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ON HARDY'S INEQUALITY IN $L^{p(x)}(0,\infty)$

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Abstract

Our aim in this paper is to obtain Hardy's inequality in variable exponent Lebesgue spaces $L^{p(x)}(0,\infty)$, where the test function $u(x)$ vanishes at infinity. We use a local Dini-Lipschitz condition and its the natural analogue at infinity, which play a central role in our proof.

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Contents

1. Introduction

Over the last decades the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and the corresponding Sobolev space $W^{m,p(\cdot)}(\Omega)$ have been a subject of active research stimulated by development of the studies of problems in elasticity, fluid dynamics, calculus of variations, and differential equations with $p(x)-$ growth [10, 12]. These spaces are a special case of the Musielak-Orlicz spaces [8]. If p is the constant, then $L^{p(\cdot)}(\Omega)$ coincides with the classical Lebesgue spaces. We refer to [4, 7] for fundamental properties of these spaces and to [5, 6, 11] for Hardy type inequalities.

The classical Hardy inequality [9] is

$$\int_0^\infty |u(x)|^p \, x^\beta dx \le \left(\frac{p}{\beta+1}\right)^p \int_0^\infty |u'(x)|^p \, x^{\beta+p} dx, \tag{1.1}$$

where $1 < p < \infty, -1 < \beta < \infty$, u is an absolutely continuous function on $(0,\infty)$ and $u(\infty) = \lim_{x\to\infty} u(x) = 0$.

Kokilashvili and Samko [6] gave the boundedness of Hardy operators with fixed singularity in the spaces $L^{p(\cdot)}(\rho,\Omega)$ over a bounded open set in $\mathbb{R}^n$ with a power weight $\rho(x) = |x - x_0|^\beta$, $x_0 \in \bar{\Omega}$ and an exponent $p(x)$ satisfying the Dini-Lipschitz condition. The Hardy type inequality can be derived

$$\left\| x^{\frac{\beta}{p(x)}} u \right\|_{p(x),(0,\ell)} \le C(p(x),\ell) \left\| x^{\frac{\beta}{p(x)}+1} u' \right\|_{p(x),(0,\ell)}, \tag{1.2}$$

where $\beta > -1, 1 < p^- \le p^+ < \infty, \ell$ is a positive finite number, and u is an absolutely continuous function on $(0,\ell)$ in the Lebesgue space with variable exponent for bounded domains from Theorem E in [6].

Recently, Harjulehto, Hästö and Koskenoja [5] have obtained the norm version of Hardy's inequality using Diening's corollaries in the variable exponent Sobolev space. Also they have given a necessary and sufficient condition for Hardy's inequality to hold.

We consider the problem of the extension of Hardy's inequality to the case of variable $p(x)$. Such inequalities with variable $p(x)$ are already known for a finite interval $(0,\ell)$ in the one-dimensional case. Our aim in this paper is to obtain a Hardy type inequality in a one-dimensional Lebesgue space $L^{p(x)}(0,\infty)$ using a distinct method, by considering relevant studies in [1] and [6].

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open set, $p(\cdot) : \Omega \to [1, \infty)$ be a measurable bounded function and be denoted as $p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x)$ and $p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consisting of all measurable functions $f : \Omega \to \mathbb{R}$ such that the modular

$$A_p(f) := \int_\Omega |f(x)|^{p(x)} dx$$

is finite. If $p^+ < \infty$ then we call p a bounded exponent and we can introduce the norm on $L^{p(\cdot)}(\Omega)$ by

$$\|f\|_{p(\cdot),\Omega} := \inf \left\{ \lambda > 0 : A_p\left(\frac{f}{\lambda}\right) \le 1 \right\} \tag{2.1}$$

and $L^{p(\cdot)}(\Omega)$ becomes a Banach space. The norm $\|f\|_{p(\cdot),\Omega}$ is in close relation with the modular $A_p(f)$.

