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INTEGRAL MEAN ESTIMATES FOR POLYNOMIALS WHOSE ZEROS ARE WITHIN A CIRCLE

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ABSTRACT. Let $p(z)$ be a polynomial of degree n. Zygmund [11] has shown that for $s \geq 1$

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^s d\theta\right)^{1/s} \leq n\left(\int_0^{2\pi} |p(e^{i\theta})|^s d\theta\right)^{1/s}.$$

In this paper, we have obtained inequalities in the reverse direction for the polynomials having a zero of order m at the origin. We also consider a problem for the class of polynomials $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$ not vanishing outside the disk $|z| < k$, $k \leq 1$ and obtain a result which, besides yielding some interesting results as corollaries, includes a result due to Aziz and Shah [*Indian J. Pure Appl. Math.*, 28 (1997), 1413–1419] as a special case.

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1. Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree n and $p'(z)$ its derivative. It was shown by Turán [10] that if $p(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.1}$$

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More generally, if the polynomial $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, it was proved by Malik [5] that the inequality (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)| . \tag{1.2}$$

Malik [6] obtained a L^p analogue of (1.1) by proving that if $p(z)$ has all its zeros in $|z| \leq 1$, then for each $r > 0$

$$n \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |p'(z)|. \tag{1.3}$$

As an extension of (1.3) and a generalization of (1.2), Aziz [1] proved that if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for each $r > 0$

$$n \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \max_{|z|=1} |p'(z)|. \tag{1.4}$$

If we let $r \to \infty$ in (1.3) and (1.4) and make use of the well known fact from analysis (see for example [8, p. 73] or [9, p. 91]) that

$$\left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \to \max_{0 \leq \theta < 2\pi} \left| p(e^{i\theta}) \right| \qquad \text{as } r \to \infty , \tag{1.5}$$

we get inequalities (1.1) and (1.2) respectively.

In this paper, we will first obtain a Zygmund [11] type integral inequality, but in the reverse direction, for polynomials having a zero of order m at the origin. More precisely, we prove

Theorem 1.1. *Let $p(z) = z^m \sum_{j=0}^{n-m} a_j z^j$ be a polynomial of degree n, having all its zeros in $|z| \leq k$, $k \leq 1$, with a zero of order m at $z = 0$. Then for β with $|\beta| < k^{n-m}$ and $s \geq 1$*

$$\left(\int_0^{2\pi} \left| p'(e^{i\theta}) + \frac{mm'}{k^n} \bar{\beta} e^{i(m-1)\theta} \right|^s d\theta \right)^{\frac{1}{s}} \geq \left\{ n - (n-m) C_s^{(k)} \right\} \left(\int_0^{2\pi} \left| p(e^{i\theta}) + \frac{m'}{k^n} \bar{\beta} e^{im\theta} \right|^s d\theta \right)^{\frac{1}{s}} , \tag{1.6}$$

where $m' = \min_{|z|=k} |p(z)|$,

$$C_s^{(k)} = \left(\frac{1}{2\pi} \int_0^{2\pi} |S_c + e^{i\theta}|^s d\theta \right)^{-\frac{1}{s}} \quad \text{and} \quad S_c = \frac{\left(\frac{1}{n-m}\right) \left| \frac{a_{n-m-1}}{a_{n-m}} \right| + 1}{k^2 + \left(\frac{1}{n-m}\right) \left| \frac{a_{n-m-1}}{a_{n-m}} \right|} .$$

By taking $k = 1$ and $\beta = 0$ in Theorem 1.1, we obtain:

Corollary 1.2. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, with a zero of order m at $z = 0$, then for $s \geq 1$*

$$\left(\int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right)^{\frac{1}{s}} \geq \left\{ n - (n-m) C_s^{(1)} \right\} \left(\int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right)^{\frac{1}{s}} , \tag{1.7}$$

where

$$C_s^{(1)} = \frac{1}{\left(\frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\theta}|^s d\theta \right)^{\frac{1}{s}}} .$$

By letting $s \to \infty$ in Theorem 1.1, we obtain

Corollary 1.3. *Let $p(z) = z^m \sum_{j=0}^{n-m} a_j z^j$ be a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$, with a zero of order m at $z = 0$. Then for β with $|\beta| < k^{n-m}$*

$$\max_{|z|=1} \left| p'(z) + \frac{mm'}{k^n} \bar{\beta} z^{m-1} \right| \ge \left(\frac{m + nS_c}{1 + S_c} \right) \max_{|z|=1} \left| p(z) + \frac{m'}{k^n} \bar{\beta} z^m \right|, \tag{1.8}$$

where m' and S_c are as defined in Theorem 1.1.

