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# EXTENDED STABILITY PROBLEM FOR ALTERNATIVE CAUCHY–JENSEN MAPPINGS

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ABSTRACT. In 1940 S.M. Ulam proposed the famous Ulam stability problem. In 1941 D.H. Hyers solved the well-known Ulam stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In this paper we introduce generalized additive mappings of Jensen type mappings and establish new theorems about the Ulam stability of additive and alternative additive mappings.

*Key words and phrases:* Stability problem; Cauchy–Jensen mappings; Euler–Lagrange mappings; Fixed point alternative.

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## 1. INTRODUCTION

In 1940 and in 1964 S.M. Ulam [34] proposed the famous Ulam stability problem:

"When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

For very general functional equations, the concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is: Do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we

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would say that the equation is stable. These stability results can be applied in stochastic analysis [17], financial and actuarial mathematics, as well as in psychology and sociology. We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [35] used a stability property of the functional equation $f(x-y)+f(x+y)=2f(x)$ to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials.

In 1941 D.H. Hyers [8] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In 1978 P.M. Gruber [7] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. Th.M. Rassias [31] and then P. Găvruta [5] obtained generalized results of Hyers' Theorem which allow the Cauchy difference to be unbounded. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. In 1982–2006 J.M. Rassias [20, 21, 23, 24, 25, 26, 27] established the Hyers–Ulam stability of linear and nonlinear mappings. In 2003-2006 J.M. Rassias and M.J. Rassias [28, 29] and J.M. Rassias [30] solved the above Ulam problem for Jensen and Jensen type mappings. In 1999 P. Găvruta [6] answered a question of J.M. Rassias [22] concerning the stability of the Cauchy equation.

We note that J.M. Rassias introduced the *Euler–Lagrange quadratic mappings*, motivated from the following pertinent algebraic equation

$$|a_1x_1+a_2x_2|^2+|a_2x_1-a_1x_2|^2=(a_1^2+a_2^2)\left[|x_1|^2+|x_2|^2\right]. \tag{1.1}$$

Thus the third author of this paper introduced and investigated the stability problem of Ulam for the relative *Euler–Lagrange functional equation*

$$f(a_1x_1+a_2x_2)+f(a_2x_1-a_1x_2)=(a_1^2+a_2^2)\left[f(x_1)+f(x_2)\right]. \tag{1.2}$$

in the publications [23, 24, 25]. Analogous quadratic mappings were introduced and investigated through J.M Rassias' publications [26, 29]. Before 1992 these mappings and equations were not known at all in functional equations and inequalities. However, a completely different kind of Euler–Lagrange partial differential equation is known in calculus of variations. In this paper we introduce Cauchy and Cauchy–Jensen mappings of Euler–Lagrange and thus generalize Ulam stability results controlled by more general mappings, by considering approximately Cauchy and Cauchy–Jensen mappings of Euler–Lagrange satisfying conditions much weaker than D.H. Hyers and J.M. Rassias conditions on approximately Cauchy and Cauchy–Jensen mappings of Euler–Lagrange.

Throughout this paper, let $X$ be a real normed space and $Y$ a real Banach space in the case of functional inequalities. Also, let $X$ and $Y$ be real linear spaces for functional equations. Let us denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}$ the set of all real numbers.

**Definition 1.1.** A mapping $A : X \to Y$ is called *additive* if $A$ satisfies the functional equation

$$A(x+y)=A(x)+A(y) \tag{1.3}$$

for all $x, y \in X$. We note that the equation (1.3) is equivalent to *the Jensen equation*

$$2A\left(\frac{x+y}{2}\right)=A(x)+A(y)$$

for all $x, y \in X$ and $A(0)=0$.

Now we consider a mapping $A : X \to Y$, which may be analogously called *Euler–Lagrange additive*, satisfying the functional equation

$$A(ax+by)+A(bx+ay)+(a+b)[A(-x)+A(-y)]=0 \tag{1.4}$$

for all $x, y \in X$, where $a, b \in \mathbb{R}$ are nonzero fixed reals with $a + b \neq 0$. Next, we consider a mapping $A : X \to Y$ of Euler–Lagrange satisfying the functional equation

$$A(ax + by) + A(ax - by) + 2aA(-x) = 0 \tag{1.5}$$

which is equivalent to the equation of Jensen type

$$A(x) + A(y) + 2aA\left(-\frac{x+y}{2a}\right) = 0$$

for all $x, y \in X$, where $a, b \in \mathbb{R}$ are nonzero fixed reals. It is easy to see that if the equation (1.5) holds for all $x, y \in X$ and $A(0) = 0$, then equation (1.3) holds for all $x, y \in X$. However, the converse does not hold. In fact, choose $a, x_0 \in \mathbb{R}$ and an additive mapping $A : \mathbb{R} \to \mathbb{R}$ such that $A(ax_0) \neq aA(x_0)$. In this case, (1.3) holds for all $x, y \in \mathbb{R}$ and $A(0) = 0$. But we see that

