BOUNDEDNESS OF THE WAVELET TRANSFORM IN CERTAIN FUNCTION SPACES

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Abstract: Using convolution transform theory boundedness results for the wavelet transform are obtained in the Triebel space-$L_p^{\Omega,k}$, Hörmander space-$B_{p,q}(\mathbb{R}^n)$ and general function space-$L_{\infty,k}$, where k denotes a weight function possessing specific properties in each case.
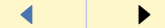
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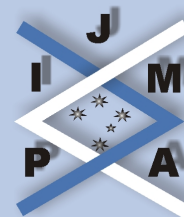
1. Introduction

The wavelet transform W of a function f with respect to the wavelet ψ is defined by

$$(1.1) \qquad \tilde{f}(a,b) = (W_\psi f)(a,b) = \int_{\mathbb{R}^n} f(t)\overline{\psi_{a,b}(t)}dt = (f * h_{a,0})(b),$$

where $\psi_{a,b} = a^{-\frac{n}{2}}\psi(\frac{x-b}{a})$, $h(x) = \overline{\psi(-x)}$, $b \in \mathbb{R}^n$ and $a > 0$, provided the integral exists. In view of (1.1) the wavelet transform $(W_\psi f)(a,b)$ can be regarded as the convolution of f and $h_{a,0}$. The existence of convolution $f * g$ has been investigated by many authors. For this purpose Triebel [6] defined the space $L_p^{\Omega,k}$ and showed that for certain weight functions k, $f * g \in L_p^{\Omega,k}$, where $f,g \in L_p^{\Omega,k}$, $0 < p \leq 1$. Convolution theory has also been developed by Hörmander in the generalized Sobolev space $B_{p,q}(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

In Section 2 of the paper, a definition and properties of the space $L_p^{\Omega,k}$ are given and a boundedness result for the wavelet transform $W_\psi f$ is obtained. In Section 3 we recall the definition and properties of the generalized Sobolev space $B_{p,q}(\mathbb{R}^n)$ due to Hörmander [1] and obtain a certain boundedness result for $W_\psi f$. Finally, using Young's inequality a third boundedness result is also obtained.
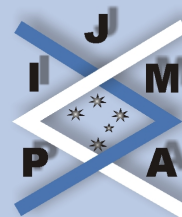
2. Boundedness of W in $L_p^{\Omega,k}$

Let us recall the definition of the space $L_p^{\Omega,k}$ by Triebel [6].

Definition 2.1. Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$. If $k(x)$ is a non-negative weight function in $\mathbb{R}^n$ and $0 < p \le \infty$, then

$$(2.1) \quad L_p^{\Omega,k} = \left\{ f \mid f \in \mathscr{S}', \operatorname{supp} Ff \subset \overline{\Omega}; \right.$$

$$\left. \| f \|_{L_p^{\Omega,k}} = \| kf \|_{L_p} = \left(\int_{\mathbb{R}^n} k^p(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

If $k(x) = 1$ then $L_p^{\Omega,k} = L_p^{\Omega}$.

We need the following theorem [6, p. 369] in the proof of our boundedness result.

Theorem 2.1 (Hans Triebel). *If k is one of the following weight functions:*
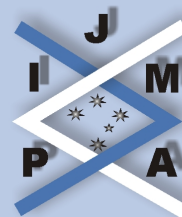
$$(2.2) \qquad k(x) = |x|^\alpha, \qquad \alpha \ge 0$$

$$(2.3) \qquad k(x) = \prod_{j=1}^{n} |x_j|^{\alpha_j}, \qquad \alpha_j \ge 0$$

$$(2.4) \qquad k(x) = k_{\beta,\gamma}(x) = e^{\beta|x|^\gamma}, \qquad \beta \ge 0, 0 \le \gamma \le 1$$

and $0 < p \le 1$, then

$$(2.5) \qquad L_p^{\Omega,k} * L_p^{\Omega,k} \subset L_p^{\Omega,k}$$

and there exists a positive number C such that for all $f, g \in L_p^{\Omega,k}$,

(2.6)
$$\|f * g\|_{L_p^k} \le C \|f\|_{L_p^k} \|g\|_{L_p^k}.$$

Using the above theorem we obtain the following boundedness result for the wavelet transform $W_\psi f$.