By choosing the argument of β suitably and letting $|\beta| \to k^{n-m}$ in Corollary 1.3, we obtain the following result.

Corollary 1.4. *Let $p(z) = z^m \sum_{j=0}^{n-m} a_j z^j$ be a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$, with a zero of order m at $z = 0$. Then*

$$\max_{|z|=1} |p'(z)| \ge \left(\frac{m + nS_c}{1 + S_c} \right) \max_{|z|=1} |p(z)| + \frac{(n-m)S_c}{1 + S_c} \frac{m'}{k^m}, \tag{1.9}$$

where m' and S_c are as defined in Theorem 1.1.

Let $\mathcal{D}_\alpha p(z)$ denote the polar derivative of the polynomial $p(z)$ of degree n with respect to the point α. Then

$$\mathcal{D}_\alpha p(z) = np(z) + (\alpha - z)p'(z) \, .$$

The polynomial $\mathcal{D}_\alpha p(z)$ is of degree at most $(n-1)$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{\mathcal{D}_\alpha p(z)}{\alpha} = p'(z) \, . \tag{1.10}$$

Our next result generalizes as well as improving upon the inequality (1.4), which in turns, gives a generalization as well as improvements of inequalities (1.3), (1.2) and (1.1) in terms of the polar derivatives of L^p inequalities.

Theorem 1.5. *If $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, is a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$, then for every real or complex numbers α and β with $|\alpha| \ge k^\mu$ and $|\beta| \le 1$ and for each $r > 0$*

$$\max_{|z|=1} |\mathcal{D}_\alpha p(z)| \ge \frac{n(|\alpha| - k^\mu)}{\left(\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta \right)^{\frac{1}{r}}} \left(\int_0^{2\pi} \left| p(e^{i\theta}) + \frac{\beta m'}{k^{n-\mu}} e^{i(n-1)\theta} \right|^r d\theta \right)^{\frac{1}{r}} + \frac{n}{k^{n-\mu}} m', \tag{1.11}$$

where $m' = \min_{|z|=k} |p(z)|$.

Dividing both sides of (1.11) by $|\alpha|$, letting $|\alpha| \to \infty$ and noting that (1.10), we obtain

Corollary 1.6. *If $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, is a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$, then for every real or complex number β with $|\beta| \le 1$, for each $r > 0$*

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{\left(\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta \right)^{\frac{1}{r}}} \left(\int_0^{2\pi} \left| p(e^{i\theta}) + \frac{\beta m'}{k^{n-\mu}} e^{i(n-1)\theta} \right|^r d\theta \right)^{\frac{1}{r}}, \tag{1.12}$$

where $m' = \min_{|z|=k} |p(z)|$.

Remark 1. Letting $r \to \infty$ in (1.12) and choosing the argument of β suitably with $|\beta| = 1$, it follows that, if $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, is a polynomial of degree n, having all its zeros in $|z| \le k$, $k \le 1$, then

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{(1+k^\mu)} \left[\max_{|z|=1} |p(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=k} |p(z)| \right]. \tag{1.13}$$

Inequality (1.13) was already proved by Aziz and Shah [2].

2. Lemmas

For the proofs of these theorems we need the following lemmas.

Lemma 2.1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n having no zeros in $|z| < k$, $k \ge 1$. Then for $s \ge 1$*

$$\left\{ \int_0^{2\pi} |p'(e^{i\theta})|^s d\theta \right\}^{\frac{1}{s}} \le n S_s \left\{ \int_0^{2\pi} |p(e^{i\theta})|^s d\theta \right\}^{\frac{1}{s}}, \tag{2.1}$$

where

$$S_s = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |S'_c + e^{i\theta}|^s d\theta \right\}^{\frac{1}{s}} \quad \textit{and} \quad S'_c = \frac{k^2 \left[\frac{1}{n} \left| \frac{a_1}{a_0} \right| + 1 \right]}{1 + \frac{1}{n} \left| \frac{a_1}{a_0} \right| k^2}.$$

The above lemma is due to Dewan, Bhat and Pukhta [3].
The following lemma is due to Rather [7].

