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MULTIPLICATIVE PRINCIPAL-MINOR INEQUALITIES FOR A CLASS OF OSCILLATORY MATRICES

XIAO PING LIU AND SHAUN M. FALLAT

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF REGINA
REGINA, SASKATCHEWAN, S4S 0A2.
liuxp@math.uregina.ca

sfallat@math.uregina.ca

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ABSTRACT. A square matrix is said to be totally nonnegative (respectively, positive) if all of its minors are nonnegative (respectively, positive). Determinantal inequalities have been a popular and important subject, especially for positivity classes of matrices such as: positive semidefinite matrices, $M-$matrices, and totally nonnegative matrices. Our main interest lies in characterizing all of the inequalities that exist among products of both principal and non-principal minors of certain subclasses of invertible totally nonnegative matrices. This description is accomplished by providing a complete list of associated multiplicative generators.

Key words and phrases: Totally positive matrices; Determinant; Principal minor; Bidiagonal factorization, Determinantal inequalities; Generators.

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1. INTRODUCTION

An $n \times n$ matrix A is called *totally positive*, TP (*totally nonnegative*, TN) if **every** minor of A is positive (nonnegative) (see [1, 10, 13]). An $n \times n$ matrix A is called *an oscillatory matrix*, OSC, if A is totally nonnegative and there exists a positive integer k, so that A^k is a totally positive matrix. Such matrices arise in a variety of applications [11], have been studied most of the 20^{th} century, and continue to be a topic of current interest.

Relationships among principal minors, particularly inequalities that occur among products of principal minors, for all matrices in a given class of square matrices have been studied for various classes of matrices (see [7] and references therein). In the case of positive definite matrices and M-matrices, many classical inequalities are known to hold (see standard submatrix

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notation below):

Hadamard: $\det A \leq \prod_{i=1}^{n} a_{ii}$;

Fischer: $\det A \leq \det A[S] \cdot \det A[S^c]$, for $S \subseteq \{1, 2, \ldots, n\}$;

Koteljanskii: $\det A[S \cup T] \cdot \det A[S \cap T] \leq \det A[S] \cdot \det A[T]$ for $S,\, T \subseteq \{1, 2, \ldots, n\}$.

These inequalities also hold for $n \times n$ totally nonnegative matrices, e.g. [1, 10, 14].

For the past few years, multiplicative inequalities have been studied in great detail (see, for example, the survey paper [7]) for classes such as: positive definite (see [2]); $M-$ and inverse $M-$ matrices (see [6]); tridiagonal $P-$matrices (see [8]). In [5] a complete description of all such inequalities (in the principal case) for $n \times n$ totally nonnegative matrices was given for $n \leq 5$.

In this paper our purpose is to better understand all inequalities among products of minors that hold for general TP matrices. We note that since TN is the closure of the TP matrices (see [1]), the inequalities in TN are the same as those in the class TP or invertible TN. Thus it suffices to consider inequalities within the class of invertible TN matrices, and there, because of the positivity of minors, we may consider ratios of products of minors and ask which are bounded by a constant independent of n throughout the entire class of invertible TN matrices. The classification of all such inequalities for general TP matrices is currently unresolved. Our plan is to restrict ourselves to a subclass of invertible TN matrices, which we conveniently refer to as STEP 1.

Here we identify all ratios of products of principal and non-principal minors bounded, and give a complete description of the generators for the class STEP 1, for each n. All bounded ratios that we identify are actually bounded by 1, and thus are all inequalities.

The paper is organized into two parts: In the first part we investigate the principal case, while the latter part studies the non-principal case.

2. PRELIMINARIES AND BACKGROUND

For an $n \times n$ matrix $A = [a_{ij}]$ and $\alpha, \beta \subseteq N \equiv \{1, 2, \ldots, n\}$, the submatrix of A lying in rows indexed by α and columns indexed by β will be denoted by $A[\alpha|\beta]$. If $\alpha = \beta$, then the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. For brevity, we may also let (S) denote $\det A[S]$.

Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ denote a collection of index sets (repeats allowed), where $\alpha_i \subseteq N$, $i = 1, 2, \ldots, p$. Then we define $\alpha(A) = \det A[\alpha_1] \det A[\alpha_2] \cdots \det A[\alpha_p]$. If, further, $\beta = \{\beta_1, \beta_2, \ldots, \beta_q\}$ is another collection of index sets with $\beta_i \subseteq N$, for all i, then we write $\alpha \leq \beta$ with respect to $\mathcal{C}$ if $\alpha(A) \leq \beta(A)$, for every $n \times n$ matrix A in $\mathcal{C}$.

We also consider ratios of products of principal minors. For two given collections α and β of index sets we interpret $\frac{\alpha}{\beta}$ as both a numerical ratio $\frac{\alpha(A)}{\beta(A)}$ for a given matrix A in $\mathcal{C}$ and as a formal ratio to be manipulated according to natural rules. Since, by convention, $\det A[\phi] = 1$, we also assume, without loss of generality, that in any ratio $\frac{\alpha}{\beta}$ both collections α and β have the same number of index sets.

Each of the classical inequalities discussed in the introduction may be written in our form $\alpha \leq \beta$. For example, Koteljanskii's inequality has the collections $\alpha = \{S \cup T, S \cap T\}$ and $\beta = \{S, T\}$. Our main problem of interest is to characterize, via set-theoretic conditions, all pairs of collections of index sets such that

$$\frac{\alpha(A)}{\beta(A)} \leq K,$$

for some constant $K \geq 0$ (which depends on n) and for all $n \times n$ matrices A in $\mathcal{C}$. If such a constant exists for all matrices A in $\mathcal{C}$, we say that the ratio $\frac{\alpha}{\beta}$ is *bounded* with respect to the class of $\mathcal{C}$ matrices.

Let α be any given collection of index sets. For $i \in \{1, 2, \ldots, n\}$, let $f_\alpha(i)$ be the number of index sets in α that contain the element i (see also [2, 6]). The next result gives a simple necessary (but by no means sufficient) condition for a given ratio of principal minors to be bounded with respect to the TN matrices.

Lemma 2.1. *Let α and β be two collections of index sets. If $\frac{\alpha}{\beta}$ is bounded with respect to any subclass of invertible TN matrices that includes all positive diagonal matrices, then $f_\alpha(i) = f_\beta(i)$, for every $i = 1, 2, \ldots, n$.*

If a given ratio $\frac{\alpha}{\beta}$ satisfies the condition $f_\alpha(i) = f_\beta(i)$, then we say the ratio satisfies ST0 (set-theoretic) (see also [2, 6]).

The fact that a TN matrix has an elementary bidiagonal factorization (see [3, 4]) seems to be a very useful fact for verifying when a ratio is bounded. By definition, an *elementary bidiagonal matrix* is an $n \times n$ matrix whose main diagonal entries are all equal to one, and there is at most one nonzero off-diagonal entry and this entry must occur on the super- or subdiagonal. To this end, we denote by $E_k(\mu) = [c_{ij}]$ ($2 \leq k \leq n$), the lower elementary bidiagonal matrix whose elements are given by

$$c_{ij} = \begin{cases} 1, & \text{if } i = j, \\ \mu, & \text{if } i = k,\ j = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

The next result can be found in [12].

Theorem 2.2. *Let A be an $n \times n$ invertible TN matrix. Then A can be written as*

$$A = (E_2(l_k))(E_3(l_{k-1})E_2(l_{k-2})) \cdots (E_n(l_{n-1}) \cdots E_3(l_2)E_2(l_1))D \\ (E_2^T(u_1)E_3^T(u_2) \cdots E_n^T(u_{n-1})) \cdots (E_2^T(u_{k-2})E_3^T(u_{k-1}))(E_2^T(u_k)), \tag{2.1}$$

where $k = \binom{n}{2}$; $l_i, u_j \geq 0$ for all $i, j \in \{1, 2, \ldots, k\}$; and D is a positive diagonal matrix.

Further, given the factorization above (2.1), we introduce the following notation:

$$\begin{aligned}
D &= \operatorname{diag}(d_1, d_2, \ldots, d_n), \\
L_1 &= (E_n(l_{n-1}) \cdots E_3(l_2)E_2(l_1)), \quad U_1 = (E_2^T(u_1)E_3^T(u_2) \cdots E_n^T(u_{n-1})), \\
&\vdots \\
L_{n-2} &= (E_3(l_{k-1})E_2(l_{k-2})), \quad U_{n-2} = (E_2^T(u_{k-2})E_3^T(u_{k-1})), \\
L_{n-1} &= (E_2(l_k)), \quad U_{n-1} = (E_2^T(u_k)),
\end{aligned}$$

where $k = \binom{n}{2}$. Then (2.1) is equivalent to

$$A = L_{n-1}L_{n-2} \cdots L_1 D U_1 \cdots U_{n-2}U_{n-1},$$

and observe that each L_i and U_j are themselves invertible bidiagonal TN matrices.

A collection of bounded ratios with respect to a fixed class of matrices is referred to as *generators* if any bounded ratio with respect to that fixed class of matrices can be written as products of positive powers of ratios from this collection.

For $n = 3, 4, 5$ there is a complete description of the generators for the bounded ratios of principal minors of TP matrices in [5]. For clarity of exposition we state this characterization in the case $n = 4$.

