journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 1, Article 5, 4 pp.

A TRACE INEQUALITY FOR POSITIVE DEFINITE MATRICES

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Received 09 November, 2008; accepted 24 January, 2009
Communicated by F. Zhang

ABSTRACT. In this note we prove that $\mathrm{Tr}\left\{\mathbf{MN}+\mathbf{PQ}\right\} \geq 0$ when the following two conditions are met: (i) the matrices $\mathbf{M}, \mathbf{N}, \mathbf{P}, \mathbf{Q}$ are structured as follows $\mathbf{M} = \mathbf{A} - \mathbf{B}$, $\mathbf{N} = \mathbf{B}^{-1} - \mathbf{A}^{-1}$, $\mathbf{P} = \mathbf{C} - \mathbf{D}$, $\mathbf{Q} = (\mathbf{B}+\mathbf{D})^{-1} - (\mathbf{A}+\mathbf{C})^{-1}$ (ii) $\mathbf{A}, \mathbf{B}$ are positive definite matrices and $\mathbf{C}, \mathbf{D}$ are positive semidefinite matrices.

Key words and phrases: Trace inequality, positive definite matrices, positive semidefinite matrices.

2000 *Mathematics Subject Classification.* 15A45.

1. INTRODUCTION

Trace inequalities are useful in many applications. For example, trace inequalities naturally arise in control theory (see e.g., [1]) and in communication systems with multiple input and multiple output (see e.g., [2]). In this paper, the authors prove an inequality for which one application has already been identified: the uniqueness of a pure Nash equilibrium in concave games. Indeed, the reader will be able to check that the proposed inequality allows one to generalize the diagonally strict concavity condition introduced by Rosen in [3] to concave communication games with matrix strategies [4].

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Let us start with the scalar case. Let $\alpha, \beta, \gamma, \delta$ be four reals such that $\alpha > 0, \beta > 0, \gamma \geq 0, \delta \geq 0$. Then, it can be checked that we have the following inequality:

$$(\alpha - \beta)\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) + (\gamma - \delta)\left(\frac{1}{\beta + \delta} - \frac{1}{\alpha + \gamma}\right) \geq 0. \tag{1.1}$$

The main issue addressed here is to show that this inequality has a matrix counterpart, i.e., we want to prove the following theorem.

Theorem 1.1. *Let* $\mathbf{A}$, $\mathbf{B}$ *be two positive definite matrices and* $\mathbf{C}$, $\mathbf{D}$, *two positive semidefinite matrices. Then*

$$\mathcal{T} = \mathrm{Tr}\left\{(\mathbf{A} - \mathbf{B})(\mathbf{B}^{-1} - \mathbf{A}^{-1}) + (\mathbf{C} - \mathbf{D})[(\mathbf{B} + \mathbf{D})^{-1} - (\mathbf{A} + \mathbf{C})^{-1}]\right\} \geq 0. \tag{1.2}$$

The closest theorem available in the literature corresponds to the case $\mathbf{C} = \mathbf{D} = \mathbf{0}$, in which case the above theorem is quite easy to prove. There are many proofs possible, the most simple of them is probably the one provided by Abadir and Magnus in [5]. In order to prove Theorem 1.1 in Sec. 3 we will use some intermediate results which are provided in the following section.

2. AUXILIARY RESULTS

Here we state three lemmas. The first two lemmas are available in the literature and the last one is easy to prove. The first lemma is the one mentioned in the previous section and corresponds to the case $\mathbf{C} = \mathbf{D} = \mathbf{0}$.

Lemma 2.1 ([5])**.** *Let* $\mathbf{A}$, $\mathbf{B}$ *be two positive definite matrices. Then*

$$\mathrm{Tr}\left\{(\mathbf{A} - \mathbf{B})(\mathbf{B}^{-1} - \mathbf{A}^{-1})\right\} \geq 0. \tag{2.1}$$

The second lemma is very simple and can be found, for example, in [5]. It is as follows.

