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INEQUALITIES ON THE VARIANCES OF CONVEX FUNCTIONS OF RANDOM VARIABLES

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ABSTRACT. We develop inequalities relating to the variances of convex decreasing functions of random variables using information on the functions and the distribution functions of the random variables.

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1. INTRODUCTION

Inequalities relating to the variances of convex functions of real-valued random variables are developed. Given a random variable, X, we denote its expectation by $\mathbb{E}[X]$, its variance by $\mathrm{Var}[X]$, and use F_X and F_X^{-1} to denote its (cumulative) distribution function and the inverse of its (cumulative) distribution function respectively. In this paper, we assume that all random variables are real-valued and non-degenerate.

One familiar and elementary inequality in probability (supposing the expectations exist) is:

$$\mathbb{E}[1/X] \geq 1/\mathbb{E}[X] \tag{1.1}$$

where X is a non-negative random variable. This may be proven using convexity (as an application of Jensen's inequality) or by more elementary approaches [2], [4], [5]. More generally, if one considers the expectations of convex functions of random variables, then Jensen's inequality gives:

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]), \tag{1.2}$$

where X is a random variable and f is convex over the (convex hull of the) range of X (see [6]).

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In the literature, there have been few studies on the variance of convex functions of random variables. In this note, we aim to provide some useful inequalities, in particular, for financial applications. Subsequently, we will deal with functions which are continuous, convex and decreasing. Note that $\mathrm{Var}\,[f(X)] = \mathrm{Var}\,[-f(X)]$. This means our results also apply to concave increasing functions, which characterize the utility functions of risk-adverse individuals in decision theory.

2. TECHNICAL LEMMAS

Lemma 2.1. *Let X be a random variable, and let f, g be continuous functions on $\mathbb{R}$.*

If f is monotonically increasing and g monotonically decreasing, then

$$\mathbb{E}\,[f(X)g(X)] \leq \mathbb{E}\,[f(X)]\,\mathbb{E}\,[g(X)]\,. \tag{2.1}$$

If f, g are both monotonically increasing or decreasing, then

$$\mathbb{E}\,[f(X)g(X)] \geq \mathbb{E}\,[f(X)]\,\mathbb{E}\,[g(X)]\,. \tag{2.2}$$

Moreover, in both cases, if both functions are strictly monotone, the inequality is strict (see [6] *or* [4]*).*

Lemma 2.2. *For any random variable X, if with probability 1, $f(X,\cdot)$ is a differentiable, convex decreasing function on $[a,b]$ $(a<b)$ and its derivative at a exists and is bounded, then*

$$\frac{\partial}{\partial\epsilon}\mathbb{E}\,[f(X,\epsilon)] = \mathbb{E}\left[\frac{\partial}{\partial\epsilon}f(X,\epsilon)\right].$$

Proof. Let $g(x,\epsilon) = \frac{\partial}{\partial\epsilon}f(x,\epsilon)$.

For $\epsilon \in [a,b)$, let

$$m_n(x,\epsilon) = (n+N_\epsilon)\left[f\left(x,\epsilon+\frac{1}{n+N_\epsilon}\right) - f(x,\epsilon)\right], \tag{2.3}$$

where $N_\epsilon = \lceil\frac{2}{b-\epsilon}\rceil$, and for $\epsilon = b$, let

$$m_n(x,\epsilon) = (n+N_\epsilon)\left[f(x,\epsilon) - f\left(x,\epsilon-\frac{1}{n+N_\epsilon}\right)\right], \tag{2.4}$$

where $N_\epsilon = \lceil\frac{2}{b-a}\rceil$.

Clearly the sequence $\{m_n\}_{n\geq 1}$ converges point-wise to g. Since with probability 1, $f(X,\cdot)$ is convex and decreasing, and (by the hypothesis of boundedness) $|m_n(X,\epsilon)| \leq |g(X,a)| \leq M$ for all $\epsilon \in [a,b]$.

