

ON A GENERALIZATION OF ALPHA CONVEXITY

KHALIDA INAYAT NOOR

Mathematics Department,
COMSATS Institute of Information Technology,
Islamabad, Pakistan.
EMail: khalidanoor@hotmail.com

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Abstract: In this paper, we introduce and study a class $\tilde{M}_k(\alpha, \beta, \gamma)$, $k \geq 2$ of analytic functions defined in the unit disc. This class generalizes the concept of alpha-convexity and include several other known classes of analytic functions. Inclusion results, an integral representation and a radius problem is discussed for this class.

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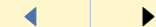
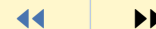
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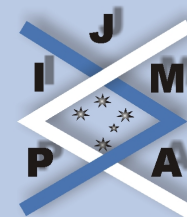
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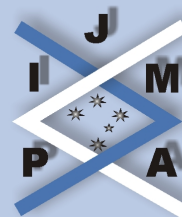
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1. Introduction

Let \tilde{P} denote the class of functions of the form

$$(1.1) \quad p(z) = 1 + c_1 z + c_2 z^2 + \dots,$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Let $\tilde{P}(\gamma)$ be the subclass of \tilde{P} consisting of functions p which satisfy the condition

$$(1.2) \quad |\arg p(z)| \leq \frac{\pi\gamma}{2}, \quad \text{for some } \gamma(\gamma > 0), \quad z \in E.$$

We note that $\tilde{P}(1) = P$ is the class of analytic functions with positive real part. We introduce the class $\tilde{P}_k(\gamma)$ as follows:

An analytic function p given by (1.1) belongs to $\tilde{P}_k(\gamma)$, for $z \in E$, if and only if there exist $p_1, p_2 \in \tilde{P}(\gamma)$ such that

$$(1.3) \quad p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad k \geq 2.$$

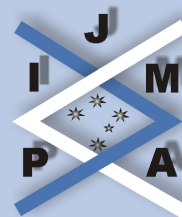
We now define the class $\tilde{M}_k(\alpha, \beta, \gamma)$ as follows:

Definition 1.1. Let $\alpha \geq 0$, $\beta \geq 0$ ($\alpha + \beta \neq 0$) and let f be analytic in E with $f(0) = 0$, $f'(0) = 1$ and $\frac{f'(z)f(z)}{z} \neq 0$. Then $f \in \tilde{M}_k(\alpha, \beta, \gamma)$ if and only if, for $z \in E$,

$$\left\{ \frac{\alpha}{\alpha + \beta} \frac{z f'(z)}{f(z)} + \frac{\beta}{\alpha + \beta} \frac{(z f'(z))'}{f'(z)} \right\} \in \tilde{P}_k(\gamma).$$

We note that, for $k = 2$, $\beta = (1 - \alpha)$, we have the class $\tilde{M}_2(\alpha, 1 - \alpha, \gamma) = \tilde{M}_\alpha(\gamma)$ of strongly alpha-convex functions introduced and studied in [4].

We also have the following special cases.



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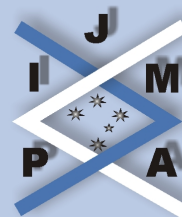
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(i) $\tilde{M}_2(\alpha, 0, 1) = S^*$, $\tilde{M}_2(0, \beta, 1) = C$, where S^* and C are respectively the well-known classes of starlike and convex functions. It is known [3] that $\tilde{M}_\alpha(\gamma) \subset S^*$ and $\tilde{M}_2(\alpha, 0, \gamma)$ coincides with the class of strongly starlike functions of order γ , see [1, 7, 8].

(ii) $\tilde{M}_k(\alpha, 0, 1) = R_k$, $\tilde{M}_k(0, \beta, 1) = V_k$, where R_k is the class of functions of bounded radius rotation and V_k is the class of functions of bounded boundary rotation.

Also $\tilde{M}_k(0, \beta, \gamma) = \tilde{V}_k(\gamma) \subset V_k$ and $\tilde{M}_k(\alpha, 0, \gamma) = \tilde{R}_k(\gamma) \subset R_k$.



