# ON A GENERALIZATION OF ALPHA CONVEXITY

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*Abstract:* In this paper, we introduce and study a class $\tilde{M}_k(\alpha, \beta, \gamma)$, $k \geq 2$ of analytic functions defined in the unit disc. This class generalizes the concept of alpha-convexity and include several other known classes of analytic functions. Inclusion results, an integral representation and a radius problem is discussed for this class.

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## 1. Introduction

Let $\tilde{P}$ denote the class of functions of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let $\tilde{P}(\gamma)$ be the subclass of $\tilde{P}$ consisting of functions $p$ which satisfy the condition

$$|\arg p(z)| \leq \frac{\pi\gamma}{2}, \quad \text{for some} \quad \gamma(\gamma > 0), \quad z \in E. \tag{1.2}$$

We note that $\tilde{P}(1) = P$ is the class of analytic functions with positive real part. We introduce the class $\tilde{P}_k(\gamma)$ as follows:

An analytic function $p$ given by (1.1) belongs to $\tilde{P}_k(\gamma)$, for $z \in E$, if and only if there exist $p_1, p_2 \in \tilde{P}(\gamma)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad k \geq 2. \tag{1.3}$$

We now define the class $\tilde{M}_k(\alpha, \beta, \gamma)$ as follows:

*Definition* 1.1. Let $\alpha \geq 0$, $\beta \geq 0$ $(\alpha + \beta \neq 0)$ and let $f$ be analytic in $E$ with $f(0) = 0$, $f'(0) = 1$ and $\frac{f'(z)f(z)}{z} \neq 0$. Then $f \in \tilde{M}_k(\alpha, \beta, \gamma)$ if and only if , for $z \in E$,

$$\left\{ \frac{\alpha}{\alpha + \beta} \frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha + \beta} \frac{(zf'(z))'}{f'(z)} \right\} \in \tilde{P}_k(\gamma).$$

We note that, for $k = 2$, $\beta = (1 - \alpha)$, we have the class $\tilde{M}_2(\alpha, 1 - \alpha, \gamma) = \tilde{M}_\alpha(\gamma)$ of strongly alpha-convex functions introduced and studied in [4].

We also have the following special cases.



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(i) $\tilde{M}_2(\alpha,0,1) = S^\star$, $\tilde{M}_2(0,\beta,1) = C$, where $S^\star$ and $C$ are respectively the well-known classes of starlike and convex functions. It is known [3] that $\tilde{M}_\alpha(\gamma) \subset S^\star$ and $\tilde{M}_2(\alpha,0,\gamma)$ coincides with the class of strongly starlike functions of order $\gamma$, see [1, 7, 8].

(ii) $\tilde{M}_k(\alpha,0,1) = R_k$, $\tilde{M}_k(0,\beta,1) = V_k$, where $R_k$ is the class of functions of bounded radius rotation and $V_k$ is the class of functions of bounded boundary rotation.

Also $\tilde{M}_k(0,\beta,\gamma) = \tilde{V}_k(\gamma) \subset V_k$ and $\tilde{M}_k(\alpha,0,\gamma) = \tilde{R}_k(\gamma) \subset R_k$.



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## 2. Main Results

**Theorem 2.1.** *A function* $f \in \tilde{M}_k(\alpha, \beta, \gamma), \quad \alpha, \beta > 0$, *if and only if, there exists a function* $F \in \tilde{R}_k(\gamma)$ *such that*

$$f(z) = \left[\frac{\alpha+\beta}{\alpha}\int_0^z \frac{(F(t))^{\frac{\alpha+\beta}{\beta}}}{t}dt\right]^{\frac{\beta}{\alpha+\beta}}. \tag{2.1}$$

*Proof.* A simple calculation yields

$$\frac{\alpha}{\alpha+\beta}\frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha+\beta}\frac{(zf'(z))'}{f'(z)} = \frac{zF'(z)}{F(z)}.$$

If the right hand side belongs to $\tilde{P}_k(\gamma)$ so does the left and conversely, and the result follows. ∎

**Theorem 2.2.** *Let* $f \in \tilde{M}_k(\alpha, \beta, \gamma)$. *Then the function*

$$g(z) = f(z)\left(\frac{zf'(z)}{f(z)}\right)^{\frac{\beta}{\alpha+\beta}} \tag{2.2}$$

*belongs to* $\tilde{R}_k(\gamma)$ *for* $z \in E$.

