ASYMPTOTIC BEHAVIOR FOR DISCRETIZATIONS OF A SEMILINEAR PARABOLIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION

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Abstract: This paper concerns the study of the numerical approximation for the following initial-boundary value problem:

$$(P)\qquad \begin{cases} u_t = u_{xx} - a|u|^{p-1}u, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0 \quad u_x(1,t) + b|u(1,t)|^{q-1}u(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \end{cases}$$

where $a > 0$, $b > 0$ and $q > p > 1$. We show that the solution of a semidiscrete form of (P) goes to zero as t goes to infinity and give its asymptotic behavior. Using some nonstandard schemes, we also prove some estimates of solutions for discrete forms of (P). Finally, we give some numerical experiments to illustrate our analysis.

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1. Introduction

Consider the following initial-boundary value problem:

$$u_t = u_{xx} - a|u|^{p-1}u, \quad 0 < x < 1, t > 0, \tag{1.1}$$

$$u_x(0,t) = 0 \quad u_x(1,t) + b|u(1,t)|^{q-1}u(1,t) = 0, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{1.3}$$

where $a > 0$, $b > 0$, $q > p > 1$, $u_0 \in C^1([0,1])$, $u_0'(0) = 0$ and $u_0'(1) + b|u_0(1)|^{q-1}u_0(1) = 0$.

The theoretical study of the asymptotic behavior of solutions for semilinear parabolic equations has been the subject of investigation for many authors (see [2], [4] and the references cited therein). In particular, in [4], when $b = 0$, the authors have shown that the solution u of (1.1) – (1.3) goes to zero as t tends to infinity and satisfies the following :

$$0 \le \|u(x,t)\|_\infty \le \frac{1}{(\|u_0(x)\|_\infty + a(p-1)t)^{\frac{1}{p-1}}} \quad \text{for} \quad t \in [0,+\infty), \tag{1.4}$$

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \|u(x,t)\|_\infty = C_0, \tag{1.5}$$

where $C_0 = \left(\frac{1}{a(p-1)}\right)^{\frac{1}{p-1}}$. The same results have been obtained in [2] in the case where $b > 0$ and $q > p > 1$.

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In this paper we are interested in the numerical study of (1.1) – (1.3). At first, using a semidiscrete form of (1.1) – (1.3), we prove similar results for the semidiscrete solution. We also construct two nonstandard schemes and show that these schemes allow the discrete solutions to obey an estimation as in (1.4). Previously, authors have used numerical methods to study the phenomenon of blow-up and the one of extinction (see [1] and [3]). This paper is organized as follows. In the next section, we prove some results about the discrete maximum principle. In the third section, we take a semidiscrete form of (1.1) – (1.3), and show that the semidiscrete solution goes to zero as t tends to infinity and give its asymptotic behavior. In the fourth section, we show that the semidiscrete scheme of the third section converges. In Section 5, we construct two nonstandard schemes and obtain some estimates as in (1.4). Finally, in the last section, we give some numerical results.

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2. Semidiscretizations Scheme

In this section, we give some lemmas which will be used later. Let I be a positive integer, and define the grid $x_i = ih$, $0 \leq i \leq I$, where $h = 1/I$. We approximate the solution u of the problem (1.1) – (1.3) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the semidiscrete equations

$$\frac{d}{dt}U_i(t) = \delta^2 U_i(t) - a|U_i(t)|^{p-1}U_i(t), \quad 0 \leq i \leq I-1, t > 0, \tag{2.1}$$

$$\frac{d}{dt}U_I(t) = \delta^2 U_I(t) - a|U_I(t)|^{p-1}U_I(t) - \frac{2b}{h}|U_I(t)|^{q-1}U_I(t), \quad t > 0, \tag{2.2}$$

$$U_i(0) = U_i^0 > 0, \quad 0 \leq i \leq I, \tag{2.3}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1. *Let $a_h(t) \in C^0([0,T], \mathbb{R}^{I+1})$ and let $V_h(t) \in C^1([0,T], \mathbb{R}^{I+1})$ such that*

$$\frac{d}{dt}V_i(t) - \delta^2 V_i(t) + a_i(t)V_i(t) \geq 0, \quad 0 \leq i \leq I, t \in (0,T), \tag{2.4}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{2.5}$$

Then we have $V_i(t) \geq 0$ for $0 \leq i \leq I$, $t \in (0,T)$.

