\phi'(x)}{\phi(x)}$$

are in $[1,\infty]$. The function ϕ is called *moderate* if $p_\phi^* < \infty$, or equivalently, if for some $\lambda > 1$, there exists a finite constant c_λ such that

$$\phi(\lambda x) \le c_\lambda \phi(x), \quad x \ge 0.$$

An example of such a function is $\phi(x) = x^\alpha$ for $\alpha \in [1,\infty)$. An example of a nonmoderate Orlicz function is $\phi(x) = \exp(x^\alpha) - 1$ for $\alpha \ge 1$.

Let $\mathcal{C}^*$ denote the set of all differentiable $\phi \in \mathcal{C}$ whose derivative is concave or convex and $\mathcal{C}'$ denote the set of $\phi \in \mathcal{C}$ such that $\phi'(x)/x$ is integrable at 0, and thus, in particular $\phi'(0) = 0$. Let $\mathcal{C}_0^* = \mathcal{C}' \cap \mathcal{C}^*$.

Given $\phi \in \mathcal{C}$ and $a \ge 0$, define

$$\Phi_a(x) = \int_a^x \int_a^s \frac{\phi'(r)}{r} dr ds, \quad x > 0.$$

It can be seen that the function $\Phi_a I_{[a,\infty)} \in \mathcal{C}$ for any $a > 0$, where I_A denotes the indicator function of the set A. If $\phi \in \mathcal{C}'$, the same holds for $\Phi \equiv \Phi_0$. If $\phi \in \mathcal{C}_0^*$, then $\Phi \in \mathcal{C}_0^*$. Furthermore, if ϕ' is concave or convex, the same holds for

$$\Phi'(x) = \int_0^x \frac{\phi'(r)}{r} dr,$$

and hence $\phi \in \mathcal{C}_0^*$ implies that $\Phi \in \mathcal{C}_0^*$. It can be checked that ϕ and Φ are related through the diferential equation

$$x\Phi'(x) - \Phi(x) = \phi(x), \quad x \geq 0$$

under the initial conditions $\phi(0) = \phi'(0) = \Phi(0) = \Phi'(0) = 0$. If $\phi(x) = x^p$ for some $p > 1$, then $\Phi(x) = x^p/(p-1)$. For instance, if $\phi(x) = x^2$, then $\Phi(x) = x^2$. If $\phi(x) = x$, then $\Phi(x) \equiv \infty$ but $\Phi_1(x) = x \log x - x + 1$. It is known that if $\phi \in \mathcal{C}'$ with $p_\phi > 1$, then the function ϕ satisfies the inequalities

$$\Phi(x) \leq \frac{1}{p_\phi - 1} \phi(x), \quad x \geq 0.$$

Furthermore, if ϕ is moderate, that is $p_\phi^* < \infty$, then

$$\Phi(x) \geq \frac{1}{p_\phi^* - 1} \phi(x), \quad x \geq 0.$$

The brief introduction for properties of Orlicz functions given here is based on [2].

We now prove some maximal ϕ-inequalities for nonnegative demisubmartingales following the techniques in [2].

Theorem 3.1. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale and let $\phi \in \mathcal{C}$. Then*

$$P\left(S_n^{\max} \geq t\right) \leq \frac{\lambda}{(1-\lambda)t} \int_t^\infty P(S_n > \lambda s) ds = \frac{\lambda}{(1-\lambda)t} E\left(\frac{S_n}{\lambda} - t\right)^+ \tag{3.1}$$

for all $n \geq 1, t > 0$ and $0 < \lambda < 1$. Furthermore,

$$E[\phi(S_n^{\max})] \leq \phi(b) + \frac{\lambda}{1-\lambda} \int_{[S_n > \lambda b]} \left(\Phi_a\left(\frac{S_n}{\lambda}\right) - \Phi_a(b) - \Phi_a'(b)\left(\frac{S_n}{\lambda} - b\right)\right) dP \tag{3.2}$$

for all $n \geq 1, a > 0, b > 0$ and $0 < \lambda < 1$. If $\phi'(x)/x$ is integrable at 0, that is, $\phi \in \mathcal{C}'$, then the inequality (3.2) holds for $b = 0$.

