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AN INEQUALITY FOR THE CLASS NUMBER

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Abstract

We prove in an elementary way a new inequality for the average order of the Piltz divisor function with application to class number of number fields.

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I would like to thank my wife Véronique for her help.

Contents

1. Introduction

It could be interesting to use tools from analytic number theory to solve problems of algebraic number theory. For example, let $\mathbb{K}$ be a number field of degree n, signature (r_1, r_2), class number $h_{\mathbb{K}}$, regulator $\mathcal{R}_{\mathbb{K}}$, and $w_{\mathbb{K}}$ is the number of roots of unity in $\mathbb{K}$, $\zeta_{\mathbb{K}}$ the Dedekind zeta function, $A_{\mathbb{K}} := 2^{-r_2}\pi^{-n/2}d_{\mathbb{K}}^{1/2}$ where $d_{\mathbb{K}}$ is the absolute value of the discriminant of $\mathbb{K}$. The following formula, valid for any real number $\sigma > 1$,

$$(1.1)\quad A_{\mathbb{K}}^{\sigma}\Gamma^{r_1}\left(\frac{\sigma}{2}\right)\Gamma^{r_2}(\sigma)\,\zeta_{\mathbb{K}}(\sigma) = \frac{2^{r_1}h_{\mathbb{K}}\mathcal{R}_{\mathbb{K}}}{\sigma(\sigma-1)\,w_{\mathbb{K}}} + \sum_{\mathfrak{a}\neq 0}\int_{\|y\|\geq 1}\left\{\|y\|^{\sigma/2} + \|y\|^{\frac{1-\sigma}{2}}\right\}e^{-g(\mathfrak{a},y)}\frac{dy}{y},$$

where $g(\mathfrak{a}, y)$ is a certain function depending on a nonzero integral ideal $\mathfrak{a}$ and vector $y := (y_1, \ldots, y_{r_1+r_2}) \in (\mathbb{R}_+)^{r_1+r_2}$ (here $\|y\| := \max|y_i|$), is the generalization of the well-known formula

$$\pi^{-\sigma/2}\Gamma\left(\frac{\sigma}{2}\right)\zeta(\sigma) = \frac{1}{\sigma(\sigma-1)} + \sum_{n=1}^{\infty}\int_1^{\infty}\left\{y^{\sigma/2} + y^{\frac{1-\sigma}{2}}\right\}e^{-\pi n^2 y}\frac{dy}{y}$$

for the classical Riemann zeta function. Since the integrand in (1.1) is positive, we get

$$(1.2)\qquad h_{\mathbb{K}}\mathcal{R}_{\mathbb{K}} \leq \sigma(\sigma-1)\,w_{\mathbb{K}}2^{-r_1}A_{\mathbb{K}}^{\sigma}\Gamma^{r_1}\left(\frac{\sigma}{2}\right)\Gamma^{r_2}(\sigma)\,\zeta_{\mathbb{K}}(\sigma)$$

for any real number $\sigma > 1$. The study of the function on the right-hand side of (1.2) provides upper bounds for $h_{\mathbb{K}}\mathcal{R}_{\mathbb{K}}$ (see [3] for example) .

In a more elementary way, one can connect the class number $h_{\mathbb{K}}$ with the Piltz divisor function τ_n by using the following result ([1]):

Lemma 1.1. *Let $b_{\mathbb{K}} > 0$ be a real number such that every class of ideals of $\mathbb{K}$ contains a nonzero integral ideal with norm $\leq b_{\mathbb{K}}$. If τ_n is the Piltz divisor function, then:*

$$h_{\mathbb{K}} \leq \sum_{m \leq b_{\mathbb{K}}} \tau_n(m).$$

Recall that τ_n is defined by the relations $\tau_1(m) = m$ and $\tau_n(m) = \sum_{d|m} \tau_{n-1}(d)$ $(n \geq 2)$. This function has been studied by many authors (see [6] for a good survey of its properties). A standard argument from analytic number theory gives if $n \geq 4$