Theorem 2.2. *Let $f \in L_p^{\Omega,k}$ and $\psi \in L_p^{\Omega,k}, 0 < p \le 1$, then for the wavelet transform $W_\psi f$ we have the estimates:*

(2.7)
$$\| (W_\psi f)(a, b) \|_{L_p^k} \le C a^{\alpha + \frac{n}{2}} \| f \|_{L_p^k} \| \psi \|_{L_p^k} \quad \text{for (2.2)};$$

(2.8)
$$\| (W_\psi f)(a, b) \|_{L_p^k} \le C a^{|\alpha| + \frac{n}{2}} \| f \|_{L_p^k} \| \psi \|_{L_p^k} \quad \text{for (2.3)};$$

(2.9)
$$\| (W_\psi f)(a, b) \|_{L_p^{k_{\beta,\gamma}}} \le C a^{\frac{n}{2}} e^{\frac{1}{2}\beta a^2 \gamma} \| f \|_{L_p^{k_{\beta,\gamma}}} \| \psi \|_{L_p^{k_{\beta,2\gamma}}} \quad \text{for (2.4)},$$

where $b \in \mathbb{R}^n$ and $a > 0$.

Proof. For $k(x) = |x|^\alpha$, $\alpha > 0$, we have $k(az) = a^\alpha k(z)$ and

$$\|h_{a,0}\|_{L_p^k} = \left(\int_{\mathbb{R}^n} k^p(x)(a^{-\frac{n}{2}}|h(\frac{x}{a})|)^p dx \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}} \left(\int_{\mathbb{R}^n} k^p(az)|h(z)|^p dz \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}} \left(\int_{\mathbb{R}^n} a^{p\alpha} k^p(z)|h(z)|^p dz \right)^{\frac{1}{p}}$$

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$$= a^{\frac{n}{2}+\alpha} \left(\int_{\mathbb{R}^n} k^p(z)|h(z)|^p dz \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}+\alpha} \parallel h \parallel_{L_p^k}$$

$$= a^{\frac{n}{2}+\alpha} \parallel \psi \parallel_{L_p^k} .$$

For $k(x) = \prod_{j=1}^n |x_j|^{\alpha_j}$, $\alpha_j \geq 0$, we have $k(az) = a^{|\alpha|}k(z)$ and

$$\|h_{a,0}\|_{L_p^k} = \left(\int_{\mathbb{R}^n} k^p(x)(a^{-\frac{n}{2}}|h(\frac{x}{a})|)^p dx \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}} \left(\int_{\mathbb{R}^n} k^p(az)|h(z)|^p dz \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}} \left(\int_{\mathbb{R}^n} a^{p|\alpha|} k^p(z)|h(z)|^p dz \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}+|\alpha|} \left(\int_{\mathbb{R}^n} k^p(z)|h(z)|^p dz \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}+|\alpha|} \parallel h \parallel_{L_p^k}$$

$$= a^{\frac{n}{2}+|\alpha|} \parallel \psi \parallel_{L_p^k} .$$

Next, for $k(x) = k_{\beta,\gamma}(x) = e^{\beta|x|^\gamma}$, $\beta \geq 0, 0 \leq \gamma \leq 1$, we have

$$k_{\beta,\gamma}(az) = e^{\beta|az|^\gamma} = e^{\beta a^\gamma |z|^\gamma} \leq e^{\beta \frac{a^{2\gamma}+|z|^{2\gamma}}{2}} = e^{\frac{1}{2}\beta a^{2\gamma}} e^{\frac{1}{2}\beta|z|^{2\gamma}} = e^{\frac{1}{2}\beta a^{2\gamma}} k_{\beta,2\gamma}(z),$$
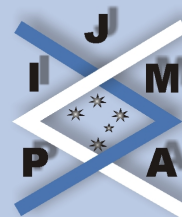
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and