$$A(ax_0 + 0) + A(ax_0 - 0) + 2aA(-x_0) = 2A(ax_0) - 2aA(ax_0) \neq 0,$$

and thus (1.5) does not hold. However we can show that if (1.3) holds for all $x, y \in X$ and $A(ax) = aA(x)$, then (1.5) holds for all $x, y \in X$. Alternatively, we investigate the functional equation of Euler–Lagrange

$$A(ax + by) - A(ax - by) + 2bA(-y) = 0 \tag{1.6}$$

for all $x, y \in X$. We note that the equation (1.6) is equivalent to

$$A(x) - A(y) + 2bA\left(-\frac{x-y}{2b}\right) = 0 \tag{1.7}$$

for all $x, y \in X$, where $a, b \in \mathbb{R}$ are nonzero fixed reals. It follows that (1.6) implies (1.3). However we can show that if (1.3) holds for all $x, y \in X$ and $A(bx) = bA(x)$, then (1.5) holds for all $x, y \in X$.

## 2. STABILITY OF EULER–LAGRANGE ADDITIVE MAPPINGS

We will investigate the conditions under which it is possible to find a true Euler–Lagrange additive mapping near an approximate Euler–Lagrange additive mapping with small error. We note that if $\lambda = 1$ in the next two theorems, then the mapping $\varphi_1$ is identically zero by the convergence of series and thus $f$ is itself the solution of the equation (1.4). Thus we may assume without loss of generality that $\lambda \neq 1$ in these theorems.

**Theorem 2.1.** *Assume that there exists a mapping $\varphi_1 : X^2 \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(ax + by) + f(bx + ay) + (a + b)[f(-x) + f(-y)]\| \leq \varphi_1(x, y) \tag{2.1}$$

*and the series*

$$\sum_{i=1}^{\infty} |\lambda|^i \varphi_1\left(\frac{x}{\lambda^i}, \frac{y}{\lambda^i}\right) < \infty \tag{2.2}$$

*for all $x, y \in X$, where $\lambda := -(a+b) \neq 0$. Then there exists a unique Euler–Lagrange additive mapping $A : X \to Y$ which satisfies the equation (1.4) and the inequality*

$$\|f(x) - A(x)\| \leq \frac{1}{2|\lambda|} \sum_{i=1}^{\infty} |\lambda|^i \varphi_1\left(\frac{-x}{\lambda^i}, \frac{-x}{\lambda^i}\right) \tag{2.3}$$

*for all $x \in X$.*

*Proof.* Substituting $x$ for $y$ in the functional inequality (2.1), we obtain

$$2\|f(-\lambda x) - \lambda f(-x)\| \le \varphi_1(x,x), \qquad \left\|f(x) - \lambda f\left(\frac{x}{\lambda}\right)\right\| \le \frac{1}{2}\varphi_1\left(\frac{-x}{\lambda}, \frac{-x}{\lambda}\right) \tag{2.4}$$

for all $x \in X$. Therefore from (2.4) with $\frac{x}{\lambda^i}$ in place of $x$ $(i = 1, \dots, n-1)$ and iterating, one gets

$$\left\|f(x) - \lambda^n f\left(\frac{x}{\lambda^n}\right)\right\| \le \frac{1}{2|\lambda|}\sum_{i=1}^{n} |\lambda|^i \varphi_1\left(\frac{-x}{\lambda^i}, \frac{-x}{\lambda^i}\right) \tag{2.5}$$

for all $x \in X$ and all $n \in \mathbb{N}$. By (2.5), for any $n > m \ge 0$ we have

$$\begin{aligned}\left\|\lambda^m f\left(\frac{x}{\lambda^m}\right) - \lambda^n f\left(\frac{x}{\lambda^n}\right)\right\| &= |\lambda|^m \left\|f\left(\frac{x}{\lambda^m}\right) - \lambda^{n-m} f\left(\frac{x}{\lambda^{n-m}\lambda^m}\right)\right\| \\ &\le \frac{1}{2|\lambda|}\sum_{i=1}^{n-m} |\lambda|^{i+m}\varphi_1\left(\frac{-x}{\lambda^{i+m}}, \frac{-x}{\lambda^{i+m}}\right)\end{aligned}$$