2. Main Results

Theorem 2.1. A function $f \in \tilde{M}_k(\alpha, \beta, \gamma)$, $\alpha, \beta > 0$, if and only if, there exists a function $F \in \tilde{R}_k(\gamma)$ such that

$$(2.1) \quad f(z) = \left[\frac{\alpha + \beta}{\alpha} \int_0^z \frac{(F(t))^{\frac{\alpha+\beta}{\beta}}}{t} dt \right]^{\frac{\beta}{\alpha+\beta}}.$$

Proof. A simple calculation yields

$$\frac{\alpha}{\alpha + \beta} \frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha + \beta} \frac{(zf'(z))'}{f'(z)} = \frac{zF'(z)}{F(z)}.$$

If the right hand side belongs to $\tilde{P}_k(\gamma)$ so does the left and conversely, and the result follows. ■

Theorem 2.2. Let $f \in \tilde{M}_k(\alpha, \beta, \gamma)$. Then the function

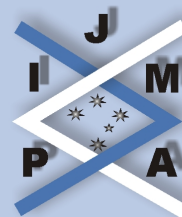
$$(2.2) \quad g(z) = f(z) \left(\frac{zf'(z)}{f(z)} \right)^{\frac{\beta}{\alpha+\beta}}$$

belongs to $\tilde{R}_k(\gamma)$ for $z \in E$.

Proof. Differentiating (2.2) logarithmically, we have

$$\frac{zg'(z)}{g(z)} = \frac{\alpha}{\alpha + \beta} \frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha + \beta} \frac{(zf'(z))'}{f'(z)},$$

and, since $f \in \tilde{M}_k(\alpha, \beta, \gamma)$, we obtain the required result. ■



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Theorem 2.3. Let $f \in \tilde{M}_k(\alpha, \beta, \gamma)$, $\beta > 0, 0 < \gamma \leq 1$. Then $f \in \tilde{R}_k(\gamma)$ for $z \in E$.

Proof. Let $\frac{zf'(z)}{f(z)} = p(z)$. Then

$$\frac{(zf'(z))'}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

Therefore, for $z \in E$,

$$(2.3) \quad \frac{\alpha}{\alpha + \beta} \frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha + \beta} \frac{(zf'(z))'}{f'(z)} = \left\{ p(z) + \frac{\beta}{\alpha + \beta} \frac{zp'(z)}{p(z)} \right\} \in \tilde{P}_k(\gamma).$$

Let

$$(2.4) \quad \phi(\alpha, \beta) = \frac{\alpha}{\alpha + \beta} \frac{z}{1 - z} + \frac{\beta}{\alpha + \beta} \frac{z}{(1 - z)^2}.$$

Then, using (1.3) and (2.4), we have

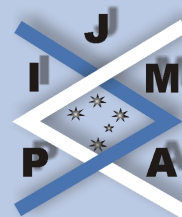
$$\left(p \star \frac{\phi(\alpha, \beta)}{z} \right) = \left(\frac{k}{4} + \frac{1}{2} \right) \left(p_1 \star \frac{\phi(\alpha, \beta)}{z} \right) - \left(\frac{k}{4} - \frac{1}{2} \right) \left(p_2 \star \frac{\phi(\alpha, \beta)}{z} \right),$$

where \star denotes the convolution (Hadamard product). This gives us

$$p(z) + \frac{\beta}{\alpha + \beta} \frac{zp'(z)}{p(z)} = \left(\frac{k}{4} + \frac{1}{2} \right) \left\{ p_1(z) + \frac{\beta}{\alpha + \beta} \frac{zp'_1(z)}{p_1(z)} \right\} - \left(\frac{k}{4} - \frac{1}{2} \right) \left\{ p_2(z) + \frac{\beta}{\alpha + \beta} \frac{zp'_2(z)}{p_2(z)} \right\}.$$

From (2.3), it follows that

$$\left\{ p_i + \frac{\beta}{\alpha + \beta} \frac{zp'_i}{p_i} \right\} \in \tilde{P}(\gamma), \quad i = 1, 2,$$



and, using a result due to Nunokawa and Owa [6], we conclude that $p_i \in \tilde{P}(\gamma)$ in E , $i = 1, 2$. Consequently $p \in \tilde{P}_k(\gamma)$ and hence $f \in \tilde{R}_k(\gamma)$ for $z \in E$. ■

Theorem 2.4. Let, for $(\alpha_1 + \beta_1) \neq 0$,

$$\frac{\alpha_1}{\alpha_1 + \beta_1} < \frac{\alpha}{\alpha + \beta}, \quad \frac{\beta_1}{\alpha_1 + \beta_1} < \frac{\beta}{\alpha + \beta} \quad \text{and} \quad 0 \leq \gamma < 1.$$

Then

$$\tilde{M}_k(\alpha, \beta, \gamma) \subset \tilde{M}_K(\alpha_1, \beta_1, \gamma), \quad z \in E.$$