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Proof. Let $T_0 < T$ and let $m = \min_{0 \le i \le I, 0 \le t \le T_0} V_i(t)$. Since for $i \in \{0, \dots, I\}$, $V_i(t)$ is a continuous function, there exists $t_0 \in [0, T_0]$ such that $m = V_{i_0}(t_0)$ for a certain $i_0 \in \{0, \dots, I\}$. It is not hard to see that

$$\frac{dV_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{V_{i_0}(t_0) - V_{i_0}(t_0 - k)}{k} \le 0, \tag{2.6}$$

$$\delta^2 V_{i_0}(t_0) = \frac{V_1(t_0) - V_0(t_0)}{h^2} \ge 0 \quad \text{if} \quad i_0 = 0, \tag{2.7}$$

$$\delta^2 V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - 2V_{i_0}(t_0) + V_{i_0-1}(t_0)}{h^2} \ge 0 \quad \text{if} \quad 1 \le i_0 \le I - 1, \tag{2.8}$$

$$\delta^2 V_{i_0}(t_0) = \frac{V_{I-1}(t_0) - V_I(t_0)}{h^2} \ge 0 \quad \text{if} \quad i_0 = I. \tag{2.9}$$

Define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is large enough such that $a_{i_0}(t_0) - \lambda > 0$. A straightforward computation reveals:

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \ge 0. \tag{2.10}$$

We observe from (2.6) – (2.9) that $\frac{dZ_{i_0}(t_0)}{dt} \le 0$ and $\delta^2 Z_{i_0}(t_0) \ge 0$. Using (2.10), we arrive at $(a_{i_0}(t) - \lambda) Z_{i_0}(t_0) \ge 0$, which implies that $Z_{i_0}(t_0) \ge 0$. Therefore, $V_{i_0}(t_0) = m \ge 0$ and we have the desired result. $\square$

Another form of the maximum principle is the following comparison lemma.

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Lemma 2.2. *Let $V_h(t), U_h(t) \in C^1([0,\infty), \mathbb{R}^{I+1})$ and $f \in C^0(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ such that for $t \in (0,\infty)$,*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t), t) < \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_i(t), t), \quad 0 \le i \le I, \tag{2.11}$$

$$V_i(0) < U_i(0), \quad 0 \le i \le I. \tag{2.12}$$

Then we have $V_i(t) < U_i(t)$, $0 \le i \le I$, $t \in (0,\infty)$.

Proof. Define the vector $Z_h(t) = U_h(t) - V_h(t)$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $i = 0, \dots, I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0.$$

$$\delta^2 Z_{i_0}(t_0) = \begin{cases} \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0 & \text{if} \quad 1 \le i_0 \le I-1, \\ \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \ge 0 & \text{if} \quad i_0 = 0, \\ \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \ge 0 & \text{if} \quad i_0 = I, \end{cases}$$

which implies:

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(U_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) \le 0.$$

But this inequality contradicts (2.11). □

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3. Asymptotic Behavior

In this section, we show that the solution U_h of (2.1) – (2.3) goes to zero as $t \to +\infty$ and give its asymptotic behavior. Firstly, we prove that the solution tends to zero as $t \to +\infty$ by the following:

Theorem 3.1. *The solution $U_h(t)$ of (2.1) – (2.3) goes to zero as $t \to \infty$ and we have the following estimate*

$$0 \leq \|U_h(t)\|_\infty \leq \frac{1}{(\|U_h(0)\|_\infty^{1-p} + a(p-1)t)^{\frac{1}{p-1}}} \quad \textit{for} \quad t \in [0, +\infty).$$

Proof. We introduce the function $\alpha(t)$ which is defined as

$$\alpha(t) = \frac{1}{(\|U_h(0)\|_\infty^{1-p} + a(p-1)t)^{\frac{1}{p-1}}}$$

and let W_h be the vector such that $W_i(t) = \alpha(t)$. It is not hard to see that

$$\frac{dW_i(t)}{dt} - \delta^2 W_i(t) + a|W_i(t)|^{p-1}W_i(t) = 0, \quad 0 \leq i \leq I-1, t \in (0,T),$$

$$\frac{dW_I(t)}{dt} - \delta^2 W_I(t) + a|W_I(t)|^{p-1}W_I(t) + \frac{2b}{h}|W_I(t)|^{q-1}W_I(t) \geq 0, \quad t \in (0,T),$$

$$W_i(0) \geq U_i(0), \quad 0 \leq i \leq I,$$

where $(0,T)$ is the maximal time interval on which $\|U_h(t)\|_\infty < \infty$. Setting $Z_h(t) = W_h(t) - U_h(t)$ and using the mean value theorem, we see that

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + ap|\theta_i(t)|^{p-1}Z_i(t) = 0, \quad 0 \leq i \leq I-1, t \in (0,T)$$

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$$\frac{dZ_I(t)}{dt} - \delta^2 Z_I(t) + \left(ap|\theta_I(t)|^{p-1} + \frac{2b}{h}|\theta_I(t)|^{q-1} \right) Z_I(t) \geq 0, \quad t \in (0,T),$$

$$Z_i(0) \geq 0, \quad 0 \leq i \leq I,$$

where θ_i is an intermediate value between $U_i(t)$ and $W_i(t)$. From Lemma 2.1, we have $0 \leq U_i(t) \leq W_i(t)$ for $t \in (0,T)$. If $T < \infty$, we have

$$\|U_h(T)\|_\infty \leq \frac{1}{\left(\|U_h(0)\|_\infty^{1-p} + a(p-1)T\right)^{\frac{1}{p-1}}} < \infty,$$

which leads to a contradiction. Hence $T = \infty$ and we have the desired result. $\square$

Remark 1. The estimate of Theorem 3.1 is a semidiscrete version of the result established in (1.4) for the continuous problem.