$$\sum_{m \leq x} \tau_n(m) = x\mathcal{P}_{n-1}(\log x) + O_{\varepsilon}\left(x^{\frac{n-1}{n+2}+\varepsilon}\right),$$

where $\mathcal{P}_{n-1}$ is a polynomial of degree $n-1$ and leading coefficient $\frac{1}{(n-1)!}$. For some improvements of the error term and related results, see [4]. Note that the Lindelöf Hypothesis is equivalent to $\alpha_n = (n-1)/(2n)$ for any $n = 2, 3, \ldots$ where α_n is the least number such that

$$\sum_{m \leq x} \tau_n(m) - x\mathcal{P}_{n-1}(\log x) = O_{\varepsilon}\left(x^{\alpha_n+\varepsilon}\right).$$

If we are interested in finding upper bounds of the form

$$\sum_{m \leq x} \tau_n(m) \ll_n x(\log x)^{n-1},$$

one mostly uses arguments based upon induction and the following inequality:

Lemma 1.2. *We set* $S_n(x) := \sum_{m \le x} \tau_n(m)$ *. Then:*

$$S_{n+1}(x) \le S_n(x) + x \int_1^x t^{-2} S_n(t)\, dt.$$

Proof. It suffices to use the definition above, interchange the summations and integrate by parts. □

Using this lemma, it is easy to show by induction the following bound:

$$\sum_{m \le x} \tau_n(m) \le \frac{x}{(n-1)!} (\log x + n - 1)^{n-1}$$

which enables us to obtain Lenstra's bound again (see [2]), namely:

$$h_{\mathbb{K}} \le \frac{b_{\mathbb{K}}}{(n-1)!} (\log b_{\mathbb{K}} + n - 1)^{n-1}. \tag{1.3}$$

In what follows, n is a positive integer and we set

$$S_n(x) := \sum_{m \le x} \tau_n(m)$$

for any real number $x \ge 1$. $b_{\mathbb{K}}$ is a positive real number always satisfying the hypothesis of Lemma 1.1. $\mathbb{K}$ is a number field of degree n and class number $h_{\mathbb{K}}$. $d_{\mathbb{K}}$ is the absolute value of the discriminant of $\mathbb{K}$. For some tables giving values of $b_{\mathbb{K}}$, see [7]. The functions ψ and ψ_2 are defined by

$$\psi(t) = t - [t] - \frac{1}{2},$$

$$\psi_2(t) = \int_0^t \psi(u)\, du + \frac{1}{8} = \frac{\psi^2(t)}{2},$$

where $[t]$ denotes the integral part of t. Recall that we have for all real numbers t :

$$|\psi(t)| \leq \frac{1}{2},$$
$$0 \leq \psi_2(t) \leq \frac{1}{8}.$$

We denote by γ and γ_1 the Euler-Mascheroni constant and the first Stieltjes constant, defined respectively by:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right),$$
$$\gamma_1 = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\log k}{k} - \frac{(\log n)^2}{2}\right).$$

The following results are well-known (see [5] for example):

$$0.577215 < \gamma < 0.577216,$$
$$-0.072816 < \gamma_1 < -0.072815,$$

and

(1.4) $$\gamma = \frac{1}{2} - 2\int_1^{\infty}\frac{\psi_2(t)}{t^3}dt$$

and

(1.5) $$\gamma_1 = -\int_1^{\infty}\frac{2\log t - 3}{t^3}\psi_2(t)\,dt.$$

2. Results

Theorem 2.1. *Let $n \geq 3$ be an integer. For any real number $x \geq 13$, we have:*

$$\sum_{m \leq x} \tau_n(m) \leq \frac{x}{(n-1)!} (\log x + n - 2)^{n-1} .$$

Applying this result with Lemma 1.1 allows us to improve upon (1.3) :

Theorem 2.2. *Let $\mathbb{K}$ be a number field of degree $n \geq 3$. If $b_{\mathbb{K}} \geq 13$ satisfies the hypothesis of Lemma 1.1, then:*

$$h_{\mathbb{K}} \leq \frac{b_{\mathbb{K}}}{(n-1)!} (\log b_{\mathbb{K}} + n - 2)^{n-1} .$$