Theorem 2.3 ([5]). *Suppose α/β is a ratio of principal minors for TP matrices with $n = 4$. Then α/β is bounded with respect to the totally positive matrices if and only if α/β can be written as a product of positive powers of the generators listed below:*

$$\frac{(14)(\emptyset)}{(1)(4)}, \frac{(2)(124)}{(12)(24)}, \frac{(3)(134)}{(13)(34)}, \frac{(23)(1234)}{(123)(234)}, \frac{(12)(3)}{(13)(2)}, \frac{(1)(24)}{(2)(14)}, \frac{(2)(34)}{(3)(24)},$$
$$\frac{(4)(13)}{(3)(14)}, \frac{(12)(134)}{(13)(124)}, \frac{(13)(234)}{(23)(134)}, \frac{(34)(124)}{(24)(134)}, \frac{(24)(123)}{(23)(124)}, \frac{(14)(23)}{(13)(24)}.$$

3. STEP 1: A CLASS OF OSCILLATORY MATRICES

By Theorem 2.2, any A $n \times n$ OSC matrix A can be written as

$$A = L_{n-1}L_{n-2}\cdots L_1 D U_1 \cdots U_{n-2}U_{n-1}.$$

We consider a special class of invertible TN matrices. Let

$$\mathbf{A_1} = \{A | A = L_1 D U_1\},$$

and we refer to $\mathbf{A_1}$ as the class STEP 1.

Note that

$$L_1 = (E_n(l_{n-1})\cdots E_3(l_2)E_2(l_1)), \quad U_1 = (E_2^T(u_1)E_3^T(u_2)\cdots E_n^T(u_{n-1})),$$

where $l_i > 0, u_i > 0$ and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, $d_i > 0$. In fact, each $A \in \mathbf{A_1}$ is indeed an OSC matrix.

The next result is a straightforward computation.

Lemma 3.1. *For any $A \in \mathbf{A_1}$, A can be written as follows:*

$$\begin{pmatrix} \overline{11} & \overline{11}u_1 & \overline{11}u_1u_2 & \cdots\cdots & \overline{11}u_1u_2\ldots u_i & \cdots\cdots & \overline{11}u_1u_2\ldots u_{n-1} \\ l_1\overline{11} & \overline{12} & \overline{12}u_2 & \cdots\cdots & \overline{12}u_2\ldots u_i & \cdots\cdots & \overline{12}u_2\ldots u_{n-1} \\ l_2l_1\overline{11} & l_2\overline{12} & \overline{13} & \cdots\cdots & \overline{13}u_3\ldots u_i & \cdots\cdots & \overline{13}u_3\ldots u_{n-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ l_i\cdots l_2l_1\overline{11} & l_i\cdots l_2\overline{12} & l_i\cdots l_3\overline{13} & \cdots\cdots & \overline{1i} & \cdots\cdots & \overline{1i}u_i\ldots u_{n-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ l_{n-1}\cdots l_2l_1\overline{11} & l_{n-1}\cdots l_2\overline{12} & l_{n-1}\cdots l_3\overline{13} & \cdots\cdots & l_{n-1}\cdots l_i\overline{1i} & \cdots\cdots & \overline{1n} \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned} \overline{11} &= d_1, \\ \overline{12} &= d_2 + t_1d_1, \\ \overline{13} &= d_3 + t_2d_2 + t_2t_1d_1, \\ &\vdots \\ \overline{1j} &= d_j + t_{j-1}d_{j-1} + t_{j-1}t_{j-2}d_{j-2} + \cdots + t_{j-1}\cdots t_2t_1d_1, \\ &\vdots \\ \overline{1n} &= d_n + t_{n-1}d_{n-1} + t_{n-1}t_{n-2}d_{n-2} + \cdots + t_{n-1}\cdots t_2t_1d_1, \end{aligned}$$

and where $t_j = l_ju_j$ for $j \in \{1, 2, \ldots, n-1\}$.

Further we lay out the following notation:

$$\overline{11} = d_1, \quad \overline{22} = d_2, \quad \overline{33} = d_3, \quad \ldots, \quad \overline{nn} = d_n,$$
$$\overline{ij} = d_j + t_{j-1}d_{j-1} + t_{j-1}t_{j-2}d_{j-2} + \cdots + t_{j-1}\cdots t_id_i \quad (1 \le i < j \le n).$$

For example,

$$\overline{35} = d_5 + t_4 d_4 + t_4 t_3 d_3, \quad \overline{14} = d_4 + t_3 d_3 + t_3 t_2 d_2 + t_3 t_2 t_1 d_1.$$

Using the above notation we conclude that

Lemma 3.2. *Let $\alpha = \{i_1, i_1+1, \dots, i_1+k_{i_1}; i_2, i_2+1, \dots, i_2+k_{i_2}; \dots; i_p, i_p+1, \dots, i_p+k_{i_p}\}$ be a subset of $\{1,2,3,\dots,n\}$ with $\{i_q, i_q+1, \dots, i_q+k_{i_q}\}$ based on contiguous indices for $q \in \{1,2,3,\dots,p\}$, and $i_1 \le i_1 + k_{i_1} < i_2 \le i_2 + k_{i_2} < \cdots < i_p \le i_p + k_{i_p}$, $k_{i_q} \ge 0$ for $q \in \{1,2,3,\dots,p\}$, then for any matrix $A \in \mathbf{A_1}$, we have*

$$\begin{aligned}(\alpha) = \overline{1 i_1}\ &\left[\overline{(i_1+1)(i_1+1)}\ \cdots\ \overline{(i_1+k_{i_1})(i_1+k_{i_1})}\right]\ \overline{(i_1+k_{i_1}+1)(i_2)} \\ &\left[\overline{(i_2+1)(i_2+1)}\ \cdots\ \overline{(i_2+k_{i_2})(i_2+k_{i_2})}\right]\ \overline{(i_2+k_{i_2}+1)(i_3)}\ \cdots \\ &\overline{(i_{p-1}+k_{i_{p-1}}+1)(i_{p+1})}\ \left[\overline{(i_p+1)(i_p+1)}\ \cdots\ \overline{(i_p+k_{i_p})(i_p+k_{i_p})}\right]. \end{aligned} \tag{3.2}$$

As an illustration of Lemma 3.2, consider the following examples.

Example 3.1.

(1) $(3) = \overline{13}$, $(15) = \overline{11}\ \overline{25}$.
(2) $(124) = \overline{11}\ \overline{22}\ \overline{34}$, $(35689) = \overline{13}\ \overline{45}\ \overline{66}\ \overline{78}\ \overline{99}$.
(3) $(23\ 5\ 789) = \overline{12}\,\overline{33}\ \overline{45}\ \overline{67}\,\overline{88}\,\overline{99}$.

Remark 1. Lemma 3.2 demonstrates that any minor is a product of terms $\overline{ij}$.

It is easy to deduce the following result, from the above analysis.

Lemma 3.3. *Suppose $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_p\}$ and $\beta = \{\beta_1, \beta_2, \dots, \beta_p\}$ denote two collections of index sets (repeats allowed) and that the ratio $\frac{\alpha}{\beta}$ satisfies $ST0$. Then with respect to STEP 1, $\frac{\alpha}{\beta}$ can be written as follows:*

$$\frac{\alpha}{\beta} = \frac{\overline{i_1 j_1}\ \overline{i_2 j_2}\ \cdots\ \overline{i_k j_k}}{\overline{s_1 t_1}\ \overline{s_2 t_2}\ \cdots\ \overline{s_k t_k}},$$

where $\{j_1, j_2, \dots, j_k\} = \{t_1, t_2, \dots, t_k\}$. Here $i_u \le j_u$ for $u \in \{1,2,3,\dots,k\}$ and $s_u \le t_u$ for $u \in \{1,2,3,\dots,l\}$.

Example 3.2.

(1)

$$\begin{aligned}\frac{\alpha}{\beta} &= \frac{(23)\ (145)\ (24578)}{(124)\ (345)\ (2)(57)(8)} \\ &= \frac{\overline{12}\ \overline{33}\ \overline{11}\ \overline{24}\ \overline{55}\ \overline{12}\ \overline{34}\ \overline{55}\ \overline{67}\ \overline{88}}{\overline{11}\ \overline{22}\ \overline{34}\ \overline{13}\ \overline{44}\ \overline{55}\ \overline{12}\ \overline{15}\ \overline{67}\ \overline{18}} \\ &= \frac{\overline{12}\ \overline{33}\ \overline{24}\ \overline{55}\ \ \overline{88}}{\overline{22}\ \overline{13}\ \overline{44}\ \overline{15}\ \overline{18}}.\end{aligned}$$

(2)

$$\begin{aligned}\frac{\alpha}{\beta} &= \frac{(123)\ (2345)\ (567)(8)}{(12)\ (23)\ (3456)(5)(78)} \\ &= \frac{\overline{11}\ \overline{22}\ \overline{33}\ \overline{12}\ \overline{33}\ \overline{44}\ \overline{55}\ \overline{15}\ \overline{66}\ \overline{77}\ \overline{18}}{\overline{11}\ \overline{22}\ \overline{12}\ \overline{33}\ \overline{13}\ \overline{44}\ \overline{55}\ \overline{66}\ \overline{15}\ \overline{17}\ \overline{88}} \\ &= \frac{\overline{33}\ \overline{77}\ \overline{18}}{\overline{13}\ \overline{17}\ \overline{88}}.\end{aligned}$$

Remark 2.

(1) In Example 1 (2) we note that the sets $\{j_1, j_2, \ldots\ldots, j_k\}$ and $\{t_1, t_2, \ldots\ldots, t_k\}$ both equal $\{3, 7, 8\}$, (the fact they are equal follows from STO and Lemma 3.2) and as such $\frac{\alpha}{\beta}$ may be written as $\frac{\overline{33}\;\overline{77}\;\overline{18}}{\overline{13}\;\overline{17}\;\overline{88}}$ in which $(j_* = t_*, * = 3, 7, 8)$.