Let us give the statement of the main theorem of this section.

Theorem 3.2. *Let U_h be the solution of (2.1) – (2.2). Then we have*

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \|U_h(t)\|_\infty = C_0,$$

where $C_0 = \left(\frac{1}{a(p-1)}\right)^{\frac{1}{p-1}}$.

The proof of Theorem 3.2 is based on the following lemmas. We introduce the function

$$\mu(x) = -\lambda(C_0 + x) + (C_0 + x)^p,$$

where $C_0 = \left(\frac{1}{a(p-1)}\right)^{\frac{1}{p-1}}$.

Firstly, we establish an upper bound of the solution for the semidiscrete problem.

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Lemma 3.3. *Let U_h be the solution of (2.1) – (2.3). For any $\varepsilon > 0$, there exist positive times T and τ such that*

$$U_i(t+\tau) \le (C_0+\varepsilon)(t+T)^{-\lambda} + (t+T)^{-\lambda-1}, \quad 0 \le i \le I.$$

Proof. Define the vector W_h such that

$$W_i(t) = (C_0+\varepsilon)t^{-\lambda} + t^{-\lambda-1}.$$

A straightforward computation reveals that

$$\begin{aligned}\frac{dW_i}{dt} &- \delta^2 W_i + a|W_i|^{p-1}W_i \\ &= -\lambda(C_0+\varepsilon)t^{-\lambda-1} - (\lambda+1)t^{-\lambda-2} + a((C_0+\varepsilon)t^{-\lambda} + t^{-\lambda-1})^p \\ &= t^{-\lambda-1}(-\lambda(C_0+\varepsilon) - (\lambda+1)t^{-1} + a(C_0+\varepsilon+t^{-1})^p),\end{aligned}$$

because $\lambda p = \lambda + 1$. Using the mean value theorem, we get

$$(C_0+\varepsilon+t^{-1})^p = (C_0+\varepsilon)^p + \xi_i t^{-1},$$

where $\xi_i(t)$ is a bounded function. We deduce that

$$\frac{dW_i}{dt} - \delta^2 W_i + a|W_i|^{p-1}W_i = t^{-\lambda-1}(\mu(\varepsilon) - (\lambda+1)t^{-1} + \xi_i t^{-1}),$$

$$\begin{aligned}\frac{dW_I}{dt} &- \delta^2 W_I + a|W_I|^{p-1}W_I + \frac{2b}{h}|W_I|^{q-1}W_I \\ &= t^{-\lambda-1}\left(\mu(\varepsilon) - (\lambda+1)t^{-1} + \xi_i t^{-1} + \frac{2b}{h}t^{-q\lambda+\lambda+1}(C_0+\varepsilon+t^{-1})^q\right).\end{aligned}$$

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Obviously $-q\lambda+\lambda+1=\frac{p-q}{p-1}<0$. We also observe that $\mu(0)=0$ and $\mu'(0)=1$, which implies that $\mu(\varepsilon)>0$. Therefore there exists a positive time T such that

$$\frac{dW_i}{dt}-\delta^2 W_i+a|W_i|^{p-1}W_i>0,\quad 0\le i\le I-1, t\in[T,+\infty),$$

$$\frac{dW_I}{dt}-\delta^2 W_I+a|W_I|^{p-1}W_I+\frac{2b}{h}|W_I(t)|^{q-1}W_I(t)>0,\quad t\in[T,+\infty),$$

$$W_i(T)>\frac{T^{-\lambda}C_0}{2}.$$

Since from Theorem 3.1 $\lim_{t\to\infty}U_i(t)=0$, there exists $\tau>T$ such that $U_i(\tau)<\frac{T^{-\lambda}C_0}{2}<W_i(T)$. We introduce the vector $Z_h(t)$ such that $Z_i(t)=U_i(t+\tau-T)$, $0\le i\le I$. We obtain

$$\frac{dZ_i}{dt}-\delta^2 Z_i+a|Z_i|^{p-1}Z_i>0,\quad 0\le i\le I-1, t\ge T,$$

$$\frac{dZ_I}{dt}-\delta^2 Z_I+a|Z_I|^{p-1}Z_I+\frac{2b}{h}|Z_I(t)|^{q-1}Z_I(t)>0,\quad t\ge T,$$

$$Z_i(T)=U_i(\tau)<W_i(T).$$

We deduce from Lemma 2.2 that $Z_i(t)\le W_i(t)$, that is to say

$$U_i(t+\tau-T)\le W_i(t)\quad \text{for}\quad t\ge T, \tag{3.1}$$

which leads us to the result. □

The lemma below gives a lower bound of the solution for the semidiscrete problem.

