AN INEQUALITY FOR BI-ORTHOGONAL PAIRS

CHRISTOPHER MEANEY
Department of Mathematics
Faculty of Science
Macquarie University
North Ryde NSW 2109, Australia
EMail: chrism@maths.mq.edu.au

Received: 26 November, 2009

Accepted: 14 December, 2009

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 42C15, 46C05.

Key words: Bi-orthogonal pair, Bessel's inequality, Orthogonal expansion, Lebesgue constants.

Abstract: We use Salem's method [13, 14] to prove an inequality of Kwapień and Pełczyński concerning a lower bound for partial sums of series of bi-orthogonal vectors in a Hilbert space, or the dual vectors. This is applied to some lower bounds on L^1 norms for orthogonal expansions.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

Suppose that H is a Hilbert space, $n \in \mathbb{N}$, and that $J = \{1, \dots, n\}$ or $J = \mathbb{N}$. A pair of sets $\{v_j : j \in J\}$ and $\{w_j : j \in J\}$ in H are said to be *a bi-orthogonal pair* when

$$\langle v_j, w_k \rangle_H = \delta_{jk}, \qquad \forall j, k \in J.$$

The inequality in Theorem 2.1 below comes from Section 6 of [6], where it was proved using Grothendieck's inequality, absolutely summing operators, and estimates on the Hilbert matrix. Here we present an alternate proof, based on earlier ideas from Salem [13, 14], where Bessel's inequality is combined with a result of Menshov [10]. Following the proof of Theorem 2.1, we will describe Salem's method of using L^2 inequalities to produce L^1 estimates on maximal functions. Such estimates are related to the stronger results of Olevskiĭ [11], Kashin and Szarek [4], and Bochkarev [1]. We conclude with an observation about the statement of Theorem 2.1 in a linear algebra setting. Some of these results were discussed in [9], where it was shown that Salem's methods emphasized the universality of the Rademacher-Menshov Theorem.

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. The Kwapień-Pełczyński Inequality

Theorem 2.1. *There is a positive constant c with the following property. For every $n \geq 1$, every Hilbert space H, and every bi-orthogonal pair $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ in H,*

$$\log n \leq c \max_{1 \leq m \leq n} \|w_m\|_H \max_{1 \leq k \leq n} \left\| \sum_{j=1}^{k} v_j \right\|_H . \tag{2.1}$$

Proof. Equip $[0,1]$ with a Lebesgue measure λ and let $V = L^2([0,1], H)$ be the space of H-valued square integrable functions on $[0,1]$, with inner product

$$\langle F, G \rangle_V = \int_0^1 \langle F(x), G(x) \rangle_H \, dx$$

and norm

$$\|F\|_V = \left(\int_0^1 \|F(x)\|_H^2 \, dx \right).$$

Suppose that $\{F_1, \ldots, F_n\}$ is an orthonormal set in $L^2([0,1])$ and define vectors $p_1, \ldots, p_n$ in V by

$$p_k(x) = F_k(x) w_k, \qquad 1 \leq k \leq n, x \in [0,1].$$

Then

$$\langle p_k(x), p_j(x) \rangle_H = F_k(x) \overline{F_j(x)} \, \langle w_k, w_j \rangle_H , \qquad 1 \leq j, k \leq n,$$

and so $\{p_1, \ldots, p_n\}$ is an orthogonal set in V. For every $P \in V$, Bessel's inequality states that

$$\sum_{k=1}^{n} \frac{|\langle P, p_k \rangle_V|^2}{\|w_k\|_H^2} \leq \|P\|_V^2 . \tag{2.2}$$

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Note that here

$$\langle P, p_k \rangle_V = \int_0^1 \langle P(x), w_k \rangle_H \overline{F_k(x)} dx, \qquad 1 \leq k \leq n.$$