3. The Case $n = 3$

The aim of this section is to show that the result of Theorem 2.1 is true for $n = 3$. Hence we will prove the following inequality for S_3 :

Lemma 3.1. *For any real number $x \geq 13$, we have:*

$$S_3(x) \leq \frac{x}{2}(\log x + 1)^2 .$$

We first check this result for $13 \leq x \leq 670$ with the PARI/GP system [8], and then suppose $x > 670$. The lemma will be a direct consequence of the following estimation:

Lemma 3.2. *For any real number $x > 670$, we have:*

$$S_3(x) = x\left\{\frac{(\log x)^2}{2} + (3\gamma - 1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1\right\} + R(x)$$

where:

$$|R(x)| \leq 2.36\, x^{2/3} \log x.$$

The proof of this lemma needs some technical results:

Lemma 3.3. *Let $x, y \geq 1$ be real numbers.*

(i) If $e^{3/2} \leq y \leq x$, then we have:

$$\sum_{k \leq y} \frac{1}{k} \log\left(\frac{x}{k}\right) = \log x \log y - \frac{(\log y)^2}{2} + \gamma \log x - \gamma_1 + R_1(x, y)$$

with:

$$|R_1(x,y)| \leq \frac{\log(x/y)}{2y} + \frac{\log x}{4y^2}.$$

(ii)

$$S_2(y) = y\log y + (2\gamma - 1)\,y + R_2(y)$$

with:

$$|R_2(y)| \leq y^{1/2} + \frac{1}{2}.$$

(iii)

$$\sum_{n\leq y} \frac{\tau(n)}{n} = \frac{(\log y)^2}{2} + 2\gamma\log y + \gamma^2 - 2\gamma_1 + R_3(y)$$

with:

$$|R_3(y)| \leq \frac{1}{y^{1/2}} + \frac{1}{y}.$$

Proof. (i) By the Euler-MacLaurin summation formula, we get:

$$\begin{aligned}
&\sum_{k\leq y} \frac{1}{k}\log\left(\frac{x}{k}\right) \\
&= \frac{\log x}{2} + \int_1^y \frac{1}{t}\log\left(\frac{x}{t}\right)dt - \frac{\psi(y)}{y}\log\left(\frac{x}{y}\right) \\
&\quad - \frac{\psi_2(y)}{y^2}\left(\log\left(\frac{x}{y}\right)+1\right) - \int_1^y \frac{2\log(x/t)+3}{t^3}\psi_2(t)\,dt
\end{aligned}$$

$$= \log x \log y - \frac{(\log y)^2}{2} + \left(\frac{1}{2} - 2\int_1^{\infty} \frac{\psi_2(t)}{t^3} dt\right) \log x$$
$$+ \int_1^{\infty} \frac{2\log t - 3}{t^3} \psi_2(t)\, dt - \frac{\psi(y)}{y} \log\left(\frac{x}{y}\right) - \frac{\psi_2(y)}{y^2}\left(\log\left(\frac{x}{y}\right) + 1\right)$$
$$+ 2\log x \int_y^{\infty} \frac{\psi_2(t)}{t^3} dt - \int_y^{\infty} \frac{2\log t - 3}{t^3} \psi_2(t)\, dt$$

and using (1.4) and (1.5) we get:

$$\sum_{k \le y} \frac{1}{k} \log\left(\frac{x}{k}\right) = \log x \log y - \frac{(\log y)^2}{2} + \gamma \log x - \gamma_1 + R_1(x, y)$$

and since $e^{3/2} \le y \le x$, we have:

$$\begin{aligned} |R_1(x,y)| &\le \frac{\log(x/y)}{2y} + \frac{\log(x/y) + 1}{8y^2} + \frac{\log x}{8y^2} + \frac{\log y - 1}{8y^2} \\ &= \frac{\log(x/y)}{2y} + \frac{\log x}{4y^2}. \end{aligned}$$