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Lemma 3.4. *Let U_h be the solution of (2.1) – (2.3). For any $\varepsilon > 0$, there exists a positive time τ such that*

$$U_i(t+1) \geq (C_0 - \varepsilon)(t+\tau)^{-\lambda} + (t+\tau)^{-\lambda-1}, \quad 0 \leq i \leq I.$$

Proof. Introduce the vector V_h such that

$$V_i(t) = (C_0 - \varepsilon)t^{-\lambda} + t^{-\lambda-1}.$$

A direct calculation yields

$$\begin{aligned} \frac{dV_i}{dt} - \delta^2 V_i + a|V_i|^{p-1}V_i &= -\lambda(C_0-\varepsilon)t^{-\lambda-1} - (\lambda+1)t^{-\lambda-2} + a((C_0-\varepsilon)t^{-\lambda} + t^{-\lambda-1})^p \\ &= t^{-\lambda-1}(-\lambda(C_0-\varepsilon) - (\lambda+1)t^{-1} + a(C_0 - \varepsilon + t^{-1})^p) \end{aligned}$$

because $\lambda p = \lambda + 1$. From the mean value theorem, we have

$$(C_0 - \varepsilon + t^{-1})^p = (C_0-\varepsilon)^p + \chi_i(t)t^{-1},$$

where $\chi_i(t)$ is a bounded function. We deduce that

$$\frac{dV_i}{dt} - \delta^2 V_i + a|V_i|^{p-1}V_i = t^{-\lambda-1}(\mu(-\varepsilon) - (\lambda+1)t^{-1} + \chi_i t^{-1}),$$

$$\begin{aligned} &\frac{dV_I}{dt} - \delta^2 V_I + a|V_I|^{p-1}V_I + \frac{2b}{h}|V_I|^{q-1}V_I \\ &\quad = t^{-\lambda-1}\left(\mu(\varepsilon) - (\lambda+1)t^{-1} + \chi_i t^{-1} + \frac{2b}{h}t^{-q\lambda+\lambda+1}(C_0 - \varepsilon + t^{-1})^q\right). \end{aligned}$$

Obviously $-q\lambda + \lambda + 1 < 0$. Also, since $\mu(0) = 0$ and $\mu'(0) = 1$, it is easy to see that $\mu(-\varepsilon) < 0$. Hence there exists $T > 0$ such that

$$\frac{dV_i}{dt} - \delta^2 V_i + a|V_i|^{p-1}V_i < 0, \quad 0 \leq i \leq I-1, t \in [T, +\infty),$$

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$$\frac{dV_I}{dt} - \delta^2 V_I + a|V_I|^{p-1}V_I + \frac{2b}{h}|V_I|^{q-1}V_I < 0, \quad t \in [T, +\infty).$$

Since $V_i(t)$ goes to zero as $t \to +\infty$, there exists $\tau > \max(T, 1)$ such that $V_i(\tau) < U_i(1)$. Setting $X_i(t) = V_i(t + \tau - 1)$, we observe that

$$\frac{dX_i}{dt} - \delta^2 X_i + a|X_i|^{p-1}X_i < 0, \quad 0 \le i \le I-1, t \ge 1,$$

$$\frac{dX_I}{dt} - \delta^2 X_I + a|X_I|^{p-1}X_I + \frac{2b}{h}|X_I|^{q-1}X_I < 0, \quad t \ge 1,$$

$$X_i(1) = V_i(\tau) < U_i(1).$$

We deduce from Lemma 2.2 that

$$U_i(t) \ge V_i(t + \tau - 1) \quad \text{for} \quad t \ge 1, \tag{3.2}$$

which leads us to the result. □

Now, we are in a position to give the proof of the main result of this section.

Proof of Theorem 3.2. From Lemma 3.3 and Lemma 3.4, we deduce

$$(C_0 - \varepsilon) \le \liminf_{t\to\infty} \left(\frac{U_i(t)}{t^\lambda}\right) \le \limsup_{t\to\infty} \left(\frac{U_i(t)}{t^\lambda}\right) \le (C_0 + \varepsilon),$$

and we have the desired result. □

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4. Convergence

In this section, we will show that for each fixed time interval $[0, T]$, where u is defined, the solution $U_h(t)$ of (2.1) – (2.3) approximates u, when the mesh parameter h goes to zero.

