

# Journal of Inequalities in Pure and Applied Mathematics

ON HARDY'S INEQUALITY IN  $L^{p(x)}(0, \infty)$

RABIL A. MASHIYEV, BILAL ÇEKIÇ AND SEZAI OGRAS

University of Dicle, Faculty of Sciences and Arts  
Department of Mathematics  
21280- Diyarbakır TURKEY

EMail: [mrabil@dicle.edu.tr](mailto:mrabil@dicle.edu.tr)

EMail: [bilalc@dicle.edu.tr](mailto:bilalc@dicle.edu.tr)

EMail: [sezaio@dicle.edu.tr](mailto:sezaio@dicle.edu.tr)

©2000 Victoria University  
ISSN (electronic): 1443-5756  
310-05



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volume 7, issue 3, article 106,  
2006.

*Received 14 October, 2005;  
accepted 07 April, 2006.*

*Communicated by: L. Pick*

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Abstract

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## Abstract

Our aim in this paper is to obtain Hardy's inequality in variable exponent Lebesgue spaces  $L^{p(x)}(0, \infty)$ , where the test function  $u(x)$  vanishes at infinity. We use a local Dini-Lipschitz condition and its the natural analogue at infinity, which play a central role in our proof.

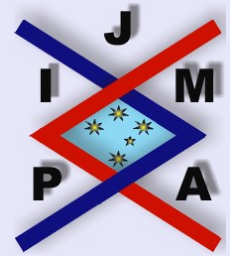
*2000 Mathematics Subject Classification:* 46E35, 26D10.

*Key words:* Variable exponent, Hardy's inequality.

The authors are thankful to the referees and Peter Hästö for their helpful suggestions and valuable contributions. This research was supported by DUAPK grant No. 04 FF 40.

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# 1. Introduction

Over the last decades the variable exponent Lebesgue space  $L^{p(\cdot)}(\Omega)$  and the corresponding Sobolev space  $W^{m,p(\cdot)}(\Omega)$  have been a subject of active research stimulated by development of the studies of problems in elasticity, fluid dynamics, calculus of variations, and differential equations with  $p(x)$ – growth [10, 12]. These spaces are a special case of the Musielak-Orlicz spaces [8]. If  $p$  is the constant, then  $L^{p(\cdot)}(\Omega)$  coincides with the classical Lebesgue spaces. We refer to [4, 7] for fundamental properties of these spaces and to [5, 6, 11] for Hardy type inequalities.

The classical Hardy inequality [9] is

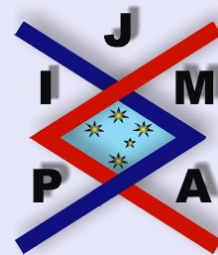
$$(1.1) \quad \int_0^\infty |u(x)|^p x^\beta dx \leq \left( \frac{p}{\beta + 1} \right)^p \int_0^\infty |u'(x)|^p x^{\beta+p} dx,$$

where  $1 < p < \infty$ ,  $-1 < \beta < \infty$ ,  $u$  is an absolutely continuous function on  $(0, \infty)$  and  $u(\infty) = \lim_{x \rightarrow \infty} u(x) = 0$ .

Kokilashvili and Samko [6] gave the boundedness of Hardy operators with fixed singularity in the spaces  $L^{p(\cdot)}(\rho, \Omega)$  over a bounded open set in  $\mathbb{R}^n$  with a power weight  $\rho(x) = |x - x_0|^\beta$ ,  $x_0 \in \bar{\Omega}$  and an exponent  $p(x)$  satisfying the Dini-Lipschitz condition. The Hardy type inequality can be derived

$$(1.2) \quad \left\| x^{\frac{\beta}{p(x)}} u \right\|_{p(x), (0, \ell)} \leq C(p(x), \ell) \left\| x^{\frac{\beta}{p(x)} + 1} u' \right\|_{p(x), (0, \ell)},$$

where  $\beta > -1$ ,  $1 < p^- \leq p^+ < \infty$ ,  $\ell$  is a positive finite number, and  $u$  is an absolutely continuous function on  $(0, \ell)$  in the Lebesgue space with variable exponent for bounded domains from Theorem E in [6].