(ii) This result is well-known (see [1] for example).
(iii) Using a result from [5], we have for any real number $y \ge 1$:

$$-y^{-1/2} - \left(\frac{3}{4} + \frac{1}{8e^3}\right) y^{-1} - \frac{y^{-3/2}}{8} - \frac{y^{-2}}{64} \le R_3(y) \le y^{-1/2} + \left(\frac{1}{2} + \frac{1}{8e^3}\right) y^{-1}$$

which concludes the proof of Lemma 3.3. □

Proof of Lemmas 3.1 and 3.2. The Dirichlet hyperbola principle and the estimations of Lemma 3.3 give, for any real number $e^{3/2} \leq T < x$:

$$S_3(x) = \sum_{n \leq T} S_2\left(\frac{x}{n}\right) + \sum_{n \leq x/T} \tau(n)\left[\frac{x}{n}\right] - [T]\, S_2\left(\frac{x}{T}\right)$$
$$= \sum_{n \leq T}\left(\frac{x}{n}\log\left(\frac{x}{n}\right) + (2\gamma - 1)\frac{x}{n} + R_4(x,n)\right) + x\sum_{n \leq x/T}\frac{\tau(n)}{n} - \frac{1}{2}S_2\left(\frac{x}{T}\right)$$
$$- \sum_{n \leq x/T}\tau(n)\,\psi\left(\frac{x}{n}\right) - TS_2\left(\frac{x}{T}\right) + \frac{1}{2}S_2\left(\frac{x}{T}\right) + \psi(T)\, S_2\left(\frac{x}{T}\right)$$
$$= \sum_{n \leq T}\left(\frac{x}{n}\log\left(\frac{x}{n}\right) + (2\gamma - 1)\frac{x}{n} + R_4(x,n)\right)$$
$$+ x\sum_{n \leq x/T}\frac{\tau(n)}{n} - TS_2\left(\frac{x}{T}\right) + R_5(x,T)$$

with

$$|R_4(x,n)| \leq \sqrt{\frac{x}{n}} + \frac{1}{2}$$
$$|R_5(x,T)| \leq S_2\left(\frac{x}{T}\right) \leq \frac{x}{T}\log\left(\frac{x}{T}\right) + (2\gamma - 1)\frac{x}{T} + \sqrt{\frac{x}{T}} + \frac{1}{2}$$

and hence:

$$S_3(x) = x\left\{\log x \log T - \frac{(\log T)^2}{2} + \gamma \log x\right.$$

$$- \gamma_1 + R_6(x,T) + (2\gamma - 1)(\log T + \gamma + R_7(T)) \Big\} + \sum_{n \le T} R_4(x,n)$$

$$+ x \left\{ \frac{(\log(x/T))^2}{2} + 2\gamma \log\left(\frac{x}{T}\right) + \gamma^2 - 2\gamma_1 + R_8(x,T) \right\}$$

$$+ R_5(x,T) - x \log\left(\frac{x}{T}\right) - (2\gamma - 1)x - TR_9(x,T)$$

with, if $e^{3/2} \le T < x$:

$$|R_6(x,T)| \le \frac{\log(x/T)}{2T} + \frac{\log x}{4T^2}$$

$$|R_7(T)| \le \frac{1}{T}$$

$$|R_8(x,T)| \le \sqrt{\frac{T}{x}} + \frac{T}{x}$$

$$|R_9(x,T)| \le \sqrt{\frac{x}{T}} + \frac{1}{2}$$

and thus:

$$S_3(x) = x \left\{ \frac{(\log x)^2}{2} + (3\gamma - 1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1 \right\}$$

$$+ xR_6(x,T) + (2\gamma - 1)xR_7(T) + R_{10}(x,T)$$

$$+ xR_8(x,T) + R_5(x,T) - TR_9(x,T)$$

with

$$\begin{aligned}|R_{10}(x,T)| &\le \sum_{n\le T} |R_4(x,n)| \\ &\le \sqrt{x}\sum_{n\le T}\frac{1}{\sqrt{n}} + \frac{T}{2} \\ &\le 2\sqrt{xT} - \sqrt{x} + \frac{T}{2}\end{aligned}$$

and therefore:

$$S_3(x) = x\left\{\frac{(\log x)^2}{2} + (3\gamma - 1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1\right\} + R_{11}(x,T)$$

with:

$$\begin{aligned}|R_{11}(x,T)| \le{}& \frac{x\log(x/T)}{2T} + \frac{x\log x}{4T^2} + 4\sqrt{xT} - \sqrt{x} \\ &+ \frac{2x}{T}\log\left(\frac{x}{T}\right) + 2(2\gamma-1)\frac{x}{T} + \sqrt{\frac{x}{T}} + 2T + \frac{1}{2}.\end{aligned}$$