Now consider a decreasing sequence $f_1 \geq f_2 \geq \cdots \geq f_n \geq f_{n+1} = 0$ of characteristic functions of measurable subsets of $[0,1]$. For each scalar-valued $G \in L^2([0,1])$ define an element of V by setting

$$P_G(x) = G(x) \sum_{j=1}^{n} f_j(x) v_j.$$

The Abel transformation shows that

$$P_G(x) = G(x) \sum_{k=1}^{n} \Delta f_k(x) \sigma_k,$$

where $\Delta f_k = f_k - f_{k+1}$ and $\sigma_k = \sum_{j=1}^{k} v_j$, for $1 \leq k \leq n$. The functions $\Delta f_1, \ldots, \Delta f_n$ are characteristic functions of mutually disjoint subsets of $[0,1]$ and for each $0 \leq x \leq 1$ at most one of the values $\Delta f_k(x)$ is non-zero. Notice that

$$\|P_G(x)\|_H^2 = |G(x)|^2 \sum_{k=1}^{n} \Delta f_k(x) \, \|\sigma_k\|_H^2 .$$

Integrating over $[0,1]$ gives

$$\|P_G\|_V^2 \leq \|G\|_2^2 \max_{1 \leq k \leq n} \|\sigma_k\|_H^2 .$$

Note that

$$\langle P_G(x), p_k(x) \rangle_H = G(x) f_k(x) \overline{F_k(x)} \, \langle v_k, w_k \rangle_H , \quad 1 \leq k \leq n,$$

and

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$$\langle P_G, p_k \rangle_V = \int_0^1 G(x) f_k(x) \overline{F_k(x)} dx \, \langle v_k, w_k \rangle_H , \quad 1 \le k \le n.$$

Combining this with Bessel's inequality (2.2), we arrive at the inequality

$$\sum_{k=1}^{n} \left| \int_{[0,1]} G f_k \overline{F_k} d\lambda \right|^2 \frac{1}{\|w_k\|_H^2} \le \|G\|_2^2 \max_{1\le k\le n} \|\sigma_k\|_H^2 . \tag{2.3}$$

This implies that

$$\left(\sum_{k=1}^{n} \left| \int_{[0,1]} G f_k \overline{F_k} d\lambda \right|^2 \right) \le \left(\max_{1\le j\le n} \|w_k\|_H^2 \right) \|G\|_2^2 \left(\max_{1\le k\le n} \|\sigma_k\|_H^2 \right) . \tag{2.4}$$

We now concentrate on the case where the functions $F_1, \dots, F_n$ are given by Menshov's result (Lemma 1 on page 255 of Kashin and Saakyan [3]). There is a constant $c_0 > 0$, independent of n, so that

$$\lambda \left(\left\{ x \in [0,1] \ : \ \max_{1\le j\le n} \left| \sum_{k=1}^{j} F_k(x) \right| > c_0 \log(n) \sqrt{n} \right\} \right) \ge \frac{1}{4}. \tag{2.5}$$

Let us use $\mathcal{M}(x)$ to denote the maximal function

$$\mathcal{M}(x) = \max_{1\le j\le n} \left| \sum_{k=1}^{j} F_k(x) \right| , \qquad 0 \le x \le 1.$$

Define an integer-valued function $m(x)$ on $[0,1]$ by

$$m(x) = \min \left\{ m \ : \ \left| \sum_{k=1}^{m} F_k(x) \right| = \mathcal{M}(x) \right\} .$$

Furthermore, let f_k be the characteristic function of the subset

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$$\{x \in [0,1] \ : \ m(x) \geq k\}.$$

Then

$$\sum_{k=1}^{n} f_k(x)F_k(x) = S_{m(x)}(x) = \sum_{k=1}^{m(x)} F_k(x), \qquad \forall 0 \leq x \leq 1.$$

For an arbitrary $G \in L^2([0,1])$ we have

$$\int_0^1 G(x)\overline{S_{m(x)}(x)}dx = \sum_{k=1}^{n} \int_0^1 G(x)f_k(x)\overline{F_k(x)}dx.$$