Lemma 2.2. *Let $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zero in $|z| \le k$, $k \le 1$. Then*

$$k^\mu |p'(z)| \ge |q'(z)| + \frac{n}{k^{n-\mu}} \min_{|z|=k} |p(z)| \quad \textit{for} \ \ |z| = 1, \tag{2.2}$$

where $q(z) = z^n \overline{p\left(\frac{1}{\bar z}\right)}$.

3. Proofs of The Theorems

Proof of Theorem 1.1. Let

$$p(z) = z^m \sum_{j=0}^{n-m} a_j z^j = z^m \phi(z), \quad \text{(say)}$$

where $\phi(z)$ is a polynomial of degree $n-m$, with the property that

$$\phi(0) \ne 0\,.$$

Then

$$q(z) = z^n \overline{p\left(\frac{1}{\bar z}\right)} = z^{n-m} \overline{\phi\left(\frac{1}{\bar z}\right)}$$

is also a polynomial of degree $n-m$ and has no zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \ge 1$. Now if

$$m_0 = \min_{|z|=\frac{1}{k}} |q(z)| = \min_{|z|=\frac{1}{k}} \left| z^n \overline{p\left(\frac{1}{\bar z}\right)} \right| = \frac{1}{k^n} \min_{|z|=k} |p(z)| = \frac{m'}{k^n},$$

then, by Rouche's theorem, the polynomial

$$q(z) + m_0 \beta z^{n-m}, \quad |\beta| < k^{n-m},$$

of degree $n-m$, will also have no zeros in $|z|<\frac{1}{k}$, $\frac{1}{k}\geq 1$. Hence, by Lemma 2.1, we have for $s\geq 1$ and $|\beta|<k^{n-m}$

$$\left(\int_0^{2\pi}\left|q'(e^{i\theta})+\frac{m'}{k^n}\beta e^{i(n-m-1)\theta}(n-m)\right|^s d\theta\right)^{\frac{1}{s}} \leq (n-m)C_s^{(k)}\left(\int_0^{2\pi}\left|q(e^{i\theta})+\frac{m'}{k^n}\beta e^{i(n-m)\theta}\right|^s d\theta\right)^{\frac{1}{s}},$$

which implies

$$\left(\int_0^{2\pi}\left|np(e^{i\theta})-e^{i\theta}p'(e^{i\theta})+\bar{\beta}\frac{m'}{k^n}(n-m)e^{im\theta}\right|^s d\theta\right)^{\frac{1}{s}} \leq (n-m)C_s^{(k)}\left(\int_0^{2\pi}\left|p(e^{i\theta})+\frac{m'}{k^n}\bar{\beta}e^{im\theta}\right|^s d\theta\right)^{\frac{1}{s}}. \tag{3.1}$$

Now by Minkowski's inequality, we have for $s\geq 1$ and $|\beta|<k^{n-m}$

$$n\left(\int_0^{2\pi}\left|p(e^{i\theta})+\frac{m'}{k^n}\bar{\beta}e^{im\theta}\right|^s d\theta\right)^{\frac{1}{s}} \leq \left(\int_0^{2\pi}\left|np(e^{i\theta})+\frac{m'}{k^n}\bar{\beta}(n-m)e^{im\theta}-e^{i\theta}p'(e^{i\theta})\right|^s d\theta\right)^{\frac{1}{s}} + \left(\int_0^{2\pi}\left|e^{i\theta}p'(e^{i\theta})+\frac{mm'}{k^n}\bar{\beta}e^{im\theta}\right|^s d\theta\right)^{\frac{1}{s}},$$

which implies, by using inequality (3.1)

$$n\left(\int_0^{2\pi}\left|p(e^{i\theta})+\frac{m'}{k^n}\bar{\beta}e^{im\theta}\right|^s d\theta\right)^{\frac{1}{s}} \leq (n-m)C_s^{(k)}\left(\int_0^{2\pi}\left|p(e^{i\theta})+\frac{m'}{k^n}\bar{\beta}e^{im\theta}\right|^s d\theta\right)^{\frac{1}{s}} + \left(\int_0^{2\pi}\left|p'(e^{i\theta})+m\frac{m'}{k^n}\bar{\beta}e^{i(m-1)\theta}\right|^s d\theta\right)^{\frac{1}{s}},$$

and the Theorem 1.1 follows. □

Proof of Theorem 1.5. Since $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$ so that $p(z)=z^n\overline{q\left(\frac{1}{\bar{z}}\right)}$, therefore, we have

$$p'(z)=nz^{n-1}\overline{q\left(\frac{1}{\bar{z}}\right)}-z^{n-2}\overline{q'\left(\frac{1}{\bar{z}}\right)}, \tag{3.2}$$

which implies

$$|p'(z)|=|nq(z)-zq'(z)| \quad \text{for } |z|=1\,. \tag{3.3}$$

Using (3.2) in (2.2), we get for $1\leq\mu\leq n$

$$|q'(z)|+\frac{m'n}{k^{n-\mu}}\leq k^{\mu}|nq(z)-zq'(z)| \quad \text{for } |z|=1\,.$$