Proof. Let $t > 0$ and $0 < \lambda < 1$. Inequality (2.2) implies that

$$\begin{aligned} P\left(S_n^{\max} \geq t\right) &\leq \frac{1}{t} \int_{[S_n^{\max} \geq t]} S_n dP \\ &= \frac{1}{t} \int_0^\infty P[S_n^{\max} \geq t, S_n > s] ds \\ &\leq \frac{1}{t} \int_0^{\lambda t} P[S_n^{\max} \geq t] ds + \frac{1}{t} \int_{\lambda t}^\infty P[S_n > s] ds \\ &\leq \lambda P[S_n^{\max} \geq t] ds + \frac{\lambda}{t} \int_t^\infty P[S_n > \lambda s] ds. \end{aligned} \tag{3.3}$$

Rearranging the last inequality, we get that

$$P\left(S_n^{\max} \geq t\right) \leq \frac{\lambda}{(1-\lambda)t}\int_t^{\infty} P(S_n > \lambda s)ds$$
$$= \frac{\lambda}{(1-\lambda)t} E\left(\frac{S_n}{\lambda} - t\right)^+$$

for all $n \geq 1$, $t > 0$ and $0 < \lambda < 1$ proving the inequality (3.1) in Theorem 3.1. Let $b > 0$. Then

$$\begin{aligned}
E[\phi(S_n^{\max})] &= \int_0^{\infty} \phi'(t)P(S_n^{\max} > t)dt \\
&= \int_0^{b} \phi'(t)P(S_n^{\max} > t)dt + \int_b^{\infty} \phi'(t)P(S_n^{\max} > t)dt \\
&\leq \phi(b) + \int_b^{\infty} \phi'(t)P(S_n^{\max} > t)dt \\
&\leq \phi(b) + \frac{\lambda}{1-\lambda}\int_b^{\infty} \frac{\phi'(t)}{t}\left[\int_t^{\infty} P(S_n > \lambda s)ds\right]dt \quad \text{(by (3.1))} \\
&= \phi(b) + \frac{\lambda}{1-\lambda}\int_b^{\infty}\left(\int_b^{s} \frac{\phi'(t)}{t}dt\right)P(S_n > \lambda s)ds \\
&= \phi(b) + \frac{\lambda}{1-\lambda}\int_b^{\infty}(\Phi_a'(s) - \Phi_a'(b))P(S_n > \lambda s)ds \\
&= \phi(b) + \frac{\lambda}{1-\lambda}\int_{[S_n > \lambda b]}\left(\Phi_a\left(\frac{S_n}{\lambda}\right) - \Phi_a(b) - \Phi_a'(b)\left(\frac{S_n}{\lambda} - b\right)\right)dP
\end{aligned}$$

for all $n \geq 1$, $b > 0$, $t > 0$, $0 < \lambda < 1$ and $a > 0$.The value of a can be chosen to be 0 if $\phi'(x)/x$ is integrable at 0. □

As special cases of the above result, we obtain the following inequalities by choosing $b = a$ in (3.2). Observe that $\Phi_a(a) = \Phi_a'(a) = 0$.

Theorem 3.2. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale and let $\phi \in \mathcal{C}$. Then*

$$E[\phi(S_n^{\max})] \leq \phi(a) + \frac{\lambda}{1-\lambda}E\left[\Phi_a\left(\frac{S_n}{\lambda}\right)\right] \tag{3.4}$$

for all $a \geq 0$, $0 < \lambda < 1$ and $n \geq 1$. Let $\lambda = \frac{1}{2}$ in (3.4). Then

$$E[\phi(S_n^{\max})] \leq \phi(a) + E[\Phi_a(2S_n)] \tag{3.5}$$

for all $a \geq 0$ and $n \geq 1$.

The following lemma is due to Alsmeyer and Rosler [2].

Lemma 3.3. *Let X and Y be nonnegative random variables satisfying the inequality*

$$t\ P(Y \geq t) \leq E(XI_{[Y \geq t]})$$

for all $t \geq 0$. Then

$$E[\phi(Y)] \leq E[\phi(q_\phi X)] \tag{3.6}$$

for any Orlicz function ϕ, where $q_\phi = \frac{p_\phi}{p_\phi - 1}$ and $p_\phi = \inf_{x>0}\frac{x\phi'(x)}{\phi(x)}$.