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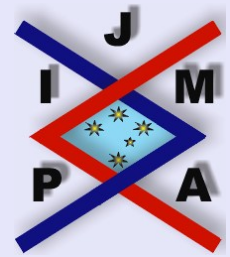
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Recently, Harjulehto, Hästö and Koskenoja [5] have obtained the norm version of Hardy's inequality using Diening's corollaries in the variable exponent Sobolev space. Also they have given a necessary and sufficient condition for Hardy's inequality to hold.

We consider the problem of the extension of Hardy's inequality to the case of variable  $p(x)$ . Such inequalities with variable  $p(x)$  are already known for a finite interval  $(0, \ell)$  in the one-dimensional case. Our aim in this paper is to obtain a Hardy type inequality in a one-dimensional Lebesgue space  $L^{p(x)}(0, \infty)$  using a distinct method, by considering relevant studies in [1] and [6].



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## 2. Preliminaries

Let  $\Omega \subset \mathbb{R}^n$  be an open set,  $p(\cdot) : \Omega \rightarrow [1, \infty)$  be a measurable bounded function and be denoted as  $p^+ = \operatorname{esssup}_{x \in \Omega} p(x)$  and  $p^- = \operatorname{essinf}_{x \in \Omega} p(x)$ . We define the variable exponent Lebesgue space  $L^{p(\cdot)}(\Omega)$  consisting of all measurable functions  $f : \Omega \rightarrow \mathbb{R}$  such that the modular

$$A_p(f) := \int_{\Omega} |f(x)|^{p(x)} dx$$

is finite. If  $p^+ < \infty$  then we call  $p$  a bounded exponent and we can introduce the norm on  $L^{p(\cdot)}(\Omega)$  by

$$(2.1) \quad \|f\|_{p(\cdot), \Omega} := \inf \left\{ \lambda > 0 : A_p \left( \frac{f}{\lambda} \right) \leq 1 \right\}$$

and  $L^{p(\cdot)}(\Omega)$  becomes a Banach space. The norm  $\|f\|_{p(\cdot), \Omega}$  is in close relation with the modular  $A_p(f)$ .

**Lemma 2.1 ([4]).** *Let  $p(x)$  be a measurable exponent such that  $1 \leq p^- \leq p(x) \leq p^+ < \infty$  and let  $\Omega$  be a measurable set in  $\mathbb{R}^n$ . Then,*

- (i)  $\|f\|_{p(x)} = \lambda \neq 0$  if and only if  $A_p \left( \frac{f}{\lambda} \right) = 1$ ;
- (ii)  $\|f\|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow A_p(f) < 1 (= 1; > 1)$ ;
- (iii) For any  $p(x)$ , the following inequalities

$$\|f\|_{p(x)}^{p^+} \leq A_p(f) \leq \|f\|_{p(x)}^{p^-}, \quad \|f\|_{p(x)} \leq 1$$



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and

$$\|f\|_{p(x)}^{p^-} \leq A_p(f) \leq \|f\|_{p(x)}^{p^+}, \quad \|f\|_{p(x)} \geq 1$$

hold.

**Lemma 2.2 ([4, 7]).** *The generalization of Hölder's inequality*

$$\left| \int_{\Omega} f(x)\varphi(x)dx \right| \leq c \|f\|_{p(x)} \|\varphi\|_{p'(x)}$$

holds, where  $p'(x) = \frac{p(x)}{p(x)-1}$  and the constant  $c > 0$  depends only on  $p(x)$ .

We say that the exponent  $p(\cdot) : \Omega \rightarrow [1, \infty)$  is Dini-Lipschitz if there exists a constant  $c > 0$  such that

$$(2.2) \quad |p(x) - p(y)| \leq \frac{c}{-\log|x - y|},$$

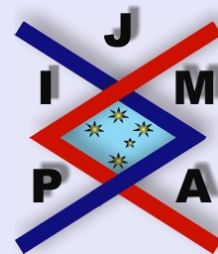
for every  $x, y \in \Omega$  with  $|x - y| \leq \frac{1}{2}$ . The natural analogue of (2.2) is

$$(2.3) \quad |p(x) - p(y)| \leq \frac{c}{\log(e + |x|)}$$

for every  $x, y \in \Omega$ ,  $|y| \geq |x|$  at infinity. Under these conditions, most of the properties of the classical Lebesgue space can be readily generalized to the Lebesgue space with variable exponent.