*Proof.* Differentiating (2.2) logarithmically, we have

$$\frac{zg'(z)}{g(z)} = \frac{\alpha}{\alpha+\beta}\frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha+\beta}\frac{(zf'(z))'}{f'(z)},$$

and, since $f \in \tilde{M}_k(\alpha, \beta, \gamma)$, we obtain the required result. ∎



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**Theorem 2.3.** *Let* $f \in \tilde{M}_k(\alpha,\beta,\gamma)$, $\beta > 0, 0 < \gamma \leq 1$. *Then* $f \in \tilde{R}_k(\gamma)$ *for* $z \in E$.

*Proof.* Let $\frac{zf'(z)}{f(z)} = p(z)$. Then

$$\frac{(zf'(z))'}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

Therefore, for $z \in E$,

$$\frac{\alpha}{\alpha+\beta}\frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha+\beta}\frac{(zf'(z))'}{f'(z)} = \left\{p(z) + \frac{\beta}{\alpha+\beta}\frac{zp'(z)}{p(z)}\right\} \in \tilde{P}_k(\gamma). \tag{2.3}$$

Let

$$\phi(\alpha,\beta) = \frac{\alpha}{\alpha+\beta}\frac{z}{1-z} + \frac{\beta}{\alpha+\beta}\frac{z}{(1-z)^2}. \tag{2.4}$$

Then, using (1.3) and (2.4), we have

$$\left(p \star \frac{\phi(\alpha,\beta)}{z}\right) = \left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1 \star \frac{\phi(\alpha,\beta)}{z}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2 \star \frac{\phi(\alpha,\beta)}{z}\right),$$

where $\star$ denotes the convolution (Hadamard product). This gives us

$$p(z) + \frac{\beta}{\alpha+\beta}\frac{zp'(z)}{p(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{p_1(z) + \frac{\beta}{\alpha+\beta}\frac{zp_1'(z)}{p_1(z)}\right\} - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{p_2(z) + \frac{\beta}{\alpha+\beta}\frac{zp_2'(z)}{p_2(z)}\right\}.$$

From (2.3), it follows that

$$\left\{p_i + \frac{\beta}{\alpha+\beta}\frac{zp_i'}{p_i}\right\} \in \tilde{P}(\gamma), \quad i = 1,2,$$


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and, using a result due to Nunokawa and Owa [6], we conclude that $p_i \in \tilde{P}(\gamma)$ in $E$, $i = 1, 2$. Consequently $p \in \tilde{P}_k(\gamma)$ and hence $f \in \tilde{R}_k(\gamma)$ for $z \in E$. ∎

**Theorem 2.4.** *Let, for* $(\alpha_1 + \beta_1) \neq 0$,

$$\frac{\alpha_1}{\alpha_1+\beta_1} < \frac{\alpha}{\alpha+\beta}, \quad \frac{\beta_1}{\alpha_1+\beta_1} < \frac{\beta}{\alpha+\beta} \quad \textit{and} \quad 0 \leq \gamma < 1.$$

*Then*

$$\tilde{M}_k(\alpha, \beta, \gamma) \subset \tilde{M}_K(\alpha_1, \beta_1, \gamma), \quad z \in E.$$