(2) Lemma 3.3 will be used to produce the form of the generators for the multiplicative bounded ratios with respect to the class STEP 1.

4. BOUNDED RATIOS AND GENERATORS FOR STEP 1

Based on Lemmas 3.2 and 3.3 above, we can construct the form of any potential generator. We begin this construction with the following key lemma.

Lemma 4.1. *Suppose i, j, u, k, s, l, are natural integers and $i \leq j, u \leq j, k \leq l, s \leq l$.*

(1) $\frac{\overline{ij}}{\overline{uj}} < 1$ *with respect to* $\mathbf{A_1}$ *if and only if* $u < i$.

(2) $\frac{\overline{ij}}{\overline{uj}} \frac{\overline{kl}}{\overline{sl}} < 1$ *with respect to* $\mathbf{A_1}$ *when* $j < l$ *if and only if* $u \leq k < s \leq i \leq j < l$.

Proof.

(1)

$$\begin{aligned}\frac{\overline{ij}}{\overline{uj}} < 1 &\Leftrightarrow \overline{ij} < \overline{uj} \\ &\Leftrightarrow d_j + t_{j-1}d_{j-1} + \cdots + t_{j-1}\cdots t_i d_i < d_j + t_{j-1}d_{j-1} + \cdots + t_{j-1}\cdots t_u d_u \\ &\Leftrightarrow u < i.\end{aligned}$$

(2) Enumerating all cases possible on the relations between u, i, k, s will result in the desired conclusion.

□

Remark 3. In particular, $\frac{\overline{(i+1)j}}{\overline{ij}} \frac{\overline{i(j+1)}}{\overline{(i+1)(j+1)}} < 1$ with respect to $\mathbf{A_1}$, for all $1 \leq i < j \leq n$.

Now consider the following terms associated with $\mathbf{A_1}$ (the class Step 1):

$$(\star)\qquad \begin{array}{cccccccccc}
\overline{11} & \overline{12} & \overline{13} & \overline{14} & \overline{15} & \overline{16} & \cdots & \overline{1(n-1)} & & \overline{1n} \\
& \overline{22} & \overline{23} & \overline{24} & \overline{25} & \overline{26} & \cdots & \overline{2(n-1)} & & \overline{2n} \\
& & \overline{33} & \overline{34} & \overline{35} & \overline{36} & \cdots & \overline{3(n-1)} & & \overline{3n} \\
& & & \overline{44} & \overline{45} & \overline{46} & \cdots & \overline{4(n-1)} & & \overline{4n} \\
& & & & \overline{55} & \overline{56} & \cdots & \overline{5(n-1)} & & \overline{5n} \\
& & & & & & \cdots & \cdots & & \cdots \\
& & & & & & & \overline{(n-1)(n-1)} & & \overline{(n-1)n} \\
& & & & & & & & & \overline{nn}
\end{array}$$

From the diagram ($\star$) above we construct a list of *special* ratios, which will be used later.

$$
\begin{array}{ccccccc}
\frac{\overline{22}}{\overline{12}}\frac{\overline{13}}{\overline{23}} & \frac{\overline{23}}{\overline{13}}\frac{\overline{14}}{\overline{24}} & \frac{\overline{24}}{\overline{14}}\frac{\overline{15}}{\overline{25}} & \frac{\overline{25}}{\overline{15}}\frac{\overline{16}}{\overline{26}} & \cdots & \frac{\overline{2(n-1)}}{\overline{1(n-1)}}\frac{\overline{1n}}{\overline{2n}} & \frac{\overline{2n}}{\overline{1n}} \\
 & \frac{\overline{33}}{\overline{23}}\frac{\overline{24}}{\overline{34}} & \frac{\overline{34}}{\overline{24}}\frac{\overline{25}}{\overline{35}} & \frac{\overline{35}}{\overline{25}}\frac{\overline{26}}{\overline{36}} & \cdots & \frac{\overline{3(n-1)}}{\overline{2(n-1)}}\frac{\overline{2n}}{\overline{3n}} & \frac{\overline{3n}}{\overline{2n}} \\
 & & \frac{\overline{44}}{\overline{34}}\frac{\overline{35}}{\overline{45}} & \frac{\overline{45}}{\overline{35}}\frac{\overline{36}}{\overline{46}} & \cdots & \frac{\overline{4(n-1)}}{\overline{3(n-1)}}\frac{\overline{3n}}{\overline{4n}} & \frac{\overline{4n}}{\overline{3n}} \\
 & & & \frac{\overline{55}}{\overline{45}}\frac{\overline{46}}{\overline{56}} & \cdots & \frac{\overline{5(n-1)}}{\overline{4(n-1)}}\frac{\overline{4n}}{\overline{5n}} & \frac{\overline{5n}}{\overline{4n}} \\
 & & & & \ddots & \cdots & \cdots \\
 & & & & & \frac{\overline{(n-1)(n-1)}}{\overline{(n-2)(n-1)}}\frac{\overline{(n-2)n}}{\overline{(n-1)n}} & \frac{\overline{n-1n}}{\overline{(n-2)n}} \\
 & & & & & & \frac{\overline{nn}}{\overline{(n-1)n}}
\end{array}
\tag{4.1}
$$

For comparison sake, the above list can be written in terms of principal minors as follows:

$$
\begin{array}{ccccccc}
\frac{(12)}{(13)}\frac{(3)}{(2)} & \frac{(13)}{(14)}\frac{(4)}{(3)} & \frac{(14)}{(15)}\frac{(5)}{(4)} & \frac{(15)}{(16)}\frac{(6)}{(5)} & \cdots & \frac{(1(n-1))}{(1n))}\frac{(n)}{(n-1)} & \frac{(1n)}{(1)}\frac{(\phi)}{(n)} \\
 & \frac{(23)}{(24)}\frac{(14)}{(13)} & \frac{(24)}{(25)}\frac{(15)}{(14)} & \frac{(25)}{(26)}\frac{(16)}{(15)} & \cdots & \frac{(2(n-1))}{(2n))}\frac{(1n)}{(1n-1)} & \frac{(2n)}{(1n)}\frac{(1)}{(2)} \\
 & & \frac{(34)}{(35)}\frac{(25)}{(24)} & \frac{(35)}{(36)}\frac{(26)}{(25)} & \cdots & \frac{(3(n-1))}{(3n))}\frac{(2n)}{(2n-1)} & \frac{(3n)}{(2n)}\frac{(2)}{(3)} \\
 & & & \frac{(45)}{(46)}\frac{(36)}{(35)} & \cdots & \frac{(4(n-1))}{(4n))}\frac{(3n)}{(3n-1)} & \frac{(4n)}{(3n)}\frac{(3)}{(4)} \\
 & & & & \cdots & \frac{(5(n-1))}{(5n))}\frac{(4n)}{(4n-1)} & \frac{(5n)}{(4n)}\frac{(4)}{(5)} \\
 & & & & \ddots & \cdots & \cdots \\
 & & & & & \frac{((n-2)(n-1))}{((n-3)(n-1))}\frac{((n-3)n)}{((n-2)n)} & \frac{((n-2)n)}{((n-3)n)}\frac{(n-3)}{(n-2)} \\
 & & & & & & \frac{((n-1)n)}{((n-2)n)}\frac{(n-2)}{(n-1)}
\end{array}
\tag{4.2}
$$

By Lemma 2, all the above ratios are bounded by 1, which we will show are, in fact, *generators*.

Lemma 4.2.

(1) *If* $\frac{\overline{ij}}{\overline{uj}} < 1$ *with respect to* $\mathbf{A_1}$, *then* $\frac{\overline{ij}}{\overline{uj}}$ *is a product of some of the ratios taken from the above list (4.1).*

(2) *Any ratio* $\frac{\alpha}{\beta}$ *over* $\mathbf{A_1}$ *that satisfies* $ST0$ *can be written as follows:*

$$
\frac{\alpha}{\beta} = \prod_{j=2}^{n} \left[\left(\frac{\overline{i_{j_1}j}\ \ \overline{i_{j_2}j}\ \ \cdots\ \ \overline{i_{j_p}j}}{\overline{u_{j_1}j}\ \ \overline{u_{j_2}j}\ \ \cdots\ \ \overline{u_{j_p}j}} \right) \left(\frac{\overline{k_{j_1}j}\ \ \overline{k_{j_2}j}\ \ \cdots\ \ \overline{k_{j_q}j}}{\overline{s_{j_1}j}\ \ \overline{s_{j_2}j}\ \ \cdots\ \ \overline{s_{j_q}j}} \right) \right], \tag{4.3}
$$

where $u_{j_t} < i_{j_t}$, for $t \in \{1, 2, \ldots, p\}$; $k_{j_t} < s_{j_t}$, for $t \in \{1, 2, \ldots, q\}$, and that satisfies the requirement that the u's are distinct from the i's and k's, the s's are distinct from the i's and k's. Therefore any ratio $\frac{\alpha}{\beta}$ that satisfies $ST0$ can be written as a ratio of products of elements from the list (4.1).