This lemma follows as an application of the Choquet decomposition

$$\phi(x) = \int_{[0,\infty)} (x-t)^+ \phi'(dt), \quad x \geq 0.$$

In view of the inequality (2.2), we can apply the above lemma to the random variables $X = S_n$ and $Y = S_n^{\max}$ to obtain the following result.

Theorem 3.4. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale and let $\phi \in \mathcal{C}$ with $p_\phi > 1$. Then*

$$E[\phi(S_n^{\max})] \leq E[\phi(q_\phi S_n)] \tag{3.7}$$

for all $n \geq 1$.

Theorem 3.5. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Suppose that the function $\phi \in \mathcal{C}$ is moderate. Then*

$$E[\phi(S_n^{\max})] \leq E[\phi(q_\phi S_n)] \leq q_\Phi^{p_\phi^*} E[\phi(S_n)]. \tag{3.8}$$

The first part of the inequality (3.8) of Theorem 3.5 follows from Theorem 3.4. The last part of the inequality follows from the observation that if $\phi \in \mathcal{C}$ is moderate, that is,

$$p_\phi^* = \sup_{x>0} \frac{x\phi'(x)}{\phi(x)} < \infty,$$

then

$$\phi(\lambda x) \leq \lambda^{p_\phi^*} \phi(x)$$

for all $\lambda > 1$ and $x > 0$ (see [2, equation (1.10)]).

Theorem 3.6. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Suppose ϕ is a nonnegative nondecreasing function on $[0, \infty)$ such that $\phi^{1/\gamma}$ is also nondecreasing and convex for some $\gamma > 1$. Then*

$$E[\phi(S_n^{\max})] \leq \left(\frac{\gamma}{\gamma - 1}\right)^\gamma E[\phi(S_n)]. \tag{3.9}$$

Proof. The inequality

$$\lambda P(S_n^{\max} \geq \lambda) \leq \int_{[S_n^{\max} \geq \lambda]} S_n dP$$

given in (2.2) implies that

$$E[(S_n^{\max})^p] \leq \left(\frac{p}{p-1}\right)^p E(S_n^p), \quad p > 1 \tag{3.10}$$

by an application of the Holder inequality (cf. [4, p. 255]). Note that the sequence $\{[\phi(S_n)]^{1/\gamma}, n \geq 1\}$ is a nonnegative demisubmartingale by Lemma 2.1 of [5]. Applying the inequality (3.10) for the sequence $\{[\phi(S_n)]^{1/\gamma}, n \geq 1\}$ and choosing $p = \gamma$ in that inequality, we get that

$$E[\phi(S_n^{\max})] \leq \left(\frac{\gamma}{\gamma - 1}\right)^\gamma E[\phi(S_n)]. \tag{3.11}$$

for all $\gamma > 1$. □

Examples of functions ϕ satisfying the conditions stated in Theorem 3.6 are $\phi(x) = x^p[\log(1+x)]^r$ for $p > 1$ and $r \geq 0$ and $\phi(x) = e^{rx}$ for $r > 0$. Applying the result in Theorem 3.6 for the function $\phi(x) = e^{rx}, r > 0$, we obtain the following inequality.

Theorem 3.7. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Then*

$$E[e^{rS_n^{\max}}] \leq eE[e^{rS_n}], \quad r > 0. \tag{3.12}$$

Proof. Applying the result stated in Theorem 3.6 to the function $\phi(x) = e^{rx}$, we get that

$$E[e^{rS_n^{\max}}] \leq \left(\frac{\gamma}{\gamma - 1}\right)^{\gamma} E[e^{rS_n}] \tag{3.13}$$

for any $\gamma > 1$. Let $\gamma \to \infty$. Then

$$\left(\frac{\gamma}{\gamma - 1}\right)^{\gamma} \downarrow e$$

and we get that

$$E[e^{rS_n^{\max}}] \leq eE[e^{rS_n}], \quad r > 0. \tag{3.14}$$

□

The next result deals with maximal inequalities for functions $\phi \in \mathcal{C}$ which are k times differentiable with the k-th derivative $\phi^{(k)} \in \mathcal{C}$ for some $k \geq 1$.