*Proof.* We can write

$$\begin{aligned}
\frac{\alpha_1}{\alpha_1+\beta_1}\frac{zf'(z)}{f(z)} + \frac{\beta_1}{\alpha_1+\beta_1}\frac{(zf'(z))'}{f'(z)}
&= \left(1 - \frac{\beta_1(\alpha+\beta)}{\beta(\alpha_1+\beta_1)}\right)\frac{zf'(z)}{f(z)} \\
&\quad + \left(\frac{\beta_1(\alpha+\beta)}{\beta(\alpha_1+\beta_1)}\right)\left[\frac{\alpha}{\alpha+\beta}\frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha+\beta}\frac{(zf'(z))'}{f'(z)}\right] \\
&= \left(1 - \frac{\beta_1(\alpha+\beta)}{\beta(\alpha_1+\beta_1)}\right)H_1(z) + \frac{\beta_1(\alpha+\beta)}{\beta(\alpha_1+\beta_1)}H_2(z),
\end{aligned}$$

where $H_1, H_2 \in \tilde{P}_k(\gamma)$ by using Definition 1.1 and Theorem 2.3. Since $0 < \gamma \leq 1$, the class $\tilde{P}(\gamma)$ is a convex set and consequently, by (1.3), the class $\tilde{P}_k(\gamma)$ is a convex set. This implies $H \in \tilde{P}_k(\gamma)$ and therefore $f \in \tilde{M}_k(\alpha_1, \beta_1, \gamma)$. This completes the proof. ∎

**Theorem 2.5.** *Let* $f \in \tilde{M}_k(\alpha, \beta, \gamma)$. *Then*

$$h(z) = \int_0^z (f'(t))^{\frac{\beta}{\alpha+\beta}} \left(\frac{f(t)}{t}\right)^{\frac{\alpha}{\alpha+\beta}} dt \quad \textit{belongs to} \quad \tilde{V}_k(\gamma) \quad \textit{for} \quad z \in E. \tag{2.5}$$



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*Proof.* From (2.5), we have

$$h'(z) = (f'(z))^{\frac{\beta}{\alpha+\beta}} \left(\frac{f(z)}{z}\right)^{\frac{\alpha}{\alpha+\beta}}.$$

Now the proof is immediate when we differentiate both sides logarithmically and use the fact that $f \in \tilde{M}_k(\alpha, \beta, \gamma)$. ∎

In the following we study the converse case of Theorem 2.3 with $\gamma = 1$.

**Theorem 2.6.** *Let $f \in \tilde{R}_k(1)$. Then $f \in \tilde{M}_k(\alpha, \beta, 1)$, $\beta > 0$ for $|z| < r(\alpha, \beta)$, where*

$$r(\alpha, \beta) = \left(1 - \rho^2\right)^{\frac{1}{2}} - \rho, \quad \textit{with} \quad \rho = \frac{\beta}{\alpha+\beta}. \tag{2.6}$$

*This result is best possible.*

*Proof.* Since $f \in \tilde{R}_k(1)$, $\frac{zf'(z)}{f(z)} \in \tilde{P}_k(1) = P_k$, and

$$\frac{\alpha}{\alpha+\beta}\frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha+\beta}\frac{(zf'(z))'}{f'(z)} = p(z) + \frac{\beta}{\alpha+\beta}\frac{zp'(z)}{p(z)}.$$

Let $\phi(\alpha, \beta)$ be as given by (2.4). Now using (1.3) and convolution techniques, we have

$$\begin{aligned} p(z) + \frac{\beta}{\alpha+\beta}\frac{zp'(z)}{p(z)} &= p(z) \star \frac{\phi(\alpha,\beta)}{z} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1(z) \star \frac{\phi(\alpha,\beta)}{z}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2(z) \star \frac{\phi(\alpha,\beta)}{z}\right). \end{aligned}$$



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Since $p_i \in \tilde{P}_2(1) = P$ and it is known [2] that $\mathrm{Re}\left\{\frac{\phi(\alpha,\beta)}{z}\right\} > \frac{1}{2}$ for $|z| < r(\alpha, \beta)$, it follows from a well known result, see [5] that $\left[p_i \star \frac{\phi(\alpha,\beta)}{z}\right] \in P$ for $|z| < r(\alpha, \beta)$, $i = 1, 2$. with $r(\alpha, \beta)$ given by (2.6). The function $\phi(\alpha, \beta)$ given by (2.4) shows that the radius $r(\alpha, \beta)$ is best possible. ∎



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