which tends to zero by (2.2) as $m$ tends to infinity. Thus it follows that a sequence $\left\{\lambda^n f(\frac{x}{\lambda^n})\right\}$ is Cauchy in $Y$ and it thus converges. Therefore we see that a mapping $A : X \to Y$ defined by

$$A(x) := \lim_{n\to\infty} \lambda^n f\left(\frac{x}{\lambda^n}\right) = \lim_{n\to\infty} (-a-b)^n f\left(\frac{x}{(-a-b)^n}\right)$$

exists for all $x \in X$. In addition it is clear from (2.1) and (2.2) that the following inequality

$$\begin{aligned}&\|A(ax+by) + A(bx+ay) + (a+b)[A(-x) + A(-y)]\| \\ &= \lim_{n\to\infty} |\lambda|^n \|f(\lambda^{-n}(ax+by)) + f(\lambda^{-n}(bx+ay)) + (a+b)[f(-\lambda^{-n}x) + f(-\lambda^{-n}y)]\| \\ &\le \lim_{n\to\infty} |\lambda|^n \varphi_1(\lambda^{-n}x, \lambda^{-n}y) = 0\end{aligned}$$

holds for all $x, y \in X$. Thus taking the limit $n \to \infty$ in (2.5), we find that the mapping $A$ is an Euler–Lagrange additive mapping satisfying the equation (1.4) and the inequality (2.3) near the approximate mapping $f : X \to Y$.

To prove the afore-mentioned uniqueness, we assume now that there is another Euler–Lagrange additive mapping $\breve{A} : X \to Y$ which satisfies the equation (1.4) and the inequality (2.3). Then it follows easily that by setting $y := x$ in (1.4) we get

$$\lambda^n A(x) = A(\lambda^n x), \qquad \lambda^n \breve{A}(x) = \breve{A}(\lambda^n x)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Thus from the last equality and (2.3) one proves that

$$\begin{aligned}\|A(x) - \breve{A}(x)\| &= |\lambda|^n \|A(\lambda^{-n}x) - \breve{A}(\lambda^{-n}x)\| \\ &\le |\lambda|^n \left(\left\|A(\lambda^{-n}x) - f(\lambda^{-n}x)\right\| + \left\|f(\lambda^{-n}x) - \breve{A}(\lambda^{-n}x)\right\|\right) \\ &\le \frac{1}{|\lambda|}\sum_{i=1}^{\infty} |\lambda|^{i+n}\varphi_1(-\lambda^{-i-n}x, -\lambda^{-i-n}x)\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore from $n \to \infty$, one establishes

$$A(x) - \breve{A}(x) = 0$$

for all $x \in X$, completing the proof of uniqueness. □

**Theorem 2.2.** *Assume that there exists a mapping $\varphi_1 : X^2 \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(ax+by) + f(bx+ay) + (a+b)[f(-x) + f(-y)]\| \le \varphi_1(x,y)$$

*and the series*

$$\sum_{i=0}^{\infty} \frac{\varphi_1(\lambda^i x, \lambda^i y)}{|\lambda|^i} < \infty$$

*for all* $x, y \in X$*, where* $\lambda := -(a+b)$. *Then there exists a unique Euler–Lagrange additive mapping* $A : X \to Y$ *which satisfies the equation (1.4) and the inequality*

$$\|f(x) - A(x)\| \leq \frac{1}{2|\lambda|} \sum_{i=0}^{\infty} \frac{\varphi_1(-\lambda^i x, -\lambda^i x)}{|\lambda|^i}$$

*for all* $x \in X$.

We obtain the following corollary concerning the stability for approximately Euler–Lagrange additive mappings in terms of a product of powers of norms.

**Corollary 2.3.** *If a mapping* $f : X \to Y$ *satisfies the functional inequality*

$$\|f(ax+by) + f(bx+ay) + (a+b)[f(-x) + f(-y)]\| \leq \delta \|x\|^\alpha \|y\|^\beta,$$

*for all* $x, y \in X$ $(X \setminus \{0\}$ *if* $\alpha, \beta \leq 0)$ *and for some fixed* $\alpha, \beta \in \mathbb{R}$*, such that* $\rho := \alpha + \beta \in \mathbb{R}, \rho \neq 1$*,* $\lambda := -(a+b) \neq 1$ *and* $\delta \geq 0$*, then there exists a unique Euler–Lagrange additive mapping* $A : X \to Y$ *which satisfies the equation (1.4) and the inequality*

$$\|f(x) - A(x)\| \leq \begin{cases} \frac{\delta\|x\|^\rho}{2(|\lambda| - |\lambda|^\rho)} & \textit{if} \quad |\lambda| > 1,\ \rho < 1\ (|\lambda| < 1,\ \rho > 1); \\ \frac{\delta\|x\|^\rho}{2(|\lambda|^\rho - |\lambda|)} & \textit{if} \quad |\lambda| > 1,\ \rho > 1\ (|\lambda| < 1,\ \rho < 1) \end{cases}$$