Theorem 3.8. *Let $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Let $\phi \in \mathcal{C}$ which is differentiable k times with the k-th derivative $\phi^{(k)} \in \mathcal{C}$ for some $k \geq 1$.Then*

$$E[\phi(S_n^{\max})] \leq \left(\frac{k+1}{k}\right)^{k+1} E[\phi(S_n)]. \tag{3.15}$$

Proof. The proof follows the arguments given in [2] following the inequality (3.9). We present the proof here for completeness. Note that

$$\phi(x) = \int_{[0,\infty)} (x - t)^+ Q_\phi(dt),$$

where

$$Q_\phi(dt) = \phi'(0)\delta_0 + \phi'(dt)$$

and δ_0 is the Kronecker delta function. Hence, if $\phi' \in \mathcal{C}$, then

$$\begin{aligned} \phi(x) &= \int_0^x \phi'(y)dy \\ &= \int_0^x \int_{[0,\infty)} (y - t)^+ Q_{\phi'}(dt)dy \\ &= \int_{[0,\infty)} \int_0^x (y - t)^+ dy Q_{\phi'}(dt) \\ &= \int_{[0,\infty)} \frac{((x - t)^+)^2}{2} Q_{\phi'}(dt). \end{aligned} \tag{3.16}$$

An inductive argument shows that

$$\phi(x) = \int_{[0,\infty)} \frac{((x - t)^+)^{k+1}}{(k + 1)!} Q_{\phi^{(k)}}(dt) \tag{3.17}$$

for any $\phi \in \mathcal{C}$ such that $\phi^{(k)} \in \mathcal{C}$. Let

$$\phi_{k,t}(x) = \frac{((x - t)^+)^{k+1}}{(k + 1)!}$$

for any $k \geq 1$ and $t \geq 0$. Note that the function $[\phi_{k,t}(x)]^{1/(k+1)}$ is nonnegative, convex and nondecreasing in x for any $k \geq 1$ and $t \geq 0$. Hence the process $\{[\phi_{k,t}(S_n)]^{1/(k+1)}, n \geq 1\}$ is a nonnegative demisubmartingale by [5]. Following the arguments given to prove (3.10), we obtain that

$$E(([\phi_{k,t}(S_n^{\max})]^{1/(k+1)})^{k+1}) \leq \left(\frac{k+1}{k}\right)^{k+1} E(([\phi_{k,t}(S_n)]^{1/(k+1)})^{k+1})$$

which implies that

$$E[\phi_{k,t}(S_n^{\max})] \leq \left(\frac{k+1}{k}\right)^{k+1} E[\phi_{k,t}(S_n)]. \tag{3.18}$$

Hence

$$\begin{aligned} E[\phi(S_n^{\max}))] &= \int_{[0,\infty)} E[\phi_{k,t}(S_n^{\max})]Q_{\phi^{(k)}}(dt) \ \text{ (by (3.17))} \\ &\leq \left(\frac{k+1}{k}\right)^{k+1} \int_{[0,\infty)} E[\phi_{k,t}(S_n)]Q_{\phi^{(k)}}(dt) \ \text{ (by (3.18))} \\ &= \left(\frac{k+1}{k}\right)^{k+1} E[\phi(S_n)] \end{aligned} \tag{3.19}$$

which proves the theorem. □

We now consider a special case of the maximal inequality derived in (3.2) of Theorem 3.1. Let $\phi(x) = x$. Then $\Phi_1(x) = x\log x - x + 1$ and $\Phi_1'(x) = \log x$. The inequality (3.2) reduces to

$$\begin{aligned} E[S_n^{\max}] &\leq b + \frac{\lambda}{1-\lambda}\int_{[S_n>\lambda b]} \left(\frac{S_n}{\lambda}\log\frac{S_n}{\lambda} - \frac{S_n}{\lambda} + b - (\log b)\frac{S_n}{\lambda}\right) dP \\ &= b + \frac{\lambda}{1-\lambda}\int_{[S_n>\lambda b]} (S_n\log S_n - S_n(\log\lambda + \log b + 1) + \lambda b)dP \end{aligned}$$

for all $b > 0$ and $0 < \lambda < 1$. Let $b > 1$ and $\lambda = \frac{1}{b}$. Then we obtain the inequality

$$E\left[S_n^{\max}\right] \leq b + \frac{b}{b-1}E\left[\int_1^{\max(S_n,1)} \log x \, dx\right], \quad b>1, n\geq 1. \tag{3.20}$$