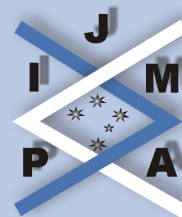
Proof. We can write

$$\begin{aligned} & \frac{\alpha_1}{\alpha_1 + \beta_1} \frac{zf'(z)}{f(z)} + \frac{\beta_1}{\alpha_1 + \beta_1} \frac{(zf'(z))'}{f'(z)} \\ &= \left(1 - \frac{\beta_1(\alpha + \beta)}{\beta(\alpha_1 + \beta_1)}\right) \frac{zf'(z)}{f(z)} \\ & \quad + \left(\frac{\beta_1(\alpha + \beta)}{\beta(\alpha_1 + \beta_1)}\right) \left[\frac{\alpha}{\alpha + \beta} \frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha + \beta} \frac{(zf'(z))'}{f'(z)} \right] \\ &= \left(1 - \frac{\beta_1(\alpha + \beta)}{\beta(\alpha_1 + \beta_1)}\right) H_1(z) + \frac{\beta_1(\alpha + \beta)}{\beta(\alpha_1 + \beta_1)} H_2(z), \end{aligned}$$

where $H_1, H_2 \in \tilde{P}_k(\gamma)$ by using Definition 1.1 and Theorem 2.3. Since $0 < \gamma \leq 1$, the class $\tilde{P}(\gamma)$ is a convex set and consequently, by (1.3), the class $\tilde{P}_k(\gamma)$ is a convex set. This implies $H \in \tilde{P}_k(\gamma)$ and therefore $f \in \tilde{M}_k(\alpha_1, \beta_1, \gamma)$. This completes the proof. ■

Theorem 2.5. Let $f \in \tilde{M}_k(\alpha, \beta, \gamma)$. Then

$$(2.5) \quad h(z) = \int_0^z (f'(t))^{\frac{\beta}{\alpha+\beta}} \left(\frac{f(t)}{t}\right)^{\frac{\alpha}{\alpha+\beta}} dt \quad \text{belongs to} \quad \tilde{V}_k(\gamma) \quad \text{for} \quad z \in E.$$

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Proof. From (2.5), we have

$$h'(z) = (f'(z))^{\frac{\beta}{\alpha+\beta}} \left(\frac{f(z)}{z} \right)^{\frac{\alpha}{\alpha+\beta}}.$$

Now the proof is immediate when we differentiate both sides logarithmically and use the fact that $f \in \tilde{M}_k(\alpha, \beta, \gamma)$. ■

In the following we study the converse case of Theorem 2.3 with $\gamma = 1$.

Theorem 2.6. *Let $f \in \tilde{R}_k(1)$. Then $f \in \tilde{M}_k(\alpha, \beta, 1)$, $\beta > 0$ for $|z| < r(\alpha, \beta)$, where*

$$(2.6) \quad r(\alpha, \beta) = (1 - \rho^2)^{\frac{1}{2}} - \rho, \quad \text{with} \quad \rho = \frac{\beta}{\alpha + \beta}.$$

This result is best possible.

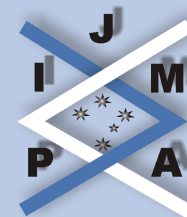
Proof. Since $f \in \tilde{R}_k(1)$, $\frac{zf'(z)}{f(z)} \in \tilde{P}_k(1) = P_k$, and

$$\frac{\alpha}{\alpha + \beta} \frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha + \beta} \frac{(zf'(z))'}{f'(z)} = p(z) + \frac{\beta}{\alpha + \beta} \frac{zp'(z)}{p(z)}.$$

Let $\phi(\alpha, \beta)$ be as given by (2.4). Now using (1.3) and convolution techniques, we have

$$\begin{aligned} p(z) + \frac{\beta}{\alpha + \beta} \frac{zp'(z)}{p(z)} &= p(z) \star \frac{\phi(\alpha, \beta)}{z} \\ &= \left(\frac{k}{4} + \frac{1}{2} \right) \left(p_1(z) \star \frac{\phi(\alpha, \beta)}{z} \right) - \left(\frac{k}{4} - \frac{1}{2} \right) \left(p_2(z) \star \frac{\phi(\alpha, \beta)}{z} \right). \end{aligned}$$

Since $p_i \in \tilde{P}_2(1) = P$ and it is known [2] that $\operatorname{Re} \left\{ \frac{\phi(\alpha, \beta)}{z} \right\} > \frac{1}{2}$ for $|z| < r(\alpha, \beta)$, it follows from a well known result, see [5] that $\left[p_i \star \frac{\phi(\alpha, \beta)}{z} \right] \in P$ for $|z| < r(\alpha, \beta)$, $i = 1, 2$, with $r(\alpha, \beta)$ given by (2.6). The function $\phi(\alpha, \beta)$ given by (2.4) shows that the radius $r(\alpha, \beta)$ is best possible. ■



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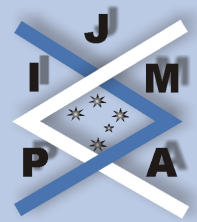
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