$$\|h_{a,0}\|_{L_p^{k_{\beta,\gamma}}} = \left(\int_{\mathbb{R}^n} k_{\beta,\gamma}^p(x) \left(a^{-\frac{n}{2}} \left| h\left(\frac{x}{a}\right) \right| \right)^p dz \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}} \left(\int_{\mathbb{R}^n} k_{\beta,\gamma}^p(az)|h(z)|^p dz \right)^{\frac{1}{p}}$$

$$\leq a^{\frac{n}{2}} \left(\int_{\mathbb{R}^n} e^{\frac{1}{2}p\beta a^{2\gamma}} k_{\beta,2\gamma}^p(z)|h(z)|^p dz \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}} e^{\frac{1}{2}\beta a^{2\gamma}} \left(\int_{\mathbb{R}^n} k_{\beta,2\gamma}^p(z)|h(z)|^p dz \right)^{\frac{1}{p}}$$

$$= a^{\frac{n}{2}} e^{\frac{1}{2}\beta a^{2\gamma}} \| h \|_{L_p^{k_{\beta,2\gamma}}}$$

$$= a^{\frac{n}{2}} e^{\frac{1}{2}\beta a^{2\gamma}} \| \psi \|_{L_p^{k_{\beta,2\gamma}}} .$$

The proofs of (2.7), (2.8) and (2.9) follow from (2.6). ∎
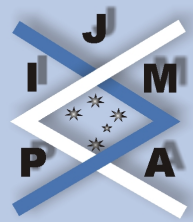
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3. Boundedness of W in $B_{p,k}$

The space $B_{p,k}(\mathbb{R}^n)$ was introduced by Hörmander [1], as a generalization of the Sobolev space $H^s(\mathbb{R}^n)$, in his study of the theory of partial differential equations. We recall its definition.

Definition 3.1. A positive function k defined in $\mathbb{R}^n$ will be called a temperate weight function if there exist positive constants C and N such that

$$(3.1) \qquad k(\xi + \eta) \le (1 + C|\xi|)^N k(\eta); \quad \xi, \eta \in \mathbb{R}^n,$$

the set of all such functions k will be denoted by $\mathscr{K}$. Certain properties of the weight function k are contained in the following theorem whose proof can be found in [1].

Theorem 3.1. *If k_1 and k_2 belong to $\mathscr{K}$,then $k_1 + k_2, k_1 k_2, \sup(k_1, k_2), \inf(k_1, k_2)$, are also in $\mathscr{K}$. If $k \in \mathscr{K}$ we have $k^s \in \mathscr{K}$ for every real s,and if μ is a positive measure we have either $\mu * k \equiv \infty$ or else $\mu * k \in \mathscr{K}$.*

Definition 3.2. If $k \in \mathscr{K}$ and $1 \le p \le \infty$, we denote by $B_{p,k}$ the set of all distributions $u \in \mathscr{S}'$ such that $\hat{u}$ is a function and

$$(3.2) \qquad \| u \|_{p,k} = (2\pi)^{-n} \left(\int |k(\xi)\hat{u}|^p d\xi \right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty;$$

$$(3.3) \qquad \| u \|_{\infty,k} = ess \sup |k(\xi)\hat{u}(\xi)|.$$

We need the following theorem [1, p.10] in the proof of our boundedness result.

Theorem 3.2 (Lars Hörmander). *If $u_1 \in B_{p,k_1} \bigcap \mathscr{E}'$ and $u_2 \in B_{\infty,k_2}$ then $u_1 * u_2 \in B_{\infty,k_1 k_2}$, and we have the estimate*

$$(3.4) \qquad \| u_1 * u_2 \|_{p,k_1 k_2} \le \| u_1 \|_{p,k_1} \| u_2 \|_{\infty,k_2}, \quad 1 \le p < \infty.$$

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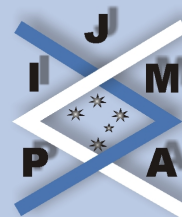
Using the above theorem we obtain the following boundedness result.