Theorem 4.1. *Assume that (1.1) – (1.3) has a solution $u \in C^{4,1}([0,1] \times [0,T])$ and the initial condition at (2.3) satisfies*

$$\|U_h^0 - u_h(0)\|_\infty = o(1) \quad \textit{as} \quad h \to 0, \tag{4.1}$$

where $u_h(t) = (u(x_0,t), \dots, u(x_I,t))^T$. Then, for h sufficiently small, the problem (2.1) – (2.3) has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ such that

$$\max_{0 \le t \le T} \|U_h(t) - u_h(t)\|_\infty = O(\|U_h^0 - u_h(0)\|_\infty + h^2) \quad \textit{as} \quad h \to 0. \tag{4.2}$$

Proof. Let $K > 0$ and L be such that

$$\frac{2\|u_{xxx}\|_\infty}{3} \le \frac{K}{2}, \quad \frac{\|u_{xxxx}\|_\infty}{12} \le \frac{K}{2}, \quad \|u\|_\infty \le K, \quad ap(K+1)^{p-1} \le L,$$

$$2q(K+1)^{q-1} \le L. \tag{4.3}$$

The problem (2.1) – (2.3) has for each h, a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1 for t \in (0, t(h)). \tag{4.4}$$

The relation (4.1) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T\}$. By the triangular inequality, we obtain

$$\|U_h(t)\|_\infty \le \|u(x,t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad for\ t \in (0, t^*(h)),$$

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which implies that

$$\|U_h(t)\|_\infty \leq 1+K, \quad for\ t\in(0,t^*(h)). \tag{4.5}$$

Let $e_h(t)=U_h(t)-u_h(x,t)$ be the error of discretization. Using Taylor's expansion, we have for $t\in(0,t^*(h))$,

$$\frac{d}{dt}e_i(t)-\delta^2 e_i(t)=\frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t)-ap\xi_i^{p-1}e_i(t),$$

$$\frac{d}{dt}e_I(t)-\delta^2 e_I(t)=\frac{2}{h}q\theta_I^{q-1}e_I+\frac{2h^2}{3}u_{xxx}(\widetilde{x}_I,t)+\frac{h^2}{12}u_{xxxx}(\widetilde{x}_I,t)-ap\xi_I^{p-1}e_I(t),$$

where $\theta_I\in(U_I(t),u(x_I,t)$ and $\xi_i\in(U_i(t),u(x_i,t)$. Using (4.3) and (4.5), we arrive at

$$\frac{d}{dt}e_i(t)-\delta^2 e_i(t)\leq L|e_i(t)|+Kh^2, 0\leq i\leq I-1, \tag{4.6}$$

$$\frac{de_I(t)}{dt}-\frac{(2e_{I-1}(t)-2e_I(t))}{h^2}\leq\frac{L|e_I(t)|}{h}+L|e_I(t)|+Kh^2. \tag{4.7}$$

Consider the function

$$z(x,t)=e^{((M+1)t+Cx^2)}(\|U_h^0-u_h(0)\|_\infty+Qh^2)$$

where M, C, Q are constants which will be determined later. We get

$$z_t(x,t)-z_{xx}(x,t)=(M+1-2C-4C^2x^2)z(x,t),$$

$$z_x(0,t)=0, z_x(1,t)=2Cz(1,t),$$

$$z(x,0)=e^{Cx^2}(\left\|U_h^0-u_h(0)\right\|_\infty+Qh).$$

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By a semidiscretization of the above problem, we may choose M, C, Q large enough that

$$\frac{d}{dt}z(x_i,t) > \delta^2 z(x_i,t) + L|z(x_i,t)| + Kh^2, 0 \leq i \leq I-1, \tag{4.8}$$

$$\frac{d}{dt}z(x_I,t) > \delta^2 z(x_I,t) + \frac{L}{h}|z(x_I,t)| + L|z(x_I,t)| + Kh^2, \tag{4.9}$$

$$z(x_i,0) > e_i(0), 0 \leq i \leq I. \tag{4.10}$$

It follows from Lemma 3.4 that

$$z(x_i,t) > e_i(t) \quad for\ t \in (0, t^*(h)), \quad 0 \leq i \leq I.$$

By the same way, we also prove that

$$z(x_i,t) > -e_i(t) \quad for\ t \in (0, t^*(h)), \quad 0 \leq i \leq I,$$

which implies that

$$\|U_h(t) - u_h(t)\|_\infty \leq e^{(Mt+C)}(\left\|U_h^0 - u_h(0)\right\|_\infty + Qh^2), t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (4.4), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq e^{(MT+C)}(\left\|U_h^0 - u_h(0)\right\|_\infty + Qh^2). \tag{4.11}$$

Since the term in the right hand side of the inequality goes to zero as h goes to zero, we deduce from (4.11) that $1 \leq 0$, which is impossible. Consequently $t^*(h) = T$, and we obtain the desired result. □