Using the Cauchy-Schwarz inequality on the right hand side, we have

$$\left|\int_0^1 G(x)\overline{S_{m(x)}(x)}dx\right| \leq \sqrt{n}\left(\sum_{k=1}^{n}\left|\int_0^1 G f_k \overline{F_k}\, d\lambda\right|^2\right)^{1/2}, \tag{2.6}$$

for all $G \in L^2([0,1])$. We will use the function G which has $|G(x)| = 1$ everywhere on $[0,1]$, with

$$G(x)\overline{S_{m(x)}(x)} = \mathcal{M}(x), \qquad \forall 0 \leq x \leq 1.$$

In this case, the left hand side of (2.6) is

$$\|\mathcal{M}\|_1 \geq \frac{c_0}{4}\log(n)\sqrt{n},$$

because of (2.5). Combining this with (2.6) we have

$$\frac{c_0}{4}\log(n)\sqrt{n} \leq \sqrt{n}\left(\sum_{k=1}^{n}\left|\int_0^1 G f_k \overline{F_k}\, d\lambda\right|^2\right)^{1/2}.$$

This can be put back into (2.4) to obtain (2.1). Notice that $\|G\|_2 = 1$ on the right hand side of (2.3). $\square$

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3. Applications

3.1. L^1 estimates

In this section we use $H = L^2(X,\mu)$, for a positive measure space (X,μ). Suppose we are given an orthonormal sequence of functions $(h_n)_{n=1}^{\infty}$ in $L^2(X,\mu)$, and suppose that each of the functions h_n is essentially bounded on X. Let $(a_n)_{n=1}^{\infty}$ be a sequence of non-zero complex numbers and set

$$M_n = \max_{1\le j\le n} \|h_j\|_\infty \text{ and } \mathcal{S}_n^*(x) = \max_{1\le k\le n}\left|\sum_{j=1}^{k} a_j h_j(x)\right|, \qquad \text{for } x\in X,\ n\ge 1.$$

Lemma 3.1. *For a set of functions* $\{h_1,\dots,h_n\}\subset L^2(X,\mu)\cap L^\infty(X,\mu)$ *and maximal function*

$$\mathcal{S}_n^*(x) = \max_{1\le k\le n}\left|\sum_{j=1}^{k} a_j h_j(x)\right|,$$

we have

$$|a_j h_j(x)| \le 2\mathcal{S}_n^*(x), \qquad \forall x\in X, 1\le j\le n,$$

and

$$\frac{\left|\sum_{j=1}^{k} a_j h_j(x)\right|}{\mathcal{S}_n^*(x)} \le 1, \quad \forall 1\le k\le n \text{ and } x \text{ where } \mathcal{S}_n^*(x)\ne 0.$$

Proof. The first inequality follows from the triangle inequality and the fact that

$$a_j h_j(x) = \sum_{k=1}^{j} a_k h_k(x) - \sum_{k=1}^{j-1} a_k h_k(x)$$

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for $2 \le j \le n$. The second inequality is a consequence of the definition of $\mathcal{S}_n^*$.

Fix $n \ge 1$ and let

$$v_j(x) = a_j h_j(x) \left(\mathcal{S}_n^*(x)\right)^{-1/2} \quad \text{and } w_j(x) = a_j^{-1} h_j(x) \left(\mathcal{S}_n^*(x)\right)^{1/2}$$

for all $x \in X$ where $\mathcal{S}_n^*(x) \neq 0$ and $1 \le j \le n$. From their definition,

$$\{v_1, \dots, v_n\} \quad \text{and} \quad \{w_1, \dots, w_n\}$$

are a bi-orthogonal pair in $L^2(X, \mu)$. The conditions we have placed on the functions h_j give:

$$\|w_j\|_2^2 = |a_j|^{-2} \int_X |h_j|^2 \left(\mathcal{S}_n^*\right) d\mu \le \frac{M_n^2}{\min_{1\le k\le n} |a_k|^2} \|\mathcal{S}_n^*\|_1$$

and

$$\left\| \sum_{j=1}^{k} v_j \right\|_2^2 = \int_X \frac{1}{\left(\mathcal{S}_n^*\right)} \left| \sum_{j=1}^{k} a_j h_j \right|^2 d\mu \le \left\| \sum_{j=1}^{k} a_j h_j \right\|_1 .$$