**Theorem 2.3 ([5, Theorem 5.2]).** *Let  $I = [0, M)$  for  $M < \infty$ ,  $p : I \rightarrow [1, \infty)$  be bounded,  $p(0) > 1$  and*

$$\limsup_{x \rightarrow 0^+} (p(x) - p(0)) \log \frac{1}{x} < \infty$$



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and  $p_{(0,x_0)}^- = p(0)$  for some  $x_0 \in (0, 1)$ . If  $a \in \left[0, 1 - \frac{1}{p(0)}\right)$ , then Hardy's inequality

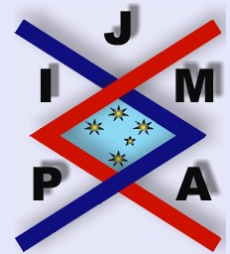
$$(2.4) \quad \left\| \frac{u(x)}{x^{1-a}} \right\|_{p(x)} \leq C \|u'(x)x^a\|_{p(x)}$$

holds for every  $u \in W^{1,p(x)}(I)$  with  $u(0) = 0$ .

Throughout this paper, we will assume that  $p(x)$  is a measurable function and use this notation

$$\|f\|_{p(x)} := \|f\|_{p(x), (0, \infty)}.$$

Moreover, we will use  $c$  and  $c_i$  as generic constants, i.e. its value may change from line to line.



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### 3. Main Result

**Theorem 3.1.** Let  $\beta > -1$  and  $p : (0, \infty) \rightarrow (1, \infty)$  be such that  $1 \leq p^- \leq p^+ < \infty$  and

$$(3.1) \quad |p(x) - p(y)| \leq \frac{c}{-\log|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \mathbb{R}^+.$$

Assume that there exists a number  $p(\infty) \in [1, \infty)$  and  $a \geq 1$  such that

$$(3.2) \quad 0 \leq p(x) - p(\infty) \leq \frac{c}{\log(e+x)}, \quad x \geq a.$$

Then, we have

$$(3.3) \quad \left\| x^{\frac{\beta}{p(x)}} u(x) \right\|_{p(x)} \leq c \left\| x^{\frac{\beta}{p(x)}+1} u'(x) \right\|_{p(x)}$$

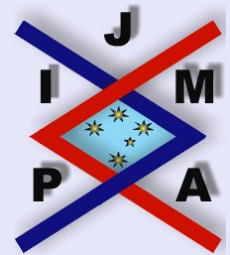
for every absolutely continuous function  $u : (0, \infty) \rightarrow \mathbb{R}$  with  $u(\infty) = 0$ .

*Proof.* To prove this inequality it suffices to consider the case

$$\left\| x^{\frac{\beta}{p(x)}+1} u'(x) \right\|_{p(x)} = 1$$

for a monotone decreasing function  $u$ . Using Hölder's inequality, we obtain

$$(3.4) \quad \begin{aligned} u(a) &= - \int_a^\infty u'(t) dt \\ &\leq c \left\| t^{\frac{\beta}{p(t)}+1} u'(t) \right\|_{p(t), (a, \infty)} \left\| t^{-\frac{\beta}{p(t)}-1} \right\|_{p'(t), (a, \infty)} \leq c_1, \end{aligned}$$



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where  $p'(x) = \frac{p(x)}{p(x)-1}$ , and the positive constant  $c_1$  depends only on  $p(x)$  and  $\beta$ . Since  $u(x) \leq c_1$  for  $(0, \infty)$ , using Hardy's inequality for the fixed exponent  $p(\infty)$  we have

$$(3.5) \quad \int_a^\infty x^\beta u(x)^{p(x)} dx \leq c_2^{p^+} \int_a^\infty x^\beta u(x)^{p(\infty)} dx \\ \leq c_3 \int_a^\infty x^\beta (-xu'(x))^{p(\infty)} dx.$$