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5. Full Discretizations

In this section, we study the asymptotic behavior, using full discrete schemes (explicit and implicit) of (1.1) – (1.3). Firstly, we approximate the solution $u(x,t)$ of (1.1) – (1.3) by the solution $U_h^{(n)} = (U_0^n, U_1^n, \ldots, U_I^n)^T$ of the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t} = \delta^2 U_i^{(n)} - a\left|U_i^{(n)}\right|^{p-1} U_i^{(n+1)}, \quad 0 \leq i \leq I-1, \tag{5.1}$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t} = \delta^2 U_I^{(n)} - a\left|U_I^{(n)}\right|^{p-1} U_I^{(n+1)} - \frac{2b}{h}\left|U_I^{(n)}\right|^{q-1} U_I^{(n+1)}, \tag{5.2}$$

$$U_i^{(0)} = \phi_i > 0, \quad 0 \leq i \leq I, \tag{5.3}$$

where $n \geq 0$, $\Delta t \leq \frac{h^2}{2}$. We need the following lemma which is a discrete form of the maximum principle for ordinary differential equations.

Lemma 5.1. *Let $f \in C^1(\mathbb{R})$ and let a_n and b_n be two bounded sequences such that*

$$\frac{a_{n+1} - a_n}{\Delta t} + f(a_n) \geq \frac{b_{n+1} - b_n}{\Delta t} + f(b_n), \quad n \geq 0, \tag{5.4}$$

$$a_0 \geq b_0. \tag{5.5}$$

Then we have $a_n \geq b_n$, $n \geq 0$ for h small enough.

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Proof. Let $Z_n = a_n - b_n$. We get

$$\frac{Z_{n+1} - Z_n}{\Delta t} + f'(\xi_n) Z_n \geq 0, \tag{5.6}$$

where ξ_n is an intermediate value between a_n and b_n. Obviously

$$Z_{n+1} \geq Z_n(1 - \Delta t f'(\xi_n)). \tag{5.7}$$

Since a_n and b_n are bounded and $f \in C^1(\mathbb{R})$, there exists a positive M such that $|f'(\xi_n)| \leq M$. Let j be the first integer such that $Z_j < 0$. From (5.5), $j \geq 0$. We have $Z_j \geq Z_{j-1}(1 - \Delta t M)$. Since $\Delta t M$ goes to zero as $h \to 0$ and $Z_{j-1} \geq 0$, we deduce that $Z_j \geq 0$ as $h \to 0$ which is a contradiction. Therefore, $Z_n \geq 0$ for any n and we have proved the lemma. $\square$

Now, we may state the following.

Theorem 5.2. *Let U_h be the solution of (5.1) – (5.3). We have $U_h^{(n)} \geq 0$ and*

$$\left\| U_h^{(n)} \right\|_\infty \leq \frac{1}{\left(\left\| U_h^{(0)} \right\|_\infty^{1-p} + A(p-1)n\Delta t \right)^{\frac{1}{p-1}}},$$

where $A = \frac{a}{1 + a\Delta t \left\| U_h^{(0)} \right\|_\infty^{p-1}}$.

Proof. A straightforward calculation yields

$$U_i^{(n+1)} = \frac{\frac{\Delta t}{h^2} U_{i+1}^{(n)} + \left(1 - \frac{2\Delta t}{h^2}\right) U_i^{(n)} + U_{i-1}^{(n)}}{1 + a\Delta t \left| U_i^{(n)} \right|^{p-1}}, \quad 1 \leq i \leq I - 1, \tag{5.8}$$

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(5.9)
$$U_0^{(n+1)} = \frac{\frac{2\Delta t}{h^2} U_1^{(n)} + \left(1 - \frac{2\Delta t}{h^2}\right) U_0^{(n)}}{1 + a\Delta t \left|U_0^{(n)}\right|^{p-1}},$$

(5.10)
$$U_I^{(n+1)} = \frac{\frac{2\Delta t}{h^2} U_{I-1}^{(n)} + \left(1 - \frac{2\Delta t}{h^2}\right) U_I^{(n)}}{1 + a\Delta t \left|U_I^{(n)}\right|^{p-1} + 2\frac{b}{h}\Delta t \left|U_I^{(n)}\right|^{q-1}}.$$

Since $1 - 2\frac{\Delta t}{h^2}$ is nonnegative, using a recursive argument, it is easy to see that $U_h^{(n)} \geq 0$. Let i_0 be such that $U_{i_0}^{(n)} = \left\|U_h^{(n)}\right\|_\infty$. From (5.8), we get

$$\left\|U_h^{(n+1)}\right\|_\infty \leq \frac{\frac{\Delta t}{h^2} U_{i_0+1}^{(n)} + \left(1 - \frac{2\Delta t}{h^2}\right) \left\|U_h^{(n)}\right\|_\infty + U_{i_0-1}^{(n)}}{1 + a\Delta t \left\|U_h^{(n)}\right\|_\infty^{p-1}} \quad \text{if} \quad 1 \leq i_0 \leq I-1.$$