We can put these estimates into (2.1) and find that

$$\log n \le c \frac{M_n}{\min_{1\le k\le n} |a_k|} \|S_n^*\|_1^{1/2} \max_{1\le k\le n} \left\| \sum_{j=1}^{k} a_j h_j \right\|_1^{1/2} .$$

We could also say that

$$\max_{1\le k\le n} \left\| \sum_{j=1}^{k} a_j h_j \right\|_1 \le \|\mathcal{S}_n^*\|_1$$

and so

$$\log(n) \le c \frac{M_n}{\min_{1\le k\le n} |a_k|} \|\mathcal{S}_n^*\|_1 .$$

$\square$

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Corollary 3.2. *Suppose that $(h_n)_{n=1}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ consisting of essentially bounded functions. For each sequence $(a_n)_{n=1}^{\infty}$ of complex numbers and each $n \geq 1$,*

$$\left(\min_{1\leq k\leq n} |a_k| \ \log n\right)^2 \leq c\left(\max_{1\leq k\leq n} \|h_k\|_\infty\right)^2 \left\|\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right|\right\|_1 \left\|\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right|\right\|_1$$

and

$$\min_{1\leq k\leq n} |a_k| \ \log n \leq c\left(\max_{1\leq k\leq n} \|h_k\|_\infty\right) \left\|\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right|\right\|_1 .$$

The constant c is independent of n, and the sequences involved here.

As observed in [4], this can also be obtained as a consequence of [11]. In addition, see [7].

The following is a paraphrase of the last page of [13]. For the special case of Fourier series on the unit circle, see Proposition 1.6.9 in [12].

Corollary 3.3. *Suppose that $(h_n)_{n=1}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ consisting of essentially bounded functions with $\|h_n\|_\infty \leq M$ for all $n \geq 1$. For each decreasing sequence $(a_n)_{n=1}^{\infty}$ of positive numbers and each $n \geq 1$,*

$$(a_n \ \log n)^2 \leq cM^2 \left\|\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right|\right\|_1 \left\|\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right|\right\|_1$$

and

$$a_n \ \log n \leq cM \left\|\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right|\right\|_1 .$$

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In particular, if $(a_n \log n)_{n=1}^{\infty}$ *is unbounded then*

$$\left(\left\| \max_{1 \le k \le n} \left| \sum_{j=1}^{k} a_j h_j \right| \right\|_1 \right)_{n=1}^{\infty} \quad \text{is unbounded.}$$

The constant c *is independent of* n*, and the sequences involved here.*

3.2. Salem's Approach to the Littlewood Conjecture

We concentrate on the case where $H = L^2(\mathbb{T})$ and the orthonormal sequence is a subset of $\{e^{inx} : n \in \mathbb{N}\}$. Let

$$m_1 < m_2 < m_3 < \cdots$$

be an increasing sequence of natural numbers and let

$$h_k(x) = e^{im_k x}$$

for all $k \ge 1$ and $x \in \mathbb{T}$. In addition, let

$$D_m(x) = \sum_{k=-m}^{m} e^{ikx}$$

be the m^{th} Dirichlet kernel. For all $N \ge m \ge 1$, there is the partial sum

$$\sum_{m_k \le m} a_k h_k(x) = D_m * \left(\sum_{m_k \le N} a_k h_k \right)(x).$$

It is a fact that D_m is an even function which satisfies the inequalities:

$$|D_m(x)| \le \begin{cases} 2m+1 & \text{for all } x, \\ 1/|x| & \text{for } \frac{1}{2m+1} < x < 2\pi - \frac{1}{2m+1}. \end{cases} \tag{3.1}$$