Theorem 3.3. *Let k_1 and k_2 belong to $\mathscr{K}$. Assume that $f \in B_{p,k_1} \bigcap \mathscr{E}'$ and $\psi \in B_{\infty,k_2}$ then the wavelet transform $(W_\psi f)(a,b) = (f * h_{a,0})(b)$, defined by (1.1) is in $B_{p,k_1 k_2}$, and*

$$(3.5) \qquad \| W_\psi f(a,b) \|_{p,k_1 k_2} \le a^{\frac{n}{2}} k_2 \left(\frac{1}{2a^2} \right) \| f \|_{p,k_1} \left\| \left(1 + \frac{C}{2} t^2 \right)^N \hat{\psi}(t) \right\|_\infty .$$
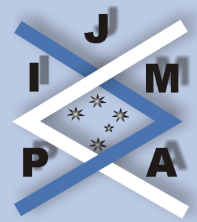
Proof. Since

$$
\begin{aligned}
\|h_{a,0}\|_{\infty,k_2} &= ess \sup \left| k_2(\xi) \hat{h}_{a,0}(\xi) \right| \\
&= ess \sup \left| k_2(\xi) a^{\frac{n}{2}} \hat{\psi}(a\xi) \right| \\
&\le a^{\frac{n}{2}} ess \sup \left| k_2(\tfrac{t}{a}) \hat{\psi}(t) \right| \\
&\le a^{\frac{n}{2}} k_2 \left(\frac{1}{2a^2} \right) ess \sup \left| \left(1 + \frac{C}{2} t^2 \right)^N \hat{\psi}(t) \right|
\end{aligned}
$$

on using (3.1). Hence by Theorem 3.2 we have

$$
\begin{aligned}
\|W_\psi f(a,b)\|_{p,k_1 k_2} &= \| (f * h_{a,0})(b) \|_{p,k_1 k_2} \\
&\le a^{\frac{n}{2}} k_2 \left(\frac{1}{2a^2} \right) \| f \|_{p,k_1} \left\| \left(1 + \frac{C}{2} t^2 \right)^N \hat{\psi}(t) \right\|_\infty .
\end{aligned}
$$

This proves the theorem. ∎

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4.   A General Boundedness Result

Using Young's inequality for convolution we obtained a general boundedness result for the wavelet transform. In the proof of our result the following theorem will be used [3, p. 90].

Theorem 4.1. *Let $p, q, r \geq 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Let $k \in L^p(\mathbb{R}^n)$, $f \in L^q(\mathbb{R}^n)$ and $g \in L^r(\mathbb{R}^n)$, then*

$$\| f * g \|_{\infty,k} = \left| \int_{\mathbb{R}^n} k(x)(f * g)(x)dx \right|$$

$$= \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k(x)f(x-y)g(y)dxdy \right|$$
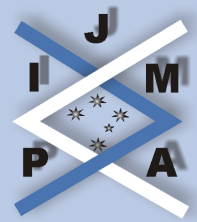
$$\leq C_{p,q,r;n} \| k \|_p \| f \|_q \| g \|_r .$$

The sharp constant $C_{p,q,r;n} = (C_p C_q C_r)^n$, where $C_p^2 = \dfrac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}$ with $(\frac{1}{p} + \frac{1}{p'} = 1)$.

Using Theorem 4.1 and following the same method of proof as for Theorem 3.3 we obtain the following boundedness result.

Theorem 4.2. *Let $p, q, r \geq 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$ and $k \in L^p(\mathbb{R}^n)$. Let $f \in L^q(\mathbb{R}^n)$ and $\psi \in L^r(\mathbb{R}^n)$, then*

$$\| W_\psi f \|_{\infty,k} \leq C_{p,q,r;n} a^{\frac{n}{r} - \frac{n}{2}} \| k \|_p \| f \|_q \| \psi \|_r$$

where $C_{p,q,r;n} = (C_p C_q C_r)^n$, $C_p^2 = \dfrac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

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