Proof. To establish (1), observe that

$$\frac{\overline{ij}}{\overline{uj}} = \left\{ \begin{array}{ccc} \left(\frac{\overline{ij}}{\overline{(i-1)j}}\frac{\overline{(i-1)(j+1)}}{\overline{i(j+1)}}\right) & \cdots\cdot & \left(\frac{\overline{in}}{\overline{(i-1)n}}\right) \\ \left(\frac{\overline{(i-1)j}}{\overline{(i-2)j}}\frac{\overline{(i-2)(j+1)}}{\overline{(i-1)(j+1)}}\right) & \cdots\cdot & \left(\frac{\overline{(i-1)n}}{\overline{(i-2)n}}\right) \\ \cdots & \cdots & \cdots \\ \left(\frac{\overline{(u+1)j}}{\overline{uj}}\frac{\overline{u(j+1)}}{\overline{(u+1)(j+1)}}\right) & \cdots\cdot & \left(\frac{\overline{(u+1)n}}{\overline{(u)n}}\right) \end{array} \right\}.$$

For item (2), we note that, by Lemma 3.3, we have (4.3). Finally observe that

$$\frac{\overline{k_{j_t}j}}{\overline{s_{j_t}j}} = \frac{1}{\frac{\overline{s_{j_t}j}}{\overline{k_{j_t}j}}},$$

which completes the proof. □

Example 4.1. The generators for $\mathbf{A_1}$ in the case $n = 9$ are as follows:

$\frac{\overline{22}}{\overline{12}}\frac{\overline{13}}{\overline{23}}$	$\frac{\overline{23}}{\overline{13}}\frac{\overline{14}}{\overline{24}}$	$\frac{\overline{24}}{\overline{14}}\frac{\overline{15}}{\overline{25}}$	$\frac{\overline{25}}{\overline{15}}\frac{\overline{16}}{\overline{26}}$	$\frac{\overline{26}}{\overline{16}}\frac{\overline{17}}{\overline{27}}$	$\frac{\overline{27}}{\overline{17}}\frac{\overline{18}}{\overline{28}}$	$\frac{\overline{28}}{\overline{18}}\frac{\overline{19}}{\overline{29}}$	$\frac{\overline{29}}{\overline{19}}$
	$\frac{\overline{33}}{\overline{23}}\frac{\overline{24}}{\overline{34}}$	$\frac{\overline{34}}{\overline{24}}\frac{\overline{25}}{\overline{35}}$	$\frac{\overline{35}}{\overline{25}}\frac{\overline{26}}{\overline{36}}$	$\frac{\overline{36}}{\overline{26}}\frac{\overline{27}}{\overline{37}}$	$\frac{\overline{37}}{\overline{27}}\frac{\overline{28}}{\overline{38}}$	$\frac{\overline{38}}{\overline{28}}\frac{\overline{29}}{\overline{39}}$	$\frac{\overline{39}}{\overline{29}}$
		$\frac{\overline{44}}{\overline{34}}\frac{\overline{35}}{\overline{45}}$	$\frac{\overline{45}}{\overline{35}}\frac{\overline{36}}{\overline{46}}$	$\frac{\overline{46}}{\overline{36}}\frac{\overline{37}}{\overline{47}}$	$\frac{\overline{47}}{\overline{37}}\frac{\overline{38}}{\overline{48}}$	$\frac{\overline{48}}{\overline{38}}\frac{\overline{39}}{\overline{49}}$	$\frac{\overline{49}}{\overline{39}}$
			$\frac{\overline{55}}{\overline{45}}\frac{\overline{46}}{\overline{56}}$	$\frac{\overline{56}}{\overline{46}}\frac{\overline{47}}{\overline{57}}$	$\frac{\overline{57}}{\overline{47}}\frac{\overline{48}}{\overline{58}}$	$\frac{\overline{58}}{\overline{48}}\frac{\overline{49}}{\overline{59}}$	$\frac{\overline{59}}{\overline{49}}$
				$\frac{\overline{66}}{\overline{56}}\frac{\overline{57}}{\overline{67}}$	$\frac{\overline{67}}{\overline{57}}\frac{\overline{58}}{\overline{68}}$	$\frac{\overline{68}}{\overline{58}}\frac{\overline{59}}{\overline{69}}$	$\frac{\overline{69}}{\overline{59}}$
					$\frac{\overline{77}}{\overline{67}}\frac{\overline{68}}{\overline{78}}$	$\frac{\overline{78}}{\overline{68}}\frac{\overline{69}}{\overline{79}}$	$\frac{\overline{79}}{\overline{69}}$
						$\frac{\overline{88}}{\overline{78}}\frac{\overline{79}}{\overline{89}}$	$\frac{\overline{89}}{\overline{79}}$
							$\frac{\overline{99}}{\overline{89}}$

Suppose

$$\frac{\alpha}{\beta} = \frac{(123)\ (2345)\ (56)(8)}{(12)\ (23)\ (3456)(58)} = \frac{\overline{33}\quad\overline{18}}{\overline{13}\quad\overline{68}}.$$

Then

$$\frac{\overline{33}}{\overline{13}} = \left\{ \begin{array}{ccccccc} \frac{\overline{23}}{\overline{13}}\frac{\overline{14}}{\overline{24}} & \frac{\overline{24}}{\overline{14}}\frac{\overline{15}}{\overline{25}} & \frac{\overline{25}}{\overline{15}}\frac{\overline{16}}{\overline{26}} & \frac{\overline{26}}{\overline{16}}\frac{\overline{17}}{\overline{27}} & \frac{\overline{27}}{\overline{17}}\frac{\overline{18}}{\overline{28}} & \frac{\overline{28}}{\overline{18}}\frac{\overline{19}}{\overline{29}} & \frac{\overline{29}}{\overline{19}} \\ \frac{\overline{33}}{\overline{23}}\frac{\overline{24}}{\overline{34}} & \frac{\overline{34}}{\overline{24}}\frac{\overline{25}}{\overline{35}} & \frac{\overline{35}}{\overline{25}}\frac{\overline{26}}{\overline{36}} & \frac{\overline{36}}{\overline{26}}\frac{\overline{27}}{\overline{37}} & \frac{\overline{37}}{\overline{27}}\frac{\overline{28}}{\overline{38}} & \frac{\overline{38}}{\overline{28}}\frac{\overline{29}}{\overline{39}} & \frac{\overline{39}}{\overline{29}} \end{array} \right\}$$

and

$$\frac{\overline{68}}{\overline{18}} = \left\{ \begin{array}{lll} \frac{\overline{28}}{\overline{18}}\frac{\overline{19}}{\overline{29}} & & \frac{\overline{29}}{\overline{19}} \\[2ex] \frac{\overline{38}}{\overline{28}}\frac{\overline{29}}{\overline{39}} & & \frac{\overline{39}}{\overline{29}} \\[2ex] \frac{\overline{48}}{\overline{38}}\frac{\overline{39}}{\overline{49}} & & \frac{\overline{49}}{\overline{39}} \\[2ex] \frac{\overline{58}}{\overline{48}}\frac{\overline{49}}{\overline{59}} & & \frac{\overline{59}}{\overline{49}} \\[2ex] \frac{\overline{68}}{\overline{58}}\frac{\overline{59}}{\overline{69}} & & \frac{\overline{69}}{\overline{59}} \end{array} \right\}.$$

Therefore

$$\frac{\alpha}{\beta} = \left[\frac{\begin{array}{ccccccc} \frac{\overline{23}}{\overline{13}}\frac{\overline{14}}{\overline{24}} & \frac{\overline{24}}{\overline{14}}\frac{\overline{15}}{\overline{25}} & \frac{\overline{25}}{\overline{15}}\frac{\overline{16}}{\overline{26}} & \frac{\overline{26}}{\overline{16}}\frac{\overline{17}}{\overline{27}} & \frac{\overline{27}}{\overline{17}}\frac{\overline{18}}{\overline{28}} & \frac{\overline{28}}{\overline{18}}\frac{\overline{19}}{\overline{29}} & \frac{\overline{29}}{\overline{19}} \\ \frac{\overline{33}}{\overline{23}}\frac{\overline{24}}{\overline{34}} & \frac{\overline{34}}{\overline{24}}\frac{\overline{25}}{\overline{35}} & \frac{\overline{35}}{\overline{25}}\frac{\overline{26}}{\overline{36}} & \frac{\overline{36}}{\overline{26}}\frac{\overline{27}}{\overline{37}} & \frac{\overline{37}}{\overline{27}}\frac{\overline{28}}{\overline{38}} & \frac{\overline{38}}{\overline{28}}\frac{\overline{29}}{\overline{39}} & \frac{\overline{39}}{\overline{29}} \end{array}}{\begin{array}{ccccc} \frac{\overline{28}}{\overline{18}}\frac{\overline{19}}{\overline{29}} & \frac{\overline{38}}{\overline{28}}\frac{\overline{29}}{\overline{39}} & \frac{\overline{48}}{\overline{38}}\frac{\overline{39}}{\overline{49}} & \frac{\overline{58}}{\overline{48}}\frac{\overline{49}}{\overline{59}} & \frac{\overline{68}}{\overline{58}}\frac{\overline{59}}{\overline{69}} \\ \frac{\overline{29}}{\overline{19}} & \frac{\overline{39}}{\overline{29}} & \frac{\overline{49}}{\overline{39}} & \frac{\overline{59}}{\overline{49}} & \frac{\overline{69}}{\overline{59}} \end{array}} \right].$$

Remark 4.