If we divide the interval  $(a, \infty)$  into three sets such that

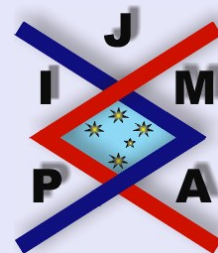
$$A = \{t \in (a, \infty) : t|u'(t)| > 1\}, \\ B = \{t \in (a, \infty) : t^{-\beta-2} < t|u'(t)| \leq 1\}, \\ C = \{t \in (a, \infty) : t|u'(t)| \leq t^{-\beta-2}\},$$

then we can write

$$\int_a^\infty t^\beta |tu'(t)|^{p(\infty)} dt \\ = \int_A t^\beta |tu'(t)|^{p(\infty)} dt + \int_B t^\beta |tu'(t)|^{p(\infty)} dt + \int_C t^\beta |tu'(t)|^{p(\infty)} dt.$$

Now, let us estimate each integral. It is easy to see that

$$\int_A t^\beta |tu'(t)|^{p(\infty)} dt \leq \int_a^\infty t^\beta |tu'(t)|^{p(t)} dt \leq 1$$



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and

$$\int_C t^\beta |tu'(t)|^{p(\infty)} dt \leq \int_C t^\beta t^{-\beta-2} dt \leq \int_a^\infty t^\beta t^{-\beta-2} dt \leq c.$$

Since

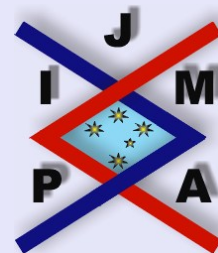
$$\begin{aligned} t^{(\beta+2)(p(t)-p(\infty))} &= (t^{p(t)-p(\infty)})^{\beta+2} \\ &\leq \left( t^{\frac{1}{\log(e+t)}} \right)^{\beta+2} \\ &\leq \left( e^{\frac{\log t}{\log(e+t)}} \right)^{\beta+2} \\ &\leq e^{\beta+2}, \end{aligned}$$

we have

$$\begin{aligned} \int_B t^\beta |tu'(t)|^{p(\infty)} dt &\leq \int_B t^\beta (t^{\beta+2} |tu'(t)|)^{p(t)-p(\infty)} |tu'(t)|^{p(\infty)} dt \\ &\leq \int_a^\infty t^{(\beta+2)(p(t)-p(\infty))} t^\beta |tu'(t)|^{p(t)} dt \\ &\leq e^{\beta+2} \int_a^\infty t^\beta |tu'(t)|^{p(t)} dt \\ &\leq e^{\beta+2}. \end{aligned}$$

Hence, we obtain

$$(3.6) \quad \int_a^\infty t^\beta |u(t)|^{p(t)} dt \leq c.$$



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On the other hand, by using inequality (1.2) and the assumption (3.1) for the interval  $(0, a)$ , we can write

$$(3.7) \quad \int_0^a t^\beta |u(t)|^{p(t)} dt \leq c.$$

Combining inequalities (3.6) and (3.7), we get

$$\int_0^\infty t^\beta |u(t)|^{p(t)} dt \leq c$$

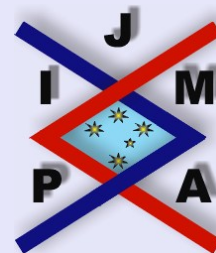
and hence from the relation between norm and modular we have

$$(3.8) \quad \left\| t^{\frac{\beta}{p(t)}} u(t) \right\|_{p(t)} \leq c.$$

Consequently, we have the required result from (3.8) for

$$\frac{u(t)}{\left\| t^{\frac{\beta}{p(t)}+1} u'(t) \right\|_{p(t)}}.$$

□



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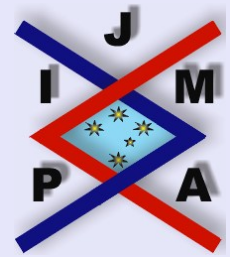
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