Lemma 2.1 ([4]). *Let $p(x)$ be a measurable exponent such that $1 \le p^- \le p(x) \le p^+ < \infty$ and let Ω be a measurable set in $\mathbb{R}^n$. Then,*

(i) $\|f\|_{p(x)} = \lambda \neq 0$ *if and only if* $A_p\left(\frac{f}{\lambda}\right) = 1$;

(ii) $\|f\|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow A_p(f) < 1 \ (= 1; > 1)$;

(iii) For any $p(x)$, the following inequalities

$$\|f\|_{p(x)}^{p^+} \le A_p(f) \le \|f\|_{p(x)}^{p^-}, \qquad \|f\|_{p(x)} \le 1$$

and

$$\|f\|_{p(x)}^{p^-} \le A_p(f) \le \|f\|_{p(x)}^{p^+}, \qquad \|f\|_{p(x)} \ge 1$$

hold.

Lemma 2.2 ([4, 7]). *The generalization of Hölder's inequality*

$$\left| \int_\Omega f(x)\varphi(x)dx \right| \le c \|f\|_{p(x)} \|\varphi\|_{p'(x)}$$

holds, where $p'(x) = \frac{p(x)}{p(x)-1}$ *and the constant* $c > 0$ *depends only on* $p(x)$*.*

We say that the exponent $p(\cdot) : \Omega \to [1, \infty)$ is Dini-Lipschitz if there exists a constant $c > 0$ such that

(2.2) $$|p(x) - p(y)| \le \frac{c}{-\log|x-y|},$$

for every $x, y \in \Omega$ with $|x - y| \le \frac{1}{2}$. The natural analogue of (2.2) is

(2.3) $$|p(x) - p(y)| \le \frac{c}{\log(e + |x|)}$$

for every $x, y \in \Omega$, $|y| \ge |x|$ at infinity. Under these conditions, most of the properties of the classical Lebesgue space can be readily generalized to the Lebesgue space with variable exponent.

Theorem 2.3 ([5, Theorem 5.2]). *Let* $I = [0, M)$ *for* $M < \infty$, $p : I \to [1, \infty)$ *be bounded,* $p(0) > 1$ *and*

$$\limsup_{x\to 0+} (p(x) - p(0)) \log \frac{1}{x} < \infty$$

and $p^-_{(0,x_0)} = p(0)$ *for some* $x_0 \in (0,1)$. *If* $a \in \left[0, 1 - \frac{1}{p(0)}\right)$, *then Hardy's inequality*

$$\left\| \frac{u(x)}{x^{1-a}} \right\|_{p(x)} \le C \left\| u'(x) x^a \right\|_{p(x)} \tag{2.4}$$

holds for every $u \in W^{1,p(x)}(I)$ *with* $u(0) = 0$.

Throughout this paper, we will assume that $p(x)$ is a measurable function and use this notation

$$\|f\|_{p(x)} := \|f\|_{p(x),(0,\infty)} .$$

Moreover, we will use c and c_i as generic constants, i.e. its value may change from line to line.

3. Main Result

Theorem 3.1. *Let* $\beta > -1$ *and* $p : (0,\infty) \to (1,\infty)$ *be such that* $1 \le p^- \le p^+ < \infty$ *and*

$$|p(x) - p(y)| \le \frac{c}{-\log|x-y|}, \qquad |x-y| \le \frac{1}{2}, \quad x, y \in \mathbb{R}^+. \tag{3.1}$$

Assume that there exists a number $p(\infty) \in [1,\infty)$ *and* $a \ge 1$ *such that*

$$0 \le p(x) - p(\infty) \le \frac{c}{\log(e+x)}, \qquad x \ge a. \tag{3.2}$$

Then, we have

$$\left\| x^{\frac{\beta}{p(x)}} u(x) \right\|_{p(x)} \le c \left\| x^{\frac{\beta}{p(x)}+1} u'(x) \right\|_{p(x)} \tag{3.3}$$

for every absolutely continuous function $u : (0,\infty) \to \mathbb{R}$ *with* $u(\infty) = 0$.