(1) Every generator in (4.1) can be written as follows:

$$g_{ij} = \frac{\overline{(i+1)j}}{\overline{ij}} \frac{\overline{i(j+1)}}{\overline{(i+1)(j+1)}} = \frac{\overline{(i+1)j}\ \overline{i(j+1)}}{\overline{(i+1)j}\ \overline{i(j+1)} + \Delta_{ij}},$$

where $\Delta_{ij} = d_{j+1}\, t_{j-1}\ \cdots\ t_i\, d_i$, and

$$\overline{(i+1)j}\ \overline{i(j+1)} = (d_j + t_{j-1}d_{j-1} + \cdots + t_{j-1}\cdots t_{i+1}d_{i+1})(d_{j+1} + t_j d_j + \cdots + t_j \cdots t_i d_i).$$

(2) The reciprocal of a generator, given by $\frac{\overline{(i+1)j}}{\overline{ij}} \frac{\overline{i(j+1)}}{\overline{(i+1)(j+1)}}$, is equal to

$$1 + \frac{\Delta_{ij}}{\overline{(i+1)j}\ \overline{i(j+1)}} = 1 + \frac{s^2}{(j-i)(2s+j-i)} = O(s).$$

If we let $d_{j+1} = d_i = s > 0$, and set all other parameters to 1 in (2), then we observe that this ratio is not bounded as $s \to +\infty$. For $1 \le i \le j \le n$, let $d_i = d_{j+1} = s,\ t_{j+1} = \frac{1}{s}$. Now we can rewrite ($\star$) as follows in terms of s:

$$\begin{array}{ccccccccccc}
1 & 2 & \dots & i-1 & s+i-1 & \dots & s+j-1 & 2s+j-1 & \frac{j-1}{s}+3 & \dots & \frac{j-1}{s}+n-j+1 \\
 & 1 & \dots & i-2 & s+i-2 & \dots & s+j-2 & 2s+j-2 & \frac{j-2}{s}+3 & \dots & \frac{j-2}{s}+n-j+1 \\
 & & \dots & i-3 & s+i-3 & \dots & s+j-3 & 2s+j-3 & \frac{j-3}{s}+3 & \dots & \frac{j-3}{s}+n-j+1 \\
 & & \dots & & & \dots & & & & \dots & \\
 & & & 1 & s+1 & \dots & s+j-i+1 & 2s+j-i+1 & \frac{j-i+1}{s}+3 & \dots & \frac{j-i+1}{s}+n-j+1 \\
 & & & & s & \dots & s+j-i & 2s+j-i & \frac{j-i}{s}+3 & \dots & \frac{j-i}{s}+n-j+1 \\
 & & & & & \dots & j-i & s+j-i & \frac{j-i}{s}+2 & \dots & \frac{j-i}{s}+n-j \\
 & & & & & & \dots & & & \dots & \\
 & & & & & & 2 & s+2 & \frac{2}{s}+2 & \dots & \frac{2}{s}+n-j \\
 & & & & & & 1 & s+1 & \frac{1}{s}+2 & \dots & \frac{1}{s}+n-j \\
 & & & & & & & s & 2 & \dots & n-j \\
 & & & & & & & & 1 & \dots & n-j-1 \\
 & & & & & & & & & \dots & n-j-2 \\
 & & & & & & & & & \dots & \\
 & & & & & & & & & & 1
\end{array}$$

From the above list, we construct a list of orders for the ratios in (4.1) or (4.2) in terms of the positive parameter s:

$$(\star b)\qquad \begin{array}{cccccccccccccc}
\frac{0}{0} & \frac{0}{0} & \cdots & \frac{1}{1} & \frac{2}{2} & \frac{2}{2} & \cdots\cdots & \frac{2}{2} & \frac{2}{2} & \frac{1}{1} & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & \frac{0}{0} & \cdots & \frac{1}{1} & \frac{2}{2} & \frac{2}{2} & \cdots\cdots & \frac{2}{2} & \frac{2}{2} & \frac{1}{1} & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & & \cdots & & & & \cdots\cdots & & & & & & \cdots & \\
 & & & \frac{1}{1} & \frac{2}{2} & \frac{2}{2} & \cdots\cdots & \frac{2}{2} & \frac{2}{2} & \frac{1}{1} & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & & & & \frac{2}{2} & \frac{2}{2} & \cdots\cdots & \frac{2}{2} & \frac{2}{2} & \frac{1}{1} & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & & & & & \frac{1}{1} & \cdots\cdots & \frac{1}{1} & \frac{\mathbf{1}}{\mathbf{2}} & \frac{1}{1} & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & & & & & & & \frac{0}{0} & \frac{0}{0} & \frac{1}{1} & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & & & & & & & & \frac{0}{0} & \frac{1}{1} & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & & & & & & & & & \frac{1}{1} & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & & & & & & & & & & \frac{0}{0} & \frac{0}{0} & \cdots & \frac{0}{0} \\
 & & & & & & & & & & & & \cdots & \\
 & & & & & & & & & & & & & \frac{0}{0}
\end{array}$$

From the above argument we conclude that

Lemma 4.3. *For any $1 \le i < j \le n$, the inverse of the generator $g_{ij} = \frac{\overline{(i+1)j}}{\overline{ij}} \frac{\overline{i(j+1)}}{\overline{(i+1)(j+1)}}$ multiplied by products of any other generators is not bounded with respect to STEP 1.*

For any ratio $\frac{\alpha}{\beta}$ that satisfies $ST0$, we can invoke Lemma 2 to deduce

$$\frac{\alpha}{\beta} = \prod_{j=2}^{n} \left[\left(\frac{\overline{i_{j_1}j} \ \ \overline{i_{j_2}j} \ \ \cdots \ \ \overline{i_{j_p}j}}{\overline{u_{j_1}j} \ \ \overline{u_{j_2}j} \ \ \cdots \ \ \overline{u_{j_p}j}} \right) \left(\frac{\overline{k_{j_1}j} \ \ \overline{k_{j_2}j} \ \ \cdots \ \ \overline{k_{j_q}j}}{\overline{s_{j_1}j} \ \ \overline{s_{j_2}j} \ \ \cdots \ \ \overline{s_{j_q}j}} \right) \right]. \tag{4.4}$$

Note that for each j the item $\left(\frac{\overline{i_{j_1}j} \ \overline{i_{j_2}j} \ \cdots \ \overline{i_{j_p}j}}{\overline{u_{j_1}j} \ \overline{u_{j_2}j} \ \cdots \ \overline{u_{j_p}j}} \right) \left(\frac{\overline{k_{j_1}j} \ \overline{k_{j_2}j} \ \cdots \ \overline{k_{j_q}j}}{\overline{s_{j_1}j} \ \overline{s_{j_2}j} \ \cdots \ \overline{s_{j_q}j}} \right)$ consists of two essential parts: The first part: $\left(\frac{\overline{i_{j_1}j} \ \overline{i_{j_2}j} \ \cdots \ \overline{i_{j_p}j}}{\overline{u_{j_1}j} \ \overline{u_{j_2}j} \ \cdots \ \overline{u_{j_p}j}} \right)$ has each successive ratio bounded. The second part: $\left(\frac{\overline{k_{j_1}j} \ \overline{k_{j_2}j} \ \cdots \ \overline{k_{j_q}j}}{\overline{s_{j_1}j} \ \overline{s_{j_2}j} \ \cdots \ \overline{s_{j_q}j}} \right)$ has each successive ratio unbounded.
Now using the above argument we define the following sets (repeats allowed) for any ratio $\frac{\alpha}{\beta}$:

$$\frac{IJ}{UJ} = \bigcup_{j=2}^{n} \left\{ \frac{ij}{uj} : \frac{ij}{uj} \text{ is a ratio from the first part in Lemma 2 (2)} \right\} \tag{4.5}$$

$$\text{(bounded ratio's index)}$$

$$\frac{KL}{SL} = \bigcup_{l=2}^{n} \left\{ \frac{kl}{sl} : \frac{kl}{sl} \text{ is a ratio from the second part in Lemma 2 (2)} \right\}$$

$$\text{(unbounded ratio's index)}$$

and let

$$I_0 = \max\left\{ i | i \in \frac{IJ}{UJ} \right\}, \qquad U_0 = \min\left\{ u | u \in \frac{IJ}{UJ} \right\},$$
$$S_0 = \max\left\{ s | s \in \frac{KL}{SL} \right\}, \qquad K_0 = \min\left\{ k | k \in \frac{KL}{SL} \right\}.$$

Lemma 4.4. *For any ratio $\frac{\alpha}{\beta}$, if $\frac{\alpha}{\beta} < 1$ with respect to $\mathbf{A_1}$, then $U_0 \leq K_0$ and $I_0 \geq S_0$.*

Proof. By the definitions above and Lemmas 2, 2, and 4.3, this result follows from an application of proof by contradiction. □

Theorem 4.5. *Any ratio $\frac{\alpha}{\beta}$ is bounded with respect to $\mathbf{A_1}$ if and only if*

(i) *$\frac{\alpha}{\beta}$ satisfies $(ST0)$; and*

(ii) *For any $\frac{kl}{sl} \in \frac{KL}{SL}$, there exists at least one ratio from the collection $\frac{IJ}{UJ}$ in (4.5) such that*

$$\bigcup_{\substack{j \leq l \\ u \leq k \\ s \leq i}} [u, i] \supset [k, s],$$

where $[u, i],\ [k, s]$ are intervals.

Proof. $(\Longrightarrow)$

(i) Follows from Lemma 2.1.

(ii) Use Lemma 4.3. (Note (ii) guarantees that all unbounded ratios $\frac{\overline{kl}}{\overline{sl}} \in \frac{KL}{SL}$ will be cancelled in the product presented in (4.4).)

$(\Longleftarrow)$ If $\frac{\alpha}{\beta}$ satisfies (i) and (ii), by Lemmas 3.3 and 2, $\frac{\alpha}{\beta}$ is a product of generators. Therefore $\frac{\alpha}{\beta}$ is bounded. □

Corollary 4.6. *For any ratio $\frac{\alpha}{\beta}$, $\frac{\alpha}{\beta}$ is bounded with respect to $\mathbf{A_1}$ if and only if*

(i) *$\frac{\alpha}{\beta}$ satisfies $(ST0)$; and*

(ii) *$\beta - \alpha$ is subtraction free expression in terms of the parameters l's and u's in (2) (that is, there are no subtraction signs in the expression).*

Proof. It is sufficient to verify that $\frac{\alpha}{\beta}$ satisfies Theorem 4.5 (ii) $\Longleftrightarrow$ $\frac{\alpha}{\beta}$ is a product of generators $\Longleftrightarrow$ $\beta - \alpha$ is subtraction free.