We choose:

$$T = x^{1/3},$$

which gives:

$$S_3(x) = x\left\{\frac{(\log x)^2}{2} + (3\gamma - 1)\log x + 3\gamma^2 - 3\gamma - 3\gamma_1 + 1\right\} + R_{12}(x),$$

where:

$$\begin{aligned}|R_{12}(x)| &\leq \frac{5}{3}x^{2/3}\log x + 2(2\gamma+1)x^{2/3} - x^{1/2} + \frac{1}{4}x^{1/3}\log x + 3x^{1/3} + \frac{1}{2} \\ &\leq 2.36\, x^{2/3}\log x\end{aligned}$$

since $x > 670$. This concludes the proof of Lemma 3.2, and then of Lemma 3.1. □

4. Proof of Theorem 2.1

We first need the following simple bounds:

Lemma 4.1. *For any integer $n \geq 3$, we have:*

$$\int_1^{13} t^{-2} S_n(t)\, dt < \frac{n^3}{4} \leq \frac{1}{n!}\left(n + \frac{1}{2}\right)^n .$$

Proof. This follows from straightforward computations which give:

$$\begin{aligned}\int_1^{13} t^{-2} S_n(t)\, dt &= \frac{7}{624}n^3 + \frac{2281}{9360}n^2 + \frac{90283}{90090}n + 1 - \frac{1}{13} \\ &< \frac{n^3}{4}\end{aligned}$$

since $n \geq 3$. The second inequality follows from studying the sequence (u_n) defined by

$$u_n = \frac{n^3 \times n!}{4\left(n + 1/2\right)^n}.$$

We get:

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{2(n+1)^4}{n^3(2n+3)}\left(\frac{2n+1}{2n+3}\right)^n \\ &\leq \frac{512}{243}\left(1 - \frac{2}{2n+3}\right)^n \leq \frac{512e^{-1}}{243} < 1\end{aligned}$$

and hence (u_n) is decreasing, and thus:

$$u_n \leq u_3 = \frac{324}{343} \leq 1,$$

which concludes the proof of Lemma 4.1. □

Proof of Theorem 2.1. We use induction, the result being true for $n = 3$ by Lemma 3.1. Now suppose the inequality is true for some integer $n \geq 3$. By Lemmas 1.2, 4.1 and the induction hypothesis, we get:

$$\begin{aligned}
&S_{n+1}(x) \\
&\leq S_n(x) + x\int_1^{13} t^{-2} S_n(t)\,dt + x\int_{13}^{x} t^{-2} S_n(t)\,dt \\
&\leq x\left\{\frac{(\log x + n - 2)^{n-1}}{(n-1)!} + \frac{1}{n!}\left(n+\frac{1}{2}\right)^n + \frac{1}{(n-1)!}\int_{13}^{x}\frac{(\log t + n - 2)^{n-1}}{t}dt\right\} \\
&= x\left\{\frac{(\log x + n - 2)^{n}}{n!} + \frac{(\log x + n - 2)^{n-1}}{(n-1)!}\right. \\
&\qquad\left. + \frac{1}{n!}\left(\left(n+\frac{1}{2}\right)^n - \left(n + \log\left(13e^{-2}\right)\right)^n\right)\right\} \\
&\leq \frac{x}{n!}\left\{(\log x + n - 2)^n + (n-1)(\log x + n - 2)^{n-1}\right\} \\
&\leq \frac{x}{n!}(\log x + n - 1)^n .
\end{aligned}$$

The proof of Theorem 2.1 is now complete. □

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