The value of b which minimizes the term on the right hand side of the equation (3.20) is

$$b^* = 1 + \left(E\left[\int_1^{\max(S_n,1)} \log x \, dx\right]\right)^{\frac{1}{2}}$$

and hence

$$E(S_n^{\max}) \leq \left(1 + E\left[\int_1^{\max(S_n,1)} \log x \, dx\right]^{\frac{1}{2}}\right)^2. \tag{3.21}$$

Since

$$\int_1^x \log y dy = x\log^+ x - (x-1), \quad x \geq 1,$$

the inequality (3.20) can be written in the form

$$E(S_n^{\max}) \leq b + \frac{b}{b-1}(E(S_n\log^+ S_n) - E(S_n-1)^+), \quad b>1, n\geq 1. \tag{3.22}$$

Let $b = E(S_n - 1)^+$ in the equation (3.22). Then we get the maximal inequality

$$E(S_n^{\max}) \le \frac{1 + E(S_n - 1)^+}{E(S_n - 1)^+} E(S_n \log^+ S_n). \tag{3.23}$$

If we choose $b = e$ in the equation (3.22), then we get the maximal inequality

$$E(S_n^{\max}) \le e + \frac{e}{e-1}(E(S_n \log^+ S_n) - E(S_n - 1)^+), \quad b > 1, n \ge 1. \tag{3.24}$$

This inequality gives a better bound than the bound obtained as a consequence of the result stated in Theorem 2.5 (cf. [16]) if $E(S_n - 1)^+ \ge e - 2$.

4. INEQUALITIES FOR DOMINATED DEMISUBMARTINGALES

Let $M_0 = N_0 = 0$ and $\{M_n, n \ge 0\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that

$$E[(M_{n+1} - M_n)f(M_0, \dots, M_n)|\zeta_n] \ge 0$$

for any nonnegative coordinatewise nondecreasing function f given a filtration $\{\zeta_n, n \ge 0\}$ contained in $\mathcal{F}$. Then the sequence $\{M_n, n \ge 0\}$ is said to be a *strong demisubmartingale* with respect to the filtration $\{\zeta_n, n \ge 0\}$. It is obvious that a strong demisubmartingale is a demisubmartingale in the sense discused earlier.

Definition 4.1. Let $M_0 = 0 = N_0$. Suppose $\{M_n, n \ge 0\}$ is a strong demisubmartingale with respect to the filtration generated by a demisubmartingale $\{N_n, n \ge 0\}$. The strong demisubmartingale $\{M_n, n \ge 0\}$ is said to be *weakly dominated* by the demisubmartingale $\{N_n, n \ge 0\}$ if for every nondecreasing convex function $\phi : \mathbb{R}_+ \to \mathbb{R}$, and for any nonnegative coordinatewise nondecreasing function $f : \mathbb{R}^{2n} \to \mathbb{R}$,

$$E[(\phi(|e_n|) - \phi(|d_n|)f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})|N_0, \dots, N_{n-1}] \ge 0 \text{ a.s.}, \tag{4.1}$$

for all $n \ge 1$ where $d_n = M_n - M_{n-1}$ and $e_n = N_n - N_{n-1}$. We write $M \ll N$ in such a case.

In analogy with the inequalities for dominated martingales developed in [12], we will now prove an inequality for domination between a strong demisubmartingale and a demisubmartingale.

Define the functions $u_{<2}(x, y)$ and $u_{>2}(x, y)$ as in Section 2.1 of [12] for $(x, y) \in \mathbb{R}^2$. We now state a weak-type inequality between dominated demisubmartingales.