Suppose the ratio $\frac{\alpha}{\beta}$ is a product of generators. Equivalently, there exist distinct indices $i_1 j_1, \ldots, i_p j_p$, such that

$$\begin{aligned}
\frac{\alpha}{\beta} \quad &= \quad \prod_{k=1}^{p} \left(\frac{\overline{(i_k+1)j_k}}{\overline{i_k j_k}} \frac{\overline{i_k(j_k+1)}}{\overline{(i_k+1)(j_k+1)}} \right) \\
&\Longleftrightarrow \quad \frac{\alpha}{\beta} = \prod_{k=1}^{p} \frac{\overline{(i_k+1)j_k}\ \overline{i_k(j_k+1)}}{\overline{(i_k+1)j_k}\ \overline{i_k(j_k+1)}\ +\ \Delta_{i_k j_k}} \\
&\Longleftrightarrow \quad \frac{\alpha}{\beta} = \frac{\prod_{k=1}^{p} \left(\overline{(i_k+1)j_k}\ \overline{i_k(j_k+1)} \right)}{\prod_{k=1}^{p} \left(\overline{(i_k+1)j_k}\ \overline{i_k(j_k+1)} \right) + \cdots + \prod_{k=1}^{p} \Delta_{i_k j_k}} \\
&\Longrightarrow \quad \beta - \alpha \quad \text{is subtraction free} .
\end{aligned}$$

If $\beta - \alpha$ is subtraction free, then $\beta - \alpha > 0$, thus $\frac{\alpha}{\beta} < 1$ is bounded. □

5. PRELIMINARIES FOR THE NON-PRINCIPAL CASE

For an $n \times n$ matrix $A = [a_{ij}] \in \mathbf{A_1}$ and $\alpha_r, \alpha_c \subseteq N \equiv \{1, 2, \ldots, n\}$, the submatrix of A lying in rows indexed by α_r and columns indexed by α_c is denoted by $A[\alpha_r|\alpha_c]$. For brevity, we may also let $(\alpha_r|\alpha_c)$ denote $\det A[\alpha_r|\alpha_c]$. For $A \in \mathbf{A_1}$, the non-principal minor $(\alpha_r|\alpha_c)$ may be zero or non-zero (see Lemma 5.1).

Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ denote a collection of multisets (repeats allowed) of the form, $\alpha_i = \{\alpha_r^i|\alpha_c^i\}$, where for each i, α_r^i denotes a row index set and α_c^i denotes a column index set ($|\alpha_r^i| = |\alpha_c^i|$, $i = 1, 2, \ldots, p$). If, further, $\beta = \{\beta_1, \beta_2, \ldots, \beta_q\}$ is another collection of such index sets with $\beta_i = \{\beta_r^i|\beta_c^i\}$ for $i = 1, 2, \ldots, q$, then, as in the principal case, we define the concepts such as: $\alpha \leq \beta$, the ratio $\frac{\alpha}{\beta}$ (assuming the denominator is not zero), bounded ratios and generators with respect to a subclass of invertible TN matrices. Since, by convention, $\det A[\phi] = 1$, we also assume, without loss of generality, that in any ratio $\frac{\alpha}{\beta}$ both collections α and β have the same number of sets. Non-principal determinantal inequalities with respect to general TN matrices have been investigated by others (see [9, 15]), although our approach is slightly different.

For $j = 1, 2, \ldots, n$ we define $f_\alpha(j|\cdot)$ to be the number of row sets in α that contain the index j, and similarly, $f_\alpha(\cdot|j)$ counts the multiplicity of j in the column sets of α. If $f_\alpha(j|\cdot) = f_\beta(j|\cdot)$ and $f_\alpha(\cdot|j) = f_\beta(\cdot|j)$ for every $j = 1, 2, \ldots, n$, we say that $\frac{\alpha}{\beta}$ satisfies (ST0).

Lemma 5.1. *For $A \in \mathbf{A_1}$, if $\alpha = \alpha(A) \neq 0$, then there exists β such that $\frac{\alpha}{\beta}$ satisfies (ST0), and $\beta = \beta(A) \neq 0$.*

With Lemma 5.1, we may assume that for any ratio $\frac{\alpha}{\beta}$ we have $\beta = \beta(A) \neq 0$.

The next lemma provides a simple necessary (but by no means sufficient) condition for a given ratio of non-principal minors to be bounded.

Lemma 5.2. *If a given ratio $\frac{\alpha}{\beta}$ is bounded with respect to $\mathbf{A_1}$, then $\frac{\alpha}{\beta}$ satisfies (ST0).*

6. COMPUTATION OF NON-PRINCIPAL MINORS AND CONSTRUCTION OF THE GENERATORS

Let $A \in \mathbf{A_1}$, and $\alpha = \{i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k\}$ be any non-principal minor of A. To evaluate $\alpha = \alpha(A)$ we have the next result. Recall the form of any $A \in \mathbf{A_1}$ in (2).

Lemma 6.1 (Computation of non-principal minors)**.** *A non-principal minor*

$$(\alpha_r|\alpha_c) = (i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k)$$

is a product of the following k factors:

(1) *Each factor has the form:*

$$\overline{xy} l_y \cdots l_{z-1}, \overline{xy} u_y \cdots u_{z-1},$$

(2) *We proceed from left to right. We first compute for the index pair $i_1|j_1$, $(s = 1)$, then the second index pair $i_2|j_2$, $(s = 2)$, and so on.*
(3) *For $s = 1$, $x = 1$, and $y = \min\{i_1, j_1\}$. If $i_1 > j_1$, multiply $\overline{1y}$ by $l_{j_1} \cdots l_{i_1-1}$; if $i_1 = j_1$, multiply $\overline{1y}$ by 1; if $i_1 < j_1$, multiply $\overline{1y}$ by $u_{i_1} \cdots u_{j_1-1}$.*
(4) *For $s = 2$, set $x = \max\{i_1, j_1\} + 1$, and $y = \min\{i_2, j_2\}$ and suppose $x \leq y$. If $i_2 > j_2$, then $\overline{xy}$ is multiplied by $l_{j_2} \cdots l_{i_2-1}$; if $i_2 = j_2$, then $\overline{xy}$ is multiplied by 1; if $i_2 < j_2$, then $\overline{xy}$ is multiplied by $u_{i_2} \cdots u_{j_2-1}$. When $x > y$, this step stops.*
(5) *Continue in this manner for $s = 3, 4, \ldots, k$, we simply multiply all of the factors together to evaluate $(\alpha_r|\alpha_c)$.*

(6) *If any of the above steps (4-5) cannot be carried out (that is, if ever $x > y$ in step 4) we conclude that* $(\alpha_r|\alpha_c) = 0$.

Consider the following illustrative examples outlining the algorithm in Lemma 6.1.

Example 6.1.

(1) Suppose $(\alpha_r|\alpha_c) = (1,2,3|2,3,4)$. For $s = 1$, we have $x = 1$, $y = \min\{1,2\} = 1$, $x = y$, which implies $\overline{1y}u_1 = \overline{11}u_1$. On the other hand for $s = 2$,

$$x = \max\{1,2\} + 1 = 3, \quad y = \min\{2,3\} = 2, \quad \text{and} \quad x > y.$$

Hence this step stops and we conclude that $(\alpha_r|\alpha_c) = (1,2,3|2,3,4) = \overline{11}u_1 \times 0 = 0$.

(2) Let $(\alpha_r|\alpha_c) = (1,4,7,10|3,5,6,8)$. For $s = 1$, $x = 1$, $y = \min\{1,3\} = 1$, $x = y$, $1 < 3$, we have a factor of $\overline{11}u_1u_2$. Similarly, $s = 2$, $x = \max\{1,3\} + 1 = 4$, $y = \min\{4,5\} = 4$, $x = y$, $4 < 5$, we have a factor of $\overline{44}u_4$. Continuing in this manner for $s = 3$, $x = \max\{4,5\} + 1 = 6$, $y = \min\{7,6\} = 6$, $x = y$, $7 > 6$, we have a factor $\overline{66}l_6$ and for $s = 4$, $x = \max\{7,6\} + 1 = 8$, $y = \min\{10,8\} = 8$, $x = y$, $10 > 8$, we have a factor $\overline{88}l_8l_9$.

Multiplying all of the factors above yields

$$(\alpha_r|\alpha_c) = (1,4,7,10|3,5,6,8) = \overline{11}u_1u_2 \times \overline{44}u_4 \times \overline{66}l_6 \times \overline{88}l_8l_9.$$

We introduce three types of ratios:
Type I_0: $\frac{\overline{xi}}{\overline{yi}}$. Example, $\frac{\overline{66}}{\overline{56}}$, $\frac{\overline{24}}{\overline{34}}$.
Type I_{up}: $\frac{l_{i_2-1}\cdots l_{i_1}\overline{xi_1}u_{i_1}\cdots u_{i_2-1}}{\overline{yi_2}}(i_1 < i_2)$. Example, $\frac{l_3l_2\overline{12}u_2u_3}{\overline{23}}$.
Type I_{dn}: $\frac{\overline{xi_1}}{l_{i_1-1}\cdots l_{i_2}\overline{yi_2}u_{i_2}\cdots u_{i_1-1}}(i_2 < i_1)$. Example, $\frac{\overline{68}}{l_7l_6l_5\overline{25}u_5u_6u_7}$.