By Lebesgue's Dominated Convergence Theorem (see, for instance, [1]),

$$\mathbb{E}\left[\frac{\partial}{\partial\epsilon}f(X,\epsilon)\right] = \mathbb{E}\,[g(X,\epsilon)] = \lim_{n\to\infty}\mathbb{E}\,[m_n(X,\epsilon)] = \frac{\partial}{\partial\epsilon}\mathbb{E}\,[f(X,\epsilon)] \tag{2.5}$$

and the proof is complete. □

3. MAIN RESULTS

Theorem 3.1. *For any random variable X, and function f such that, with probability 1,*

1. *$f(X,\cdot)$ meets the requirements of Lemma 2.2 on $[a,b]$ and is non-negative,*
2. *$f(\cdot,\epsilon)$ is decreasing $\forall\epsilon\in[a,b]$, and*
3. *$\frac{\partial}{\partial\epsilon}f(\cdot,\epsilon)$ is increasing $\forall\epsilon\in[a,b]$,*

then for $\epsilon_1, \epsilon_2 \in [a, b]$ *with* $\epsilon_1 < \epsilon_2$,

$$\operatorname{Var}[f(X, \epsilon_2)] \leq \operatorname{Var}[f(X, \epsilon_1)] \tag{3.1}$$

provided the variances exist.

Moreover, if $\exists\, \epsilon_3, \epsilon_4 \in [\epsilon_1, \epsilon_2]$*, such that* $\epsilon_3 < \epsilon_4$ *and* $\forall\, \hat{\epsilon} \in [\epsilon_3, \epsilon_4]$*,* $f(\cdot, \hat{\epsilon})$ *is strictly decreasing and* $\frac{\partial}{\partial \epsilon} f(\cdot, \epsilon)\big|_{\epsilon=\hat{\epsilon}}$ *is strictly increasing, the above inequality is strict.*

Proof. It suffices to show that $\operatorname{Var}[f(X, \epsilon)]$ is a decreasing function of ϵ. First, note that (with probability 1) $f(X, \cdot)^2$ is convex and decreasing since $f(X, \cdot)$ is convex decreasing and non-negative. We note that its derivative at a is $2f(X, a)f'(X, a)$ and hence $f(X, \cdot)^2$ meets the requirements of Lemma 2.2. Thus, we have

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \operatorname{Var}[f(X, \epsilon)] &= \frac{\partial}{\partial \epsilon} \left\{ \mathbb{E}\left[f(X, \epsilon)^2 \right] - \left(\mathbb{E}[f(X, \epsilon)] \right)^2 \right\} \\ &= \mathbb{E}\left[\frac{\partial}{\partial \epsilon} f(X, \epsilon)^2 \right] - 2\mathbb{E}[f(X, \epsilon)] \frac{\partial}{\partial \epsilon} \mathbb{E}[f(X, \epsilon)] \\ &= \mathbb{E}\left[2f(X, \epsilon) \frac{\partial}{\partial \epsilon} f(X, \epsilon) \right] - 2\mathbb{E}[f(X, \epsilon)]\, \mathbb{E}\left[\frac{\partial}{\partial \epsilon} f(X, \epsilon) \right] \\ &\leq 0, \end{aligned} \tag{3.2}$$

where the last inequality follows by applying Lemma 2.1 to the decreasing function $f(\cdot, \epsilon)$, and the increasing function $\frac{\partial}{\partial \epsilon} f(\cdot, \epsilon)$, proving the initial assertion.

If $\exists\, \epsilon_3, \epsilon_4 \in [\epsilon_1, \epsilon_2]$, such that $\epsilon_3 < \epsilon_4$ and $\forall\, \hat{\epsilon} \in [\epsilon_3, \epsilon_4]$, $f(\cdot, \hat{\epsilon})$ is strictly decreasing and $\frac{\partial}{\partial \epsilon} f(\cdot, \epsilon)\big|_{\epsilon=\hat{\epsilon}}$ is strictly increasing, Lemma 2.1 gives strict inequality. Integrating the inequality from ϵ_1 to ϵ_2, we obtain

$$\operatorname{Var}[f(X, \epsilon_2)] < \operatorname{Var}[f(X, \epsilon_1)]. \tag{3.3}$$

□

The inequality below on the variance of the reciprocals of shifted random variables follows immediately from Theorem 3.1.