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Lemma 3.4. *If p is a trigonometric polynomial of degree N, then the maximal function of its Fourier partial sums*

$$S^*p(x) = \sup_{m\geq 1} |D_m * p(x)|$$

satisfies

$$\|S^*p\|_1 \leq c\log(2N+1)\,\|p\|_1\,.$$

Proof. For such a trigonometric polynomial p, the partial sums are all partial sums of $p * D_N$, and all the Dirichlet kernels D_m for $1 \leq m \leq N$ are dominated by a function whose L^1 norm is of the order of $\log(2N+1)$. □

We can combine this with the inequalities in Corollary 3.2, since

$$\left\|\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} a_j h_j\right|\right\|_1 \leq c\log(2m_n+1)\left\|\sum_{j=1}^{m} a_j h_j\right\|_1 .$$

We then arrive at the main result in [14].

Corollary 3.5. *For an increasing sequence $(m_n)_{n=1}^{\infty}$ of natural numbers and a sequence of non-zero complex numbers $(a_n)_{n=1}^{\infty}$ the partial sums of the trigonometric series*

$$\sum_{k=1}^{\infty} a_k e^{im_k x}$$

satisfy

$$\min_{1\leq k\leq n}|a_k|\,\frac{\log n}{\sqrt{\log(2m_n+1)}} \leq c\max_{1\leq k\leq n}\left\|\sum_{j=1}^{k} a_j e^{im_j(\cdot)}\right\|_1 .$$

This was Salem's attempt at Littlewood's conjecture, which was subsequently settled in [5] and [8].

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3.3. Linearly Independent Sequences

Notice that if $\{v_1, \dots, v_n\}$ is an arbitrary linearly independent subset of H then there is a unique subset

$$\{w_j^n \ : \ 1 \le j \le n\} \subseteq \text{span}\left(\{v_1, \dots, v_n\}\right)$$

so that $\{v_1, \dots, v_n\}$ and $\{w_1^n, \dots, w_n^n\}$ are a bi-orthogonal pair. See Theorem 15 in Chapter 3 of [2]. We can apply Theorem 2.1 to the pair in either order.

Corollary 3.6. *For each $n \ge 2$ and linearly independent subset $\{v_1, \dots, v_n\}$ in an inner-product space H, with dual basis $\{w_1^n, \dots, w_n^n\}$,*

$$\log n \le c \max_{1\le k\le n} \|w_k^n\|_H \max_{1\le k\le n} \left\| \sum_{j=1}^{k} v_j \right\|_H$$

and

$$\log n \le c \max_{1\le k\le n} \|v_k\|_H \max_{1\le k\le n} \left\| \sum_{j=1}^{k} w_j^n \right\|_H .$$

The constant $c > 0$ is independent of n, H, and the sets of vectors.

3.4. Matrices

Suppose that A is an invertible $n \times n$ matrix with complex entries and columns

$$a_1, \dots, a_n \in \mathbb{C}^n.$$

Let $b_1, \dots, b_n$ be the rows of A^{-1}. From their definition

$$\sum_{j=1}^{n} b_{ij} a_{jk} = \delta_{ik}$$

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and so the two sets of vectors

$$\left\{\overline{b_1^T}, \ldots, \overline{b_n^T}\right\} \quad \text{and} \quad \{a_1, \ldots, a_n\}$$

are a bi-orthogonal pair in $\mathbb{C}^n$. Theorem 2.1 then says that

$$\log(n) \leq c \max_{1\leq k\leq n} \|b_k\| \max_{1\leq k\leq n} \left\| \sum_{j=1}^{k} a_j \right\|.$$

The norm here is the finite dimensional ℓ^2 norm. This brings us back to the material in [6]. Note that [4] has logarithmic lower bounds for ℓ^1-norms of column vectors of orthogonal matrices.

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issn: 1443-5756