For Type I_{up} and Type I_{dn} ratios the *numbers* of l's and u's attached on the left and right are the same, which we refer to as the l's and u's *in pairs*. Note that the number of l's or u's is equal to $i_2 - i_1$ for Type I_{up} (or $i_1 - i_2$ for I_{dn}), and in this case we say that the number of l's or u's is *matched*.

Lemma 6.2. *For any ratio $\frac{\alpha}{\beta}$ that satisfies (ST0) with $\alpha \neq 0, \beta \neq 0$, $\frac{\alpha}{\beta}$ can be written as:*

$$\frac{\alpha}{\beta} = \prod(I_0)\prod(I_{up})\prod(I_{dn})$$

with respect to $\mathbf{A_1}$, and the numbers of l's and u's in the above product are in pairs and matched.

Proof. By Lemma 6.1 the numbers of l's and or u's that appear in $\frac{\alpha}{\beta}$ are completely determined by the index pairs from α or β. Therefore we can split all the minors in α and β into row-column index pairs, and by induction we can establish the matched property of $\frac{\alpha}{\beta}$ for all cases. □

Example 6.2. Let

$$\alpha = \{(1,3,7|1,5,7),(5,8|3,8),(4|4)\}, \quad \beta = \{(1,3|1,3),(4,8|7,8),(5|5),(7|4)\}.$$

To evaluate $\frac{\alpha}{\beta}$ we first compute each minor in the collections α and β via Lemma 6.1. In this case:

$(1,3,7|1,5,7) = \overline{11}\ \overline{23}u_3\overline{67}$,
$(5,8|3,8) = \overline{13}l_3l_4\overline{68}$,
$(4|4) = \overline{14}$,
$(1,3|1,3) = \overline{11}\ \overline{23}$,
$(4,8|7,8) = \overline{14}u_5u_6\overline{88}$,
$(5|5) = \overline{55}$,
$(7|4) = \overline{14}l_4l_5l_6$.

Second, we put them together, simplify, and re-order to produce

$$\begin{aligned}\frac{\alpha}{\beta} &= \frac{\{(1,3,7|1,5,7),(5,8|3,8),(4|4)\}}{\{(1,3|1,3),(4,8|7,8),(5|5),(7|4)\}}\\ &= \frac{\{\overline{11}\ \overline{23}u_3\overline{67}\ \overline{13}l_3l_4\overline{68}\ \overline{14}\}}{\{\overline{11}\ \overline{23}\ \overline{14}u_5u_6\overline{88}\ \overline{55}\ \overline{14}l_4l_5l_6\}} = \frac{l_3\overline{13}u_3}{\overline{14}} \times \frac{\overline{67}}{l_6l_5\overline{15}u_5u_6} \times \frac{\overline{68}}{\overline{88}}.\end{aligned}$$

Note:

(1) For the factors $\frac{l_3\overline{13}u_3}{\overline{14}}$, $\frac{\overline{67}}{l_6l_5\overline{15}u_5u_6}$, there is exactly 1 l and 1 u (attached to $\overline{13}$) and 2 l's and u's (attached to $\overline{15}$). This is what is meant as the l's and u's are *in pairs*.

(2) For the factor $\frac{l_3\overline{13}u_3}{\overline{14}}$, the span on the indices of the l's and u's starts at 3 and there are exactly $4-3=1$ of them, which means the l's and u's are *matched*.
Similarly for $\frac{\overline{67}}{l_6l_5\overline{15}u_5u_6}$, there are $7-5=2$ l's and u's starting from 5

We are now in a position to construct a list of generators in the non-principal case. Before we proceed to this construction, we need an additional lemma on the forms of special bounded ratios.

Lemma 6.3.

(1) $\frac{\overline{xi}}{\overline{yi}} < 1$ *with respect to* $\mathbf{A_1}$ *if and only if* $x > y$.
(2) $\frac{\overline{xi}}{\overline{yj}}t_i\cdots t_{j-1} < 1$ *with respect to* $\mathbf{A_1}$ *if and only if* $x \geq y$ $(x \leq i, y \leq j, i < j)$.
(3) $\frac{\overline{xi}}{\overline{yj}t_j\cdots t_{i-1}}$ *is not bounded with respect to* $\mathbf{A_1}$ *for any* x, y $(x \leq i, y \leq j, j < i)$.

Using Lemma 6.3, we construct a list of bounded ratios (see page 16):

In the list on page 16 the left part are the generators from the principal case, which we call Type I_0 generators, while the right part are generators which are called Type I_{up} generators. We will verify that the union of Type I_0 and Type I_{up} are generators for the non-principal case.

Lemma 6.4. *For any ratio* $\frac{\alpha}{\beta}$ *that satisfies (ST0), we have*

$$\frac{\alpha}{\beta} = \frac{\prod(I_0)\prod(I_{up})\prod(I_{dn})}{\prod(I_0)\prod(I_{up})\prod(I_{dn})}$$

with respect to $\mathbf{A_1}$*, where* $\prod(I_0)$ *are products of Type* I_0 *generators:* $\frac{\overline{(i+1)ji(j+1)}}{\overline{ij(i+1)(j+1)}}$*;* $\prod(I_{up})$ *are products of Type* I_{up} *generators:* $\frac{l_i\overline{xi}u_i}{\overline{x(i+1)}}$*;* $\prod(I_{dn})$ *are products of the reciprocals of Type* I_{up} *generators.*

Lemma 6.5. *For any ratio* $\frac{\alpha}{\beta}$ *that satisfies (ST0), if* $\frac{\alpha}{\beta}$ *is bounded with respect to* $\mathbf{A_1}$*, then* $\frac{\alpha}{\beta} = \prod(I_0)\prod(I_{up})$*. That is, there are no Type* I_{dn} *ratios in* $\frac{\alpha}{\beta}$*, and Type* I_0 *and Type* I_{up} *generators do not appear in the denominator of* $\frac{\alpha}{\beta}$*.*

Proof. We establish the proof by verifying it in all the possible cases below:

$$\begin{aligned}&1)\ \frac{\alpha}{\beta} = \frac{\prod(I_{up})}{(I_{up})}, \quad 2)\ \frac{\alpha}{\beta} = \frac{\prod(I_0)}{(I_{up})}, \quad 3)\ \frac{\alpha}{\beta} = \frac{\prod(I_0)\prod(I_{up})}{(I_{up})},\\ &4)\ \frac{\alpha}{\beta} = \frac{\prod(I_{up})}{(I_0)}, \quad 5)\ \frac{\alpha}{\beta} = \frac{\prod(I_0)}{(I_0)}, \quad 6)\ \frac{\alpha}{\beta} = \frac{\prod(I_0)\prod(I_{up})}{(I_0)},\\ &7)\ \frac{\alpha}{\beta} = \frac{\prod(I_{up})}{(I_0)(I_{up})}, \quad 8)\ \frac{\alpha}{\beta} = \frac{\prod(I_0)}{(I_0)(I_{up})}, \quad 9)\ \frac{\alpha}{\beta} = \frac{\prod(I_{up})\prod(I_0)}{(I_0)(I_{up})}.\end{aligned}$$

The notation $\frac{\alpha}{\beta} = \frac{\prod(I_{up})}{(I_{up})}$ means that the numerator of the ratios are products of Type I_{up} generators and the denominator is a fixed generator from the Type I_{up}, which is necessarily different from any of the generators appearing in the numerator. Similar meanings are intended for the other cases.

Consider case 1). Suppose $\frac{\alpha}{\beta} = \frac{\prod(I_{up})}{(I_{up})} = \frac{\prod(I_{up})}{\frac{\overline{ij}t_j}{\overline{\overline{i(j+1)}}}}$, and $\frac{\overline{ij}t_j}{\overline{\overline{i(j+1)}}} \notin \bigcup(I_{up})$ (that is, there are no generators in the same column as $\frac{\overline{ij}t_j}{\overline{\overline{i(j+1)}}}$ among the union of generators $\bigcup(I_{up})$ in the numerator of the ratio $\frac{\alpha}{\beta}$).

Let $d_{j+1} = t$, and set all other parameters to be 1. Then the right part of the list on page 16 will have the following form in terms of the nonnegative parameter t:

$$
\begin{array}{ccccccccc}
 & & & & & (j \text{ column}) & & & \\
 & 0 & 0 & \cdots & 0 & \text{empty} & \frac{1}{1} & \cdots & \frac{1}{1} \\
 & & 0 & \cdots & 0 & \text{empty} & \frac{1}{1} & \cdots & \frac{1}{1} \\
 & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(i \text{ row}) & & & \cdots & 0 & \frac{1}{0} & \frac{1}{1} & \cdots & \frac{1}{1} \\
 & & & \cdots & 0 & \text{empty} & \frac{1}{1} & \cdots & \frac{1}{1} \\
 & & & & \cdots & \text{empty} & \cdots & \cdots & \cdots \\
 & & & & & & \frac{1}{1} & \cdots & \frac{1}{1} \\
 & & & & & & \cdots & \cdots & \cdots \\
 & & & & & & & & \frac{1}{1}
\end{array}
$$

From the above argument we conclude that $\frac{\alpha}{\beta} = O(t) \to \infty$ $(t \to \infty)$. Therefore $\frac{\alpha}{\beta}$ is unbounded.

Similar arguments may be applied in the other cases. $\square$

For any bounded Type I_{up} ratio $\frac{\overline{xi}t_i \cdots t_{j-1}}{\overline{\overline{yj}}}$ $(t_w = l_w u_w, w = i, \dots, j-1)$, we have

$$\frac{\overline{xi}t_i \cdots t_{j-1}}{\overline{\overline{yj}}} = \frac{\overline{xi}t_i}{\overline{\overline{xi+1}}} \frac{\overline{xi+1}t_{i+1}}{\overline{\overline{xi+2}}} \cdot \frac{\overline{xj-1}t_{j-1}}{\overline{\overline{xj}}} \frac{\overline{xj}}{\overline{\overline{yj}}},$$

when $\frac{\overline{xj}}{\overline{\overline{yj}}}$ is of Type I_0.