Example 3.1. Let X be a positive random variable, then for all $q > 0$ and $\epsilon > 0$,

$$\operatorname{Var}\left[\frac{1}{(X + \epsilon)^q} \right] < \operatorname{Var}\left[\frac{1}{X^q} \right] \tag{3.4}$$

provided the variances exist. Note that the theorem applies since $X > 0$ with probability 1.

The next result compares the variance of two different convex functions of the same random variable.

Theorem 3.2. *Let* X *be a random variable. If* f *and* g *are non-negative, differentiable, convex decreasing functions such that* $f \leq g$ *and* $f' \geq g'$ *over the convex hull of the range of* X*, then*

$$\operatorname{Var}[f(X)] \leq \operatorname{Var}[g(X)] \tag{3.5}$$

provided the variances exist. Moreover, if $0 > f' > g'$*, then the above inequality is strict.*

Proof. Consider the function h where $h(x, \epsilon) = \epsilon f(x) + (1 - \epsilon) g(x)$, $\epsilon \in [0, 1]$. We observe that

(1) $h(x, \cdot)$ is non-negative, linear over $[0, 1]$ (hence differentiable and convex decreasing), and meets the requirements of Lemma 2.2.
(2) $h(\cdot, \epsilon)$ is a decreasing function $\forall \epsilon \in [0, 1]$ (since both f and g are decreasing).

(3) $\frac{\partial}{\partial x}\frac{\partial}{\partial \epsilon}h(x,\epsilon) = \frac{\partial}{\partial x}\left(f(x) - g(x)\right) = f' - g' \geq 0$. That is, $\frac{\partial}{\partial \epsilon}h(\cdot,\epsilon)$ is an increasing function.

Therefore, by Theorem 3.1,

$$\mathrm{Var}\left[f(X)\right] = \mathrm{Var}\left[h(X,1)\right] \leq \mathrm{Var}\left[h(X,0)\right] = \mathrm{Var}\left[g(X)\right]. \tag{3.6}$$

Furthermore, if $0 > f' > g'$, $h(\cdot,\hat{\epsilon})$ is strictly decreasing and $\frac{\partial}{\partial \epsilon}h(\cdot,\epsilon)|_{\epsilon=\hat{\epsilon}}$ is strictly increasing $\forall \hat{\epsilon} \in [0,1]$. The result then holds with strict inequality by Theorem 3.1. □

Given a random variable X, the inverse of its distribution function F_X^{-1} is well defined except on a set of measure zero since the set of points of discontinuity of an increasing function is countable (see [3]). Given a uniform random variable U on $[0,1]$, X has the same distribution function as $F_X^{-1}(U)$.

We now present an inequality comparing the variance of a convex function of two different random variables.

Theorem 3.3. *Let X, Y be non-negative random variables with inverse distribution functions F_X^{-1}, F_Y^{-1} respectively. Given a non-negative convex decreasing function g, if $F_Y^{-1} - F_X^{-1}$ is*

(1) *non-negative and*
(2) *monotone decreasing*

on $[0,1]$, then

$$\mathrm{Var}\left[g(Y)\right] \leq \mathrm{Var}\left[g(X)\right] \tag{3.7}$$

provided the variances exist.

Moreover, if g is strictly convex and strictly decreasing and either of the following hold almost everywhere:

(1) $(F_Y^{-1})' - (F_X^{-1})' < 0$*, or*
(2) $F_Y^{-1} - F_X^{-1} > 0$ *and* $\hat{\epsilon}(F_Y^{-1})' + (1-\hat{\epsilon})(F_X^{-1})' > 0$ *for all* $\hat{\epsilon} \in [\epsilon_1, \epsilon_2] \subseteq [0,1]$ *with* $\epsilon_1 < \epsilon_2$*,*

then the above inequality is strict.

