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ON THE q-ANALOGUE OF GAMMA FUNCTIONS AND RELATED INEQUALITIES

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Abstract

In this paper, we obtain a q-analogue of a double inequality involving the Euler gamma function which was first proved geometrically by Alsina and Tomás [1] and then analytically by Sándor [6].

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Dedicated to H. M. Srivastava on his 65th birthday.

Contents

1. Introduction

F. H. Jackson defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1, \text{ cf. [2, 4, 5, 7]},$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}}, \quad q > 1,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1^-$, where $\Gamma(x)$ is the ordinary Euler gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0.$$

Recently Alsina and Tomás [1] have proved the following double inequality on employing a geometrical method:

Theorem 1.1. *For all $x \in [0,1]$, and for all nonnegative integers n, one has*

$$\frac{1}{n!} \le \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \le 1. \tag{1.1}$$

Sándor [6] has obtained a generalization of (1.1) by using certain simple analytical arguments. In fact, he proved that for all real numbers $a \geq 1$, and all $x \in [0,1]$,

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1. \tag{1.2}$$

But to prove (1.2), Sándor used the following result:

Theorem 1.2. *For all $x > 0$,*

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + (x-1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)}. \tag{1.3}$$

In an e-mail message, Professor Sándor has informed the authors that, relation (1.2) follows also from the log-convexity of the Gamma function (i.e. in fact, the monotonous increasing property of the ψ -function). However, (1.3) implies many other facts in the theory of gamma functions. For example, the function $\psi(x)$ is strictly increasing for $x > 0$, having as a consequence that, inequality (1.2) holds true with strict inequality (in both sides) for $a > 1$. The main purpose of this paper is to obtain a q-analogue of (1.2). Our proof is simple and straightforward.

2. Main Result

In this section, we prove our main result.

Theorem 2.1. *If* $0 < q < 1, a \geq 1$ *and* $x \in [0, 1]$, *then*

$$\frac{1}{\Gamma_q(1+a)} \leq \frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)} \leq 1.$$

Proof. We have

$$\Gamma_q(1+x) = \frac{(q;q)_\infty}{(q^{1+x};q)_\infty}(1-q)^{-x} \tag{2.1}$$

and

$$\Gamma_q(1+ax) = \frac{(q;q)_\infty}{(q^{1+ax};q)_\infty}(1-q)^{-ax}. \tag{2.2}$$

Taking the logarithmic derivatives of (2.1) and (2.2), we obtain

$$\frac{d}{dx}(\log \Gamma_q(1+x)) = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{1+x+n}}{1-q^{1+x+n}}, \text{ cf. [3, 4, 5]}, \tag{2.3}$$

and

$$\frac{d}{dx}(\log \Gamma_q(1+ax)) = -a\log(1-q) + a\log q \sum_{n=0}^{\infty} \frac{q^{1+ax+n}}{1-q^{1+ax+n}}. \tag{2.4}$$

Since $x \geq 0, a \geq 1, \;\; \log q < 0$ and

$$\frac{q^{1+ax+n}}{1-q^{1+ax+n}} - \frac{q^{1+x+n}}{1-q^{1+x+n}} = \frac{q^{1+ax+n} - q^{1+x+n}}{(1-q^{1+ax+n})(1-q^{1+x+n})} \leq 0,$$

we have

$$\frac{d}{dx}\left(\log \Gamma_q(1+ax)\right) \geq a\frac{d}{dx}\left(\log \Gamma_q(1+x)\right). \tag{2.5}$$

Let

$$g(x) = \log \frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)}, \quad a \geq 1, \; x \geq 0.$$

Then

$$g(x) = a \log \Gamma_q(1+x) - \log \Gamma_q(1+ax)$$

and

$$g'(x) = a\frac{d}{dx}\left(\log \Gamma_q(1+x)\right) - \frac{d}{dx}\left(\log \Gamma_q(1+ax)\right).$$

By (2.5), we get $g'(x) \leq 0$, so g is decreasing. Hence the function

$$f(x) = \frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)}, \quad a \geq 1$$

is a decreasing function of $x \geq 0$. Thus for $x \in [0,1]$ and $a \geq 1$, we have

$$\frac{\Gamma_q(2)^a}{\Gamma_q(1+a)} \leq \frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)} \leq \frac{\Gamma_q(1)^a}{\Gamma_q(1)}.$$

We complete the proof by noting that $\Gamma_q(1) = \Gamma_q(2) = 1$. $\square$

Remark 1. *Letting q to 1 in the above theorem. we obtain (1.2).*

Remark 2. *Letting q to 1 and then putting $a = n$ in the above theorem, we get (1.1).*

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