Lemma 2.2 ([5])**.** *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be two positive semidefinite matrices. Then*

$$\mathrm{Tr}\{\mathbf{M}\mathbf{N}\} \geq 0. \tag{2.2}$$

At last, we will need the following result.

Lemma 2.3. *Let* $\mathbf{A}$, $\mathbf{B}$ *be two positive definite matrices,* $\mathbf{C}$, $\mathbf{D}$, *two positive semidefinite matrices whereas* $\mathbf{X}$ *is only assumed to be Hermitian. Then*

$$\mathrm{Tr}\left\{\mathbf{X}\mathbf{A}^{-1}\mathbf{X}\mathbf{B}^{-1}\right\} - \mathrm{Tr}\left\{\mathbf{X}(\mathbf{A} + \mathbf{C})^{-1}\mathbf{X}(\mathbf{B} + \mathbf{D})^{-1}\right\} \geq 0. \tag{2.3}$$

Proof. First note that $\mathbf{A} + \mathbf{C} \succeq \mathbf{A}$ implies (see e.g., [6]) that $\mathbf{A}^{-1} \succeq (\mathbf{A} + \mathbf{C})^{-1} \succeq 0$ and that $\mathbf{A}^{-1} - (\mathbf{A} + \mathbf{C})^{-1} \succeq 0$. In a similar way we have $\mathbf{B}^{-1} - (\mathbf{B} + \mathbf{D})^{-1} \succeq 0$. Therefore we obtain the following two inequalities:

$$\begin{aligned} \mathrm{Tr}\left\{\mathbf{X}\mathbf{A}^{-1}\mathbf{X}\mathbf{B}^{-1}\right\} &\overset{(a)}{\geq} \mathrm{Tr}\left\{\mathbf{X}\mathbf{A}^{-1}\mathbf{X}(\mathbf{B} + \mathbf{D})^{-1}\right\} \\ \mathrm{Tr}\left\{\mathbf{A}^{-1}\mathbf{X}(\mathbf{B} + \mathbf{D})^{-1}\mathbf{X}\right\} &\overset{(b)}{\geq} \mathrm{Tr}\left\{(\mathbf{A} + \mathbf{C})^{-1}\mathbf{X}(\mathbf{B} + \mathbf{D})^{-1}\mathbf{X}\right\} \end{aligned} \tag{2.4}$$

where (a) follows by applying Lemma 2.2 with $\mathbf{M} = \mathbf{X}\mathbf{A}^{-1}\mathbf{X}$ and $\mathbf{N} = \mathbf{B}^{-1} - (\mathbf{B} + \mathbf{D})^{-1} \succeq 0$ and (b) follows by applying the same lemma with $\mathbf{M} = \mathbf{A}^{-1} - (\mathbf{A} + \mathbf{C})^{-1} \succeq 0$ and $\mathbf{N} = \mathbf{X}(\mathbf{B} + \mathbf{D})^{-1}\mathbf{X}$. Using the fact that $\mathrm{Tr}\left\{\mathbf{X}\mathbf{A}^{-1}\mathbf{X}(\mathbf{B} + \mathbf{D})^{-1}\right\} = \mathrm{Tr}\left\{\mathbf{A}^{-1}\mathbf{X}(\mathbf{B} + \mathbf{D})^{-1}\mathbf{X}\right\}$ we obtain the desired result. □

3. PROOF OF THEOREM 1.1

Let us define the auxiliary quantities $\mathcal{T}_1$ and $\mathcal{T}_2$ as $\mathcal{T}_1 \triangleq \operatorname{Tr}\{(\mathbf{A}-\mathbf{B})(\mathbf{B}^{-1}-\mathbf{A}^{-1})\}$ and $\mathcal{T}_2 \triangleq \operatorname{Tr}\{(\mathbf{C}-\mathbf{D})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\}$. Assuming $\mathcal{T}_2 \geq 0$ directly implies that $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 \geq 0$ since $\mathcal{T}_1$ is always non-negative after Lemma 2.1. As a consequence, we will only consider, from now on, the non-trivial case where $\mathcal{T}_2 < 0$ (Assumption (A)).