Theorem 4.1. *Suppose $\{M_n, n \ge 0\}$ is a strong demisubmartingale with respect to the filtration generated by the sequence $\{N_n, n \ge 0\}$ which is a demisubmartingale. Further suppose that $M \ll N$. Then, for any $\lambda > 0$,*

$$\lambda \;\; P(|M_n| \ge \lambda) \le 6 \;\; E|N_n|, \quad n \ge 0. \tag{4.2}$$

We will at first prove a Lemma which will be used to prove Theorem 4.1.

Lemma 4.2. *Suppose $\{M_n, n \ge 0\}$ is a strong demisubmartingale with respect to the filtration generated by the sequence $\{N_n, n \ge 0\}$ which is a demisubmartingale. Further suppose that $M \ll N$. Then*

$$\begin{aligned} E[u_{<2}(M_n, N_n)f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \\ \ge E[u_{<2}(M_{n-1}, N_{n-1})f(M_0, \dots, M_{n-1}; N_0, \dots, N_{n-1})] \end{aligned} \tag{4.3}$$

and

$$E[u_{>2}(M_n,N_n)f(M_0,\dots,M_{n-1};N_0,\dots,N_{n-1})] \geq E[u_{>2}(M_{n-1},N_{n-1})f(M_0,\dots,M_{n-1};N_0,\dots,N_{n-1})] \tag{4.4}$$

for any nonnegative coordinatewise nondecreasing function $f:\mathbb{R}^{2n}\to\mathbb{R}, n\geq 1.$

Proof. Define $u(x,y)$ where $u=u_{<2}$ or $u=u_{>2}$ as in Section 2.1 of [12]. From the arguments given in [12], it follows that there exist a nonnegative function $A(x,y)$ nondecreasing in x and a nonnegative function $B(x,y)$ nondecreasing in y and a convex nondecreasing function $\phi_{x,y}(\cdot):\mathbb{R}_+\to\mathbb{R}$, such that, for any h and k,

$$u(x,y)+A(x,y)h+B(x,y)k+\phi_{x,y}(|k|)-\phi_{x,y}(|h|)\leq u(x+h,y+k). \tag{4.5}$$

Let $x=M_{n-1}, y=N_{n-1}, h=d_n$ and $k=e_n$. Then, it follows that

$$u(M_{n-1},N_{n-1})+A(M_{n-1},N_{n-1})d_n + B(M_{n-1},N_{n-1})e_n+\phi_{M_{n-1},N_{n-1}}(|e_n|)-\phi_{M_{n-1},N_{n-1}}(|d_n|) \leq u(M_{n-1}+d_n,N_{n-1}+e_n)=u(M_n,N_n). \tag{4.6}$$

Note that,

$$E[A(M_{n-1},N_{n-1})d_nf(M_0,\dots,M_{n-1};N_0,\dots,N_{n-1})|N_0,\dots,N_{n-1}]\geq 0 \text{ a.s.}$$

from the fact that $\{M_n,n\geq 0\}$ is a strong demisubmartingale with respect to the filtration generated by the process $\{N_n,n\geq 0\}$ and that the function

$$A(x_{n-1},y_{n-1})f(x_0,\dots,x_{n-1};y_0,\dots,y_{n-1})$$

is a nonnegative coordinatewise nondecreasing function in $x_0,\dots,x_{n-1}$ for any fixed $y_0,\dots,y_{n-1}$. Taking expectation on both sides of the above inequality, we get that

$$E[A(M_{n-1},N_{n-1})d_nf(M_0,\dots,M_{n-1};N_0,\dots,N_{n-1})]\geq 0. \tag{4.7}$$

Similarly we get that

$$E[B(M_{n-1},N_{n-1})d_nf(M_0,\dots,M_{n-1};N_0,\dots,N_{n-1})]\geq 0. \tag{4.8}$$