Now, from the above inequality, for every complex β with $|\beta| \leq 1$, we get, for $|z| = 1$

$$\begin{aligned}\left|q'(z) + \bar{\beta}\frac{m'n}{k^{n-\mu}}\right| &\leq |q'(z)| + \frac{m'n}{k^{n-\mu}} \\ &\leq k^{\mu}|nq(z) - zq'(z)|\,. \end{aligned} \tag{3.4}$$

For every real or complex number α with $|\alpha| \geq k^{\mu}$, we have

$$\begin{aligned}|\mathcal{D}_{\alpha}p(z)| &= |np(z) + (\alpha - z)p'(z)| \\ &\geq |\alpha|\,|p'(z)| - |np(z) - zp'(z)|,\end{aligned}$$

which gives by interchanging the roles of $p(z)$ and $q(z)$ in (3.3) for $|z| = 1$

$$\begin{aligned}|\mathcal{D}_{\alpha}p(z)| &\geq |\alpha||p'(z)| - |q'(z)| \\ &\geq |\alpha||p'(z)| - k^{\mu}|p'(z)| + \frac{m'n}{k^{n-\mu}} \quad \text{(using (2.2)).}\end{aligned} \tag{3.5}$$

Since $p(z)$ has all its zeros in $|z| \leq k \leq 1$, by the Gauss-Lucas theorem, all the zeros of $p'(z)$ also lie in $|z| \leq 1$. This implies that the polynomial

$$z^{n-1}\overline{p'\left(\frac{1}{\bar{z}}\right)} = nq(z) - zq'(z)$$

has all its zeros in $|z| \geq \frac{1}{k} \geq 1$. Therefore, it follows from (3.4) that the function

$$w(z) = \frac{zq'(z) + \bar{\beta}\frac{m'n}{k^{n-\mu}}z}{k^{\mu}(nq(z) - zq'(z))} \tag{3.6}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| \leq 1$. Furthermore $w(0) = 0$. Thus the function $1 + k^{\mu}w(z)$ is a subordinate to the function $1 + k^{\mu}z$ in $|z| \leq 1$. Hence by a well-known property of subordination [4], we have for $r > 0$ and for $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi} |1 + k^{\mu}w(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + k^{\mu}e^{i\theta}|^r d\theta\,. \tag{3.7}$$

Also from (3.6), we have

$$1 + k^{\mu}w(z) = \frac{nq(z) + \bar{\beta}\frac{m'n}{k^{n-\mu}}z}{nq(z) - zq'(z)},$$

or

$$\left|nq(z) + \bar{\beta}\frac{m'n}{k^{n-\mu}}z\right| = |1 + k^{\mu}w(z)||p'(z)| \quad \text{for } |z| = 1,$$

which implies

$$n\left|p(z) + \beta\frac{m'}{k^{n-\mu}}z^{n-1}\right| = |1 + k^{\mu}w(z)||p'(z)| \quad \text{for } |z| = 1. \tag{3.8}$$

Now combining (3.7) and (3.8), we get

$$n^r \int_0^{2\pi} \left|p(e^{i\theta}) + \beta\frac{m'}{k^{n-\mu}}e^{i(n-1)\theta}\right|^r d\theta \leq \int_0^{2\pi} |1 + k^{\mu}e^{i\theta}|^r |p'(e^{i\theta})|^r d\theta\,.$$

Using (3.5) in the above inequality, we obtain

$$n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} \left|p(e^{i\theta}) + \beta \frac{m'}{k^{n-\mu}} e^{i(n-1)\theta}\right|^r d\theta$$
$$\leq \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta \left\{\max_{|z|=1} |\mathcal{D}_\alpha p(z)| - \frac{nm'}{k^{n-\mu}}\right\}^r,$$

from which we obtain the required result. □

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