First we rewrite $\mathcal{T}$ as:

$$
\begin{aligned}
\mathcal{T} &= \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}^{-1}-\mathbf{A}^{-1})\right\} \\
&\qquad + \operatorname{Tr}\left\{[(\mathbf{A}+\mathbf{C})-(\mathbf{B}+\mathbf{D})][(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\} \\
&\qquad - \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\} \\
&\overset{(c)}{\geq} \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}^{-1}-\mathbf{A}^{-1})\right\} - \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\} \\
&= \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})\mathbf{B}^{-1}(\mathbf{A}-\mathbf{B})\mathbf{A}^{-1}\right\} \\
&\qquad - \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}[(\mathbf{A}+\mathbf{C})-(\mathbf{B}+\mathbf{D})](\mathbf{B}+\mathbf{D})^{-1}\right\} \\
&= \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})\mathbf{B}^{-1}(\mathbf{A}-\mathbf{B})\mathbf{A}^{-1}\right\} - \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}\right\} \\
&\qquad - \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}\right\}
\end{aligned}
$$

where (c) follows from Lemma 2.1. We see from the last equality that if we can prove that

$$\operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}\right\} \leq 0,$$

then proving $\mathcal{T} \geq 0$ boils down to showing that

$$
\begin{aligned}
\mathcal{T}' &\triangleq \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})\mathbf{B}^{-1}(\mathbf{A}-\mathbf{B})\mathbf{A}^{-1}\right\} \\
&\qquad - \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}\right\} \geq 0.
\end{aligned} \tag{3.1}
$$

Let us show that $\operatorname{Tr}\{(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}\} \leq 0$. By assumption we have that $\operatorname{Tr}\{(\mathbf{C}-\mathbf{D})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\} < 0$ which is equivalent to

$$
\begin{aligned}
&\operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\} \\
&\qquad > \operatorname{Tr}\left\{[(\mathbf{A}+\mathbf{C})-(\mathbf{B}+\mathbf{D})][(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\}.
\end{aligned} \tag{3.2}
$$

From this inequality and Lemma 2.1 we have that

$$\operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\} > 0. \tag{3.3}$$

On the other hand, let us rewrite $\mathcal{T}_2$ as

$$
\begin{aligned}
\mathcal{T}_2 &= \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}[(\mathbf{A}-\mathbf{B})+(\mathbf{C}-\mathbf{D})](\mathbf{A}+\mathbf{C})^{-1}\right\} \\
&= \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\} \\
&\qquad + \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}\right\} \\
&= \operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\} + \operatorname{Tr}[\mathbf{Y}\mathbf{Y}^H]
\end{aligned} \tag{3.4}
$$

where $\mathbf{Y} = (\mathbf{A}+\mathbf{C})^{-1/2}(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1/2}$. Thus $\mathcal{T}_2 < 0$ implies that:

$$\operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\} < 0, \tag{3.5}$$

which is exactly the desired result since

$$
\begin{aligned}
&\operatorname{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\} \\
&\qquad = \operatorname{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}\right\}.
\end{aligned}
$$

In order to conclude the proof we only need to prove that $\mathcal{T}' \geq 0$. This is achieved on noticing that $\mathcal{T}'$ can be rewritten as

$$\mathcal{T}' \triangleq \mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})\mathbf{A}^{-1}(\mathbf{A}-\mathbf{B})\mathbf{B}^{-1}\right\} - \mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}\right\}$$

and calling for Lemma 2.3 with $\mathbf{X} = \mathbf{A} - \mathbf{B}$, concluding the proof.

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