Applying the triangle inequality and the fact that $1 - \frac{2\Delta t}{h^2}$ is nonnegative, we arrive at

(5.11)
$$\left\|U_h^{(n+1)}\right\|_\infty \leq \frac{\left\|U_h^{(n)}\right\|_\infty}{1 + a\Delta t \left\|U_h^{(n)}\right\|_\infty^{p-1}}.$$

We obtain the same estimation if $i_0 = 0$ or $i_0 = I$. The inequality (5.11) implies that $\left\|U_h^{(n+1)}\right\|_\infty \leq \left\|U_h^{(n)}\right\|_\infty$ and by iterating, we obtain $\left\|U_h^{(n)}\right\|_\infty \leq \left\|U_h^{(0)}\right\|_\infty$. From

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(5.11), we also observe that

$$\frac{\left\|U_h^{(n+1)}\right\|_\infty - \|U^{(n)}\|_\infty}{\Delta t} \le -\frac{a\left\|U_h^{(n)}\right\|_\infty^p}{1 + a\Delta t\left\|U_h^{(n)}\right\|_\infty^{p-1}}.$$

Using the fact that $\left\|U_h^{(n)}\right\|_\infty \le \left\|U_h^{(0)}\right\|_\infty$, we have

$$\frac{\left\|U_h^{(n+1)}\right\|_\infty - \left\|U_h^{(n)}\right\|_\infty}{\Delta t} \le -A\left\|U_h^{(n)}\right\|_\infty^p.$$

We introduce the function $\alpha(t)$ which is defined as follows

$$\alpha(t) = \frac{1}{\left(\left\|U_h^{(0)}\right\|_\infty^{1-p} + A(p-1)t\right)^{\frac{1}{p-1}}}.$$

We remark that $\alpha(t)$ obeys the following differential equation

$$\alpha'(t) = -A\alpha^p(t), \quad \alpha(0) = \left\|U_h^{(0)}\right\|_\infty.$$

Using a Taylor's expansion, we have

$$\alpha(t_{n+1}) = \alpha(t_n) + \Delta t\alpha'(t_n) + \frac{(\Delta t)^2}{2}\alpha''(\tilde{t_n}),$$

where $\tilde{t_n}$ is an intermediate value between t_n and t_{n+1}. It is not hard to see that $\alpha(t)$ is a convex function. Therefore, we obtain

$$\frac{\alpha(t_{n+1}) - \alpha(t_n)}{\Delta t} \ge -A\alpha^p(t_n).$$

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From Lemma 5.1, we get $\left\|U_h^{(n)}\right\|_\infty \le \alpha(t_n)$, which ensures that

$$\left\|U_h^{(n)}\right\|_\infty \le \frac{1}{\left(\left\|U_h^{(0)}\right\|_\infty^{1-p} + A(p-1)n\Delta t\right)^{\frac{1}{p-1}}},$$

and we have the desired result. $\square$

Remark 2. The estimate of Theorem 5.2 is the discrete form of the one given in (1.4) for the continuous problem.

Now, we approximate the solution $u(x,t)$ of problem (1.1) – (1.3) by the solution $U_h^{(n)}$ of the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t} = \delta^2 U_i^{(n+1)} - \left|U_i^{(n)}\right|^{p-1} U_i^{(n+1)}, \quad 0 \le i \le I-1, \tag{5.12}$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t} = \delta^2 U_I^{(n+1)} - a\left|U_I^{(n)}\right|^{p-1} U_I^{(n+1)} - \frac{2b}{h}\left|U_I^{(n)}\right|^{p-1} U_I^{(n+1)}, \tag{5.13}$$

$$U_i^{(0)} = \phi_i > 0, \quad 0 \le i \le I, \tag{5.14}$$

where $n \ge 0$. Let us note that in the above construction, we do not need a restriction on the step time.

The above equations may be rewritten in the following form:

$$U_0^{(n)} = -\frac{2\Delta t}{h^2} U_1^{(n+1)} + \left(1 + 2\frac{\Delta t}{h^2} + a\Delta t\left|U_0^{(n)}\right|^{p-1}\right) U_0^{(n+1)},$$