Since the sequence $\{M_n,n\geq 0\}$ is dominated by the sequence $\{N_n,n\geq 0\}$, it follows that

$$E[(\phi_{M_{n-1},N_{n-1}}(|e_n|)-\phi_{M_{n-1},N_{n-1}}(|d_n|)f(M_0,\dots,M_{n-1};N_0,\dots,N_{n-1})]\geq 0 \tag{4.9}$$

by taking expectation on both sides of (4.1). Combining the relations (4.6) to (4.9), we get that

$$E[u(M_n,N_n)f(M_0,\dots,M_{n-1};N_0,\dots,N_{n-1})] \geq E[u(M_{n-1},N_{n-1})f(M_0,\dots,M_{n-1};N_0,\dots,N_{n-1})]. \tag{4.10}$$

□

Remark 4.3. Let $f\equiv 1$. Repeated application of the inequality obtained in Lemma 4.2 shows that

$$E[u(M_n,N_n)]\geq E[u(M_0,N_0)]=0. \tag{4.11}$$

Proof of Theorem 4.1. Let

$$v(x,y)=18\ \ |y|-I\left[|x|\geq\frac{1}{3}\right].$$

It can be checked that (cf. [12])

$$v(x,y)\geq u_{<2}(x,y). \tag{4.12}$$

Let $\lambda > 0$. It is easy to see that the strong demisubmartingale $\left\{\frac{M_n}{3\lambda}, n \geq 0\right\}$ is weakly dominated by the demisubmartingale $\{\frac{N_n}{3\lambda}, n \geq 0\}$. In view of the inequalities (4.7) and (4.8), we get that

$$6 \ \ E|N_n| - \lambda \ \ P(|M_n| \geq \lambda) = \lambda E\left[v\left(\frac{M_n}{3\lambda}, \frac{N_n}{3\lambda}\right)\right] \geq \lambda E\left[u_{<2}\left(\frac{M_n}{3\lambda}, \frac{N_n}{3\lambda}\right)\right] \geq 0 \tag{4.13}$$

which proves the inequality

$$\lambda \ \ P(|M_n| \geq \lambda) \leq 6 \ \ E|N_n|, n \geq 0. \tag{4.14}$$

□

Remark 4.4. It would be interesting if the other results in [12] can be extended in a similar fashion for dominated demisubmartingales. We do not discuss them here.

5. $N-$DEMIMARTINGALES AND $N-$DEMISUPERMARTINGALES

The concept of a negative demimartingale, which is now termed as $N-$demimartingale, was introduced in [14] and in [6]. It can be shown that the partial sum $\{S_n, n \geq 1\}$ of mean zero negatively associated random variables $\{X_j, j \geq 1\}$ is a $N-$demimartingale (cf. [6]). This can be seen from the observation

$$E[(S_{n+1} - S_n))f(S_1, \ldots, S_n)] = E(X_{n+1}f(S_1, \ldots, S_n)] \leq 0$$

for any coordinatewise nondecreasing function f and from the observation that increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated (cf. [10]) and the fact that $\{X_n, n \geq 1\}$ are negatively associated. Suppose U_n is a U-statistic based on negatively associated random variables $\{X_n, n \geq 1\}$ and the product kernel $h(x_1, \ldots, x_m) = \prod_{i=1}^m g(x_i)$ for some nondecreasing function $g(\cdot)$ with $E(g(X_i)) = 0, 1 \leq i \leq n$. Let

$$T_n = \frac{n!}{(n-m)!m!}U_n, n \geq m.$$

Then the sequence $\{T_n, n \geq m\}$ is a $N-$demimartingale. For a proof, see [6].

The following theorem is due to Christofides [6].

Theorem 5.1. *Suppose $\{S_n, n \geq 1\}$ is a $N-$demisupermartingale. Then, for any $\lambda > 0$,*

$$\lambda \ P\left[\max_{1\leq k\leq n} S_k \geq \lambda\right] \leq E(S_1) - \int_{[\max_{1\leq k\leq n} S_k \geq \lambda]} S_n dP.$$

In particular, the following maximal inequality holds for a nonnegative $N-$demisupermartingale.

Theorem 5.2. *Suppose $\{S_n, n \geq 1\}$ is a nonnegative $N-$demisupermartingale. Then, for any $\lambda > 0$,*

$$\lambda \ P\left(\max_{1\leq k\leq n} S_k \geq \lambda\right) \leq E(S_1)$$

and

$$\lambda \ P\left(\max_{k\geq n} S_k \geq \lambda\right) \leq E(S_n).$$

Prakasa Rao [15] gives a Chow type maximal inequality for $N-$demimartingales.