Proof. Consider the function h where $h(u,\epsilon) = g\left(F_X^{-1}(u) + \epsilon\left[F_Y^{-1}(u) - F_X^{-1}(u)\right]\right)$ $\epsilon \in [0,1]$. Note that $g \geq 0$, $g' \leq 0$, $g'' \geq 0$ since g is non-negative convex and decreasing; and that the inverse distribution function of a non-negative random variable is non-negative. Hence,

(1) $h(u,\cdot)$ is non-negative, differentiable, convex and decreasing, and
$$\left.\frac{\partial}{\partial \epsilon}h(u,\epsilon)\right|_{\epsilon=0} = \left[F_Y^{-1}(u) - F_X^{-1}(u)\right] g'\left(F_X^{-1}(u)\right)$$
exists and is bounded with probability 1, so h meets the requirements of Lemma 2.2.
(2) $h(\cdot,\epsilon)$ is a decreasing function $\forall \epsilon \in [0,1]$.
(3) $\frac{\partial}{\partial \epsilon}h(\cdot,\epsilon)$ is an increasing function since

$$\begin{aligned}
&\frac{\partial}{\partial u}\frac{\partial}{\partial \epsilon}h(u,\epsilon)\\
&= \frac{\partial}{\partial u}\left\{\left[F_Y^{-1}(u) - F_X^{-1}(u)\right] g'\left(F_X^{-1}(u) + \epsilon\left[F_Y^{-1}(u) - F_X^{-1}(u)\right]\right)\right\}\\
&= \left[(F_Y^{-1})'(u) - (F_X^{-1})'(u)\right] g'\left(F_X^{-1}(u) + \epsilon\left[F_Y^{-1}(u) - F_X^{-1}(u)\right]\right)\\
&\qquad + \epsilon(F_Y^{-1})'(u)\left[F_Y^{-1}(u) - F_X^{-1}(u)\right] g''\left(F_X^{-1}(u) + \epsilon\left[F_Y^{-1}(u) - F_X^{-1}(u)\right]\right)\\
&\qquad + (1-\epsilon)(F_X^{-1})'(u)\left[F_Y^{-1}(u) - F_X^{-1}(u)\right] g''\left(F_X^{-1}(u) + \epsilon\left[F_Y^{-1}(u) - F_X^{-1}(u)\right]\right)\\
&\geq 0.
\end{aligned} \tag{3.8}$$

To justify the inequality, consider (3.8), the first term is non-negative due to condition (2) and g being a decreasing function ($g' \leq 0$), and the second (resp. third) term is non-negative since by the properties of distribution functions $(F_Y^{-1})' \geq 0$ (resp. $(F_X^{-1})' \geq 0$), condition (1) holds, and g is convex ($g'' \geq 0$).

Therefore, by Theorem 3.1,

$$\operatorname{Var}[g(Y)] = \operatorname{Var}[h(U,1)] \leq \operatorname{Var}[h(U,0)] = \operatorname{Var}[g(X)]\,. \tag{3.9}$$

If the subsidiary conditions for strict inequality are met, since $g' < 0$ and $g'' > 0$, it is then clear that Theorem 3.1 gives strict inequality. $\square$

4. APPLICATIONS

Applications of such inequalities include comparing variances of present worth of financial cash flows under stochastic interest rates. Specifically, the present worth of y dollars in q years at a interest rate of X is given by $\frac{y}{(1+X)^q}$ ($q > 0, X > 0$). When the interest rate X increases by a positive amount, ϵ, it is clear that the expected present worth decreases:

$$\mathbb{E}\left[\frac{y}{(1+X+\epsilon)^q}\right] < \mathbb{E}\left[\frac{y}{(1+X)^q}\right].$$

Example 3.1 shows that its variance decreases as well, that is,

$$\operatorname{Var}\left[\frac{y}{(1+X+\epsilon)^q}\right] < \operatorname{Var}\left[\frac{y}{(1+X)^q}\right].$$

In this example, the random variable X represents the projected interest rate (which is not known with certainty), while $X + \epsilon$ represents the interest rate should an increase of ϵ be envisaged.

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