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$$U_i^{(n)} = -\frac{\Delta t}{h^2}U_{i-1}^{(n+1)} + \left(1 + 2\frac{\Delta t}{h^2} + a\Delta t\left|U_i^{(n)}\right|^{p-1}\right)U_i^{(n+1)} - \frac{\Delta t}{h^2}U_{i+1}^{(n+1)}, \quad 1 \le i \le I-1,$$

$$U_I^{(n)} = -\frac{2\Delta t}{h^2}U_{I-1}^{(n+1)} + \left(1 + 2\frac{\Delta t}{h^2} + a\Delta t\left|U_I^{(n)}\right|^{p-1} + \frac{2b}{h}\Delta t\left|U_I^{(n)}\right|^{q-1}\right)U_I^{(n+1)},$$

which gives the following linear system

$$A^{(n)}U_h^{(n+1)} = U_h^{(n)}$$

where $A^{(n)}$ is the tridiagonal matrix defined as follows

$$A^{(n)} = \begin{pmatrix} d_0 & \frac{-2\Delta t}{h^2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-\Delta t}{h^2} & d_1 & \frac{-\Delta t}{h^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{-\Delta t}{h^2} & d_2 & \frac{-\Delta t}{h^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-\Delta t}{h^2} & d_{I-2} & \frac{-\Delta t}{h^2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{-\Delta t}{h^2} & d_{I-1} & \frac{-\Delta t}{h^2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-2\Delta t}{h^2} & d_I \end{pmatrix},$$

with

$$d_i = 1 + 2\frac{\Delta t}{h^2} + a\Delta t|U_i^{(n)}|^{p-1} \quad \text{for} \quad 0 \le i \le I-1$$

and

$$d_I = 1 + 2\frac{\Delta t}{h^2} + a\Delta t\left|U_I^{(n)}\right|^{p-1} + \frac{2b}{h}\Delta t\left|U_I^{(n)}\right|^{q-1}.$$

Let us remark that the tridiagonal matrix $A^{(n)}$ satisfies the following properties

$$A_{ii}^{(n)} > 0 \quad \text{and} \quad A_{ij}^{(n)} < 0 \quad i \ne j,$$

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$$\left|A_{ii}^{(n)}\right| > \sum_{i \neq j} \left|A_{ij}^{(n)}\right|.$$

These properties imply that U_h^n exists for any n and $U_h^{(n)} \geq 0$ (see for instance [2]). As we know that the solution of the discrete implicit scheme exists, we may state the following.

Theorem 5.3. *Let $U_h^{(n)}$ be the solution of (5.12) – (5.14). We have $U_h^{(n)} \geq 0$ and*

$$\left\|U_h^{(n)}\right\|_\infty \leq \frac{1}{\left(\left\|U_h^{(0)}\right\|_\infty^{1-p} + A(p-1)n\Delta t\right)^{\frac{1}{p-1}}},$$

where $A = \frac{a}{1+a\Delta t\left\|U_h^{(0)}\right\|_\infty^{p-1}}$.

Proof. We know that $U_h^{(n)} \geq 0$ as we have seen above. Now, let us obtain the above estimate to complete the proof. Let i_0 be such that $U_{i_0}^{(n)} = \left\|U_h^{(n)}\right\|_\infty$. Using the equality (5.12), we have

$$\left(1 + 2\frac{\Delta t}{h^2} + a\Delta t\left\|U_h^{(n)}\right\|_\infty\right)\left\|U_h^{(n+1)}\right\|_\infty \leq \left\|U_h^{(n)}\right\|_\infty + \frac{\Delta t}{h^2}U_{i_0-1}^{(n)} + \frac{\Delta t}{h^2}U_{i_0+1}^{(n)}$$
$$\text{if} \quad 1 \leq i_0 \leq I-1.$$

Applying the triangle inequality, we derive the following estimate

$$\left\|U_h^{(n+1)}\right\|_\infty \leq \frac{\left\|U_h^{(n)}\right\|_\infty}{1 + a\Delta t\left\|U_h^{(n)}\right\|_\infty^{p-1}}.$$

We obtain the same estimation if we take $i_0 = 0$ or $i_0 = I$. Reasoning as in the proof of Theorem 5.3, we obtain the desired result. □

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6. Numerical Results

In this section, we consider the explicit scheme in (5.1) – (5.3) and the implicit scheme in (5.12) – (5.14). We suppose that $p = 2$, $q = 3$, $a = 1$, $b = 1$, $U_i^0 = 0.8 + 0.8 * \cos(\pi hi)$ and $\Delta t = \frac{h^2}{2}$. In the following tables, in the rows, we give the first n when

$$\left\| n\Delta t U_h^{(n)} - 1 \right\|_\infty < \varepsilon,$$

the corresponding time $T^n = n\Delta t$, the CPU time and the order(s) of method computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Table 1: **($\varepsilon = 10^{-2}$)** Numerical times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	674.0820	345129	103	-
32	674.2632	1.380890.	660	-
64	674.3085	5.523.934	6020	2.01
128	674.3278	22095735	58290	1.24
256	674.4807	87383041	574823	2.99

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Table 2: ($\varepsilon = 10^{-2}$) Numerical times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	674.3281	345.255	90	-
32	674.3452	1.381.058	720	-
64	674.3290	5.524.102	10820	0.08
128	674.3187	22845950	323528	0.65
256	674.3098	88237375	19457811	0.21

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