Suppose ϕ is a right continuous decreasing function on $(0, \infty)$ satisfying the condition

$$\lim_{t\to\infty} \phi(t) = 0.$$

Further suppose that ϕ is also integrable on any finite interval $(0,x)$. Let

$$\Phi(x) = \int_0^x \phi(t)dt, \quad x \geq 0.$$

Then the function $\Phi(x)$ is a nonnegative nondecreasing function such that $\Phi(0) = 0$. Further suppose that $\Phi(\infty) = \infty$. Such a function is called a *concave Young function*. Properties of such functions are given in [1]. An example of such a function is $\Phi(x) = x^p$, $0 < p < 1$. Christofides [6] obtained the following maximal inequality.

Theorem 5.3. *Let $\{S_n, n \geq 1\}$ be a nonnegative $N-$demisupermartingale. Let $\Phi(x)$ be a concave Young function and define $\psi(x) = \Phi(x) - x\phi(x)$. Then*

$$E[\psi(S_n^{\max})] \leq E[\Phi(S_1)]. \tag{5.1}$$

Furthermore, if

$$\limsup_{x\to\infty} \frac{x\phi(x)}{\Phi(x)} < 1,$$

then

$$E[\Phi(S_n^{\max})] \leq c_\Phi(1 + E[\Phi(S_1)]) \tag{5.2}$$

for some constant c_Φ depending only on the function Φ.

6. REMARKS

It would be interesting to find whether an upcrossing inequality can be obtained for $N-$ demimartingales and then derive an almost sure convergence theorem for $N-$demisupermartingales. Such results are known for demisubmartingales (see Theorem 2.3).

Wood [18] extended the notion of a discrete time parameter demisubmartingale to a continuous time parameter demisubmartingale following the ideas in [7]. A stochastic process $\{S_t, 0 \leq t \leq T\}$ is said to be a demisubmartingale if for every set $\{t_j, 0 \leq j \leq k\}$, $k \geq 1$ contained in the interval $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_k = T$, the sequence $\{S_{t_j}, 0 \leq j \leq k\}$ forms a demisubmartingale.

Suppose that a stochastic process $\{S_t, 0 \leq t \leq T\}$ is a demisubmartingale in the sense defined above. One can assume that the process is separable in the sense of [7]. It is easy to check that $E(S_\alpha) \leq E(S_\beta)$ whenever $\alpha \leq \beta$ since the constant function $f \equiv 1$ is a nonnegative nondecreasing function and

$$E[(S_\beta - S_\alpha)f(S_0, S_\alpha)] \geq 0.$$

Furthermore, for any $\lambda > 0$,

$$\lambda P\left(\sup_{0\leq t\leq T} S_t \geq \lambda\right) \leq \int_{[\sup_{0\leq t\leq T} S_t \geq \lambda]} S_T dP$$

and

$$\lambda P\left(\inf_{0\leq t\leq T} S_t \leq \lambda\right) \geq \int_{[\inf_{0\leq t\leq T} S_t \leq \lambda]} S_T dP - E(S_T) + E(S_0).$$

In analogy with the above remarks, a continuous time parameter stochastic process $\{S_t, 0 \leq t \leq T\}$ is said to be a $N-$demisupermartingale if for every set $\{t_j, 0 \leq j \leq k\}$, $k \geq 1$ contained in the interval $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_k = T$, the sequence $\{S_{t_j}, 0 \leq j \leq k\}$ forms a $N-$demisupermartingale. Theorems 5.1 and 5.2 can be extended to continuous time parameter $N-$demisupermartingales.

Results on maximal inequalities stated and proved in this paper for demisubmaqrtingales and $N-$sdemisupermartingales generalize maximal inequalities for submartingales and supermartingales respectively. Recall that the class of submartingales is a *proper* subclass of demisubmartingales and the class of supermartingales is a *proper* subclass of $N-$ demisupermartingales with respect to the natural choice of σ-algebras..

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