Proof. To prove this inequality it suffices to consider the case

$$\left\| x^{\frac{\beta}{p(x)}+1} u'(x) \right\|_{p(x)} = 1$$

for a monotone decreasing function u. Using Hölder's inequality, we obtain

$$\begin{aligned} u(a) &= -\int_a^\infty u'(t)dt \\ &\le c \left\| t^{\frac{\beta}{p(t)}+1} u'(t) \right\|_{p(t),(a,\infty)} \left\| t^{-\frac{\beta}{p(t)}-1} \right\|_{p'(t),(a,\infty)} \le c_1, \end{aligned} \tag{3.4}$$

where $p'(x) = \frac{p(x)}{p(x)-1}$, and the positive constant c_1 depends only on $p(x)$ and β. Since $u(x) \le c_1$ for $(0,\infty)$, using Hardy's inequality for the fixed exponent $p(\infty)$ we have

$$\begin{aligned} (3.5)\qquad \int_a^\infty x^\beta u(x)^{p(x)} dx &\le c_2^{p^+} \int_a^\infty x^\beta u(x)^{p(\infty)} dx \\ &\le c_3 \int_a^\infty x^\beta (-xu'(x))^{p(\infty)} dx. \end{aligned}$$

If we divide the interval (a,∞) into three sets such that

$$\begin{aligned} A &= \{t \in (a,\infty) : t|u'(t)| > 1\}, \\ B &= \{t \in (a,\infty) : t^{-\beta-2} < t|u'(t)| \le 1\}, \\ C &= \{t \in (a,\infty) : t|u'(t)| \le t^{-\beta-2}\}, \end{aligned}$$

then we can write

$$\begin{aligned} &\int_a^\infty t^\beta |tu'(t)|^{p(\infty)} dt \\ &\qquad = \int_A t^\beta |tu'(t)|^{p(\infty)} dt + \int_B t^\beta |tu'(t)|^{p(\infty)} dt + \int_C t^\beta |tu'(t)|^{p(\infty)} dt. \end{aligned}$$

Now, let us estimate each integral. It is easy to see that

$$\int_A t^\beta |tu'(t)|^{p(\infty)} dt \le \int_a^\infty t^\beta |tu'(t)|^{p(t)} dt \le 1$$

and

$$\int_C t^\beta |tu'(t)|^{p(\infty)} dt \le \int_C t^\beta t^{-\beta-2} dt \le \int_a^\infty t^\beta t^{-\beta-2} dt \le c.$$

Since

$$\begin{aligned} t^{(\beta+2)(p(t)-p(\infty))} &= \left(t^{p(t)-p(\infty)}\right)^{\beta+2} \\ &\le \left(t^{\frac{1}{\log(e+t)}}\right)^{\beta+2} \\ &\le \left(e^{\frac{\log t}{\log(e+t)}}\right)^{\beta+2} \\ &\le e^{\beta+2}, \end{aligned}$$

we have

$$\begin{aligned} \int_B t^\beta |tu'(t)|^{p(\infty)} dt &\le \int_B t^\beta \left(t^{\beta+2}|tu'(t)|\right)^{p(t)-p(\infty)} |tu'(t)|^{p(\infty)} dt \\ &\le \int_a^\infty t^{(\beta+2)(p(t)-p(\infty))} t^\beta |tu'(t)|^{p(t)} dt \\ &\le e^{\beta+2} \int_a^\infty t^\beta |tu'(t)|^{p(t)} dt \\ &\le e^{\beta+2}. \end{aligned}$$

Hence, we obtain

(3.6) $$\int_a^\infty t^\beta \left|u(t)\right|^{p(t)} dt \le c.$$

On the other hand, by using inequality (1.2) and the assumption (3.1) for the interval $(0, a)$, we can write

$$\int_0^a t^\beta |u(t)|^{p(t)}\, dt \le c. \tag{3.7}$$

Combining inequalities (3.6) and (3.7), we get

$$\int_0^\infty t^\beta |u(t)|^{p(t)}\, dt \le c$$

and hence from the relation between norm and modular we have

$$\left\| t^{\frac{\beta}{p(t)}} u(t) \right\|_{p(t)} \le c. \tag{3.8}$$

Consequently, we have the required result from (3.8) for

$$\frac{u(t)}{\left\| t^{\frac{\beta}{p(t)}+1} u'(t) \right\|_{p(t)}}.$$

□

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