Lemma 6.6. *Any bounded ratio $\frac{\alpha}{\beta}$ with respect to $\mathbf{A_1}$ that satisfies (ST0), can be uniquely written as follows:*

$$\frac{\alpha}{\beta} = \prod_i \left(\frac{\overline{xi}}{\overline{\overline{yi}}}\right) \prod_j \left(\frac{\overline{zj}}{\overline{\overline{zj+1}}}\right) = \prod (I_0)' \prod (I_{up})'.$$

Proof. Apply the above factorization to each type I_{up} ratio. $\square$

As in the principal case, $\prod(I_0)'$ can be decomposed into two classes:

$$I'_{0>} : \frac{\overline{ij}}{\overline{\overline{uj}}}, i > u, \qquad I'_{0<} : \frac{\overline{kl}}{\overline{\overline{sl}}}, k < s.$$

Let

$$\left(\frac{IJ}{UJ}'\right) = \cup_{j=2}^{n} \left\{ \frac{ij}{uj}, \frac{\overline{ij}}{\overline{\overline{uj}}} \in I'_{0>} \right\}, \qquad \left(\frac{KL}{SL}'\right) = \cup_{l=2}^{n} \left\{ \frac{kl}{sl}, \frac{\overline{kl}}{\overline{\overline{sl}}} \in I'_{0<} \right\}.$$

We now come to our main observations for describing all of the multiplicative non-principal determinantal inequalities for the class STEP 1.

Theorem 6.7. *Any ratio $\frac{\alpha}{\beta}$ is bounded with respect to $\mathbf{A_1}$ if and only if:*

(1) $\frac{\alpha}{\beta}$ *satisfies (ST0),*
(2) *There are no Type* I_{dn} *ratios in* $\frac{\alpha}{\beta}$*,*
(3) *For any* $\frac{kl}{sl} \in \left(\frac{KL}{SL}'\right)$*, there exists at least one ratio from the collection* $\left(\frac{IJ}{UJ}'\right)$ *above such that*

$$\cup_{j\leq l, u\leq k, s\leq i}[u, i] \supset [k, s].$$

Proof. By Lemmas 5.2, 6.5, and 6.6 along with the definitions of $\frac{IJ}{UJ}'$ and $\frac{KL}{SL}'$ above, we can apply similar arguments as in the principal case to establish this result. $\square$

Corollary 6.8. *For any ratio* $\frac{\alpha}{\beta}$ *is bounded with respect to* $\mathbf{A_1}$ *if and only if*
(1) $\frac{\alpha}{\beta}$ *satisfies (ST0),*
(2) *There are no Type* I_{dn} *ratios in* $\frac{\alpha}{\beta}$*,*
(3) $\beta - \alpha$ *is subtraction free.*

Note that

$$\frac{l_i\overline{xi}u_i}{\overline{xi+1}} = \frac{(x-1i|x-1i+1)(x-1i+1|x-1i)}{(x-1i|x-1i)(x-1i+1|x-1i+1)} \qquad \text{for } x > 1,$$

and

$$\frac{l_i\overline{1i}u_i}{\overline{1i+1}} = \frac{(i|i+1)(i+1|i)}{(i|i)(i+1|i+1)},$$

so we can rewrite the list of generators in their original forms as minors (see page 17).

Theorem 6.9. *Any ratio* $\frac{\alpha}{\beta}$ *is bounded with respect to* $\mathbf{A_1}$ *if and only if* $\frac{\alpha}{\beta}$ *can be written as a product of generators from the lists (7.1a) and (7.1b).*

7. CONCLUSION

In this paper we set out to characterize all of the multiplicative determinantal inequalities for a certain class of invertible totally nonnegative matrices. Due to the nature of the proof techniques developed, this characterization involves constructing a complete (finite) list of multiplicative generators for all such determinantal inequalities. However, to describe these generators we needed to carefully analyze various factorizations of ratios of minors, which in turn required an exhaustive study of all possible ratio types and eventually construction of the key generators.

It is evident that these factorizations or decompositions require plenty of notation, and hence can be viewed as cumbersome to read. Notwithstanding this, we feel that an important feature of our paper is the resulting description of the complete list of generators (both in the principal and non-principal cases). This list of generators is both simple to read and naturally laid out. In our opinion, it is this remark that points to the potential applicability of our work, particularly for future research on determinantal inequalities for more general subclasses of totally nonnegative matrices. We also feel that more study is required on this important problem, not just for TN matrices but for other classes of matrices as well.

$$\begin{array}{ccccccccc}
\frac{\overline{22}}{\overline{12}}\,\frac{\overline{13}}{\overline{23}} & \frac{\overline{23}}{\overline{13}}\,\frac{\overline{14}}{\overline{24}} & \cdots & \frac{\overline{2(n-1)}}{\overline{1(n-1)}}\,\frac{\overline{1n}}{\overline{2n}} & \frac{\overline{2n}}{\overline{1n}} & \frac{l_{n-1}\overline{1n-1}u_{n-1}}{\overline{1n}} & \cdots & \frac{l_2\overline{12}u_2}{\overline{13}} & \frac{l_1\overline{11}u_1}{\overline{12}} \\
 & \frac{\overline{33}}{\overline{23}}\,\frac{\overline{24}}{\overline{34}} & \cdots & \frac{\overline{3(n-1)}}{\overline{2(n-1)}}\,\frac{\overline{2n}}{\overline{3n}} & \frac{\overline{3n}}{\overline{2n}} & \frac{l_{n-1}\overline{2n-1}u_{n-1}}{\overline{2n}} & \cdots & \frac{l_2\overline{22}u_2}{\overline{23}} & \\
 & & \cdots & \frac{\overline{4(n-1)}}{\overline{3(n-1)}}\,\frac{\overline{3n}}{\overline{4n}} & \frac{\overline{4n}}{\overline{3n}} & \frac{l_{n-1}\overline{3n-1}u_{n-1}}{\overline{3n}} & \cdots & & \\
 & & & \cdots & \cdots & \cdots & \cdots & & \\
 & & & \frac{\overline{n-1(n-1)}}{\overline{n-2(n-1)}}\,\frac{\overline{n-2n}}{\overline{n-1n}} & \frac{\overline{n-1n}}{\overline{n-2n}} & \frac{l_{n-1}\overline{n-2n-1}u_{n-1}}{\overline{n-2n}} & & & \\
 & & & & \frac{\overline{nn}}{\overline{n-1n}} & \frac{l_{n-1}\overline{n-1n-1}u_{n-1}}{\overline{n-1n}} & & &
\end{array}$$

$$
\begin{array}{cccccc}
\frac{(12)(3)}{(13)(2)} & \frac{(13)(4)}{(14)(3)} & \frac{(14)(5)}{(15)(4)} & \cdots & \frac{\{1(n-1)\}(n)}{\{(1n)(n-1)\}} & \frac{(1n)}{\{(1)(n)\}} \\[2ex]
 & \frac{(23)(14)}{(24)(13)} & \frac{(24)(15)}{(25)(14)} & \cdots & \frac{\{2(n-1)\}(1n)}{(2n)\{1(n-1)\}} & \frac{(2n)(1)}{(1n)(2)} \\[2ex]
 & & \frac{(34)(25)}{(35)(24)} & \cdots & \frac{\{3(n-1)\}(2n)}{(3n)\{2(n-1)\}} & \frac{(3n)(2)}{(2n)(3)} \\[2ex]
 & & & \cdots & \cdots & \cdots \\[2ex]
 & & & & \frac{\{(n-2)(n-1)\}\{((n-3)(n)\}}{\{(n-3)(n-1)\}\{(n-2)n\}} & \frac{\{(n-2)n\}(n-3)}{\{(n-3)n\}(n-2)} \\[2ex]
 & & & & & \frac{\{(n-1)n\}(n-2)}{\{(n-2)n\}(n-1)}
\end{array}
\tag{7.1a}
$$

$$
\begin{array}{cccc}
\frac{(n-1|n)(n|n-1)}{(n-1|n-1)(n|n)} & \frac{(n-2|n-1)(n-1|n-2)}{(n-2|n-2)(n-1|n-1)} & \cdots & \frac{(1|2)(2|1)}{(1|1)(2|2)} \\[2ex]
\frac{\{1(n-1)|1n\}\{1n|1(n-1)\}}{\{1(n-1)|1(n-1)\}(1n|1n)} & \frac{\{1(n-2)|1(n-1)\}\{1(n-1)|1(n-2)\}}{\{1(n-2)|1(n-2)\}\{1(n-1)|1(n-1)\}} & \cdots & \\[2ex]
\frac{\{2(n-1)|2n\}\{2n|2(n-1)\}}{\{2(n-1)|2(n-1)\}(2n|2n)} & \frac{\{2(n-2)|2(n-1)\}\{2(n-1)|2(n-2)\}}{\{2(n-2)|2(n-2)\}\{2(n-1)|2(n-1)\}} & & \\[2ex]
\cdots & \cdots & & \\[2ex]
\frac{\{(n-2)(n-1)|(n-2)n\}\{(n-2)n|(n-2)(n-1)\}}{\{(n-2)(n-1)|(n-2)(n-1)\}\{(n-2)n|(n-2)n\}} & & &
\end{array}
\tag{7.1b}
$$

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