*for all* $x \in X$ $(X \setminus \{0\}$ *if* $\rho \leq 0)$. *The mapping* $A$ *is defined by the formula*

$$A(x) = \begin{cases} \lim\limits_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n}, & \textit{if} \quad |\lambda| > 1,\ \rho < 1\ (|\lambda| < 1, \rho > 1); \\ \lim\limits_{n\to\infty} \lambda^n f\left(\frac{x}{\lambda^n}\right), & \textit{if} \quad |\lambda| > 1,\ \rho > 1\ (|\lambda| < 1,\ \rho < 1). \end{cases}$$

Now we are going to investigate the stability problem of the Euler–Lagrange type equation (1.5), [23, 24, 25], by using either Banach's contraction principle or fixed points. For explicit later use, we state the following theorem (*The alternative of fixed point*) [18, 32] : Suppose that we are given a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant $L$. Then for each given $x \in \Omega$, *either*

$$d(T^n x, T^{n+1} x) = \infty \quad \text{for all } n \geq 0,$$

*or* there exists a nonnegative integer $n_0$ such that

(1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
(2) the sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$;
(3) $y^*$ is the unique fixed point of $T$ in the set $\Delta := \{y \in \Omega | d(T^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

The reader is referred to the book of D.H. Hyers, G. Isac and Th.M. Rassias [10] for an extensive theory of fixed points with a large variety of applications. In recent years, L. Cădariu and V. Radu [3, 4] applied the fixed point method to the investigation of the Cauchy and Jensen functional equations. Using such an elegant idea, they could present a short and simple proof for the stability of these equations [19, 19]. The reader can be referred to the references [11, 12, 13, 14].

Utilizing the above mentioned fixed point alternative, we now obtain our main stability result for the equation (1.5).

**Theorem 2.4.** *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\|f(ax+by) + f(ax-by) + 2af(-x)\| \leq \varphi_2(x,y) \tag{2.6}$$

*and $\varphi_2 : X^2 \to [0,\infty)$ is a mapping satisfying*

$$\lim_{n\to\infty} \frac{\varphi_2(\lambda^n x, \lambda^n y)}{|\lambda|^n} = 0 \quad \left(\lim_{n\to\infty} |\lambda|^n \varphi_2\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0, \ \textit{respectively}\right) \tag{2.7}$$

*for all $x, y \in X$, where $|\lambda := -2a| \neq 1$. If there exists a constant $L < 1$ such that the mapping*

$$x \mapsto \psi_2(x) := \varphi_2\left(-\frac{x}{\lambda}, -\frac{ax}{b\lambda}\right)$$

*has the property*

$$\psi_2(x) \leq L|\lambda|\psi_2\left(\frac{x}{\lambda}\right), \tag{2.8}$$

$$\left(\psi_2(x) \leq L\frac{\psi_2(\lambda x)}{|\lambda|}, \quad \textit{respectively}\right) \tag{2.9}$$

*for all $x \in X$, then there exists a unique additive mapping $A : X \to Y$ of Euler–Lagrange which satisfies the equation (1.5) and the inequality*

$$\|f(x) - A(x)\| \leq \frac{L}{1-L}\psi_2(x)$$

$$\left(\|f(x) - A(x)\| \leq \frac{1}{1-L}\psi_2(x), \quad \textit{respectively}\right)$$

*for all $x \in X$. If, moreover, $f$ is measurable or $f(tx)$ is continuous in $t$ for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.*

*Proof.* Consider the function space

$$\Omega := \{g \mid g : X \to Y, \ g(0) = 0\}$$

equipped with the generalized metric $d$ on $\Omega$,

$$d(g,h) := \inf\{K \in [0,\infty] \mid \|g(x) - h(x)\| \leq K\psi_2(x), \ \ x \in X\}.$$

It is easy to see that $(\Omega, d)$ is complete generalized metric space.

Now we define an operator $T : \Omega \to \Omega$ by

$$Tg(x) := \frac{g(\lambda x)}{\lambda} \quad \left(Tg(x) := \lambda g\left(\frac{x}{\lambda}\right), \quad \text{respectively}\right)$$

for all $x \in X$. Note that for all $g, h \in \Omega$ with $d(g,h) \leq K$, one has

$$\|g(x) - h(x)\| \leq K\psi_2(x), \quad x \in X,$$

which implies by (2.8)

$$\left\|\frac{g(\lambda x)}{\lambda} - \frac{h(\lambda x)}{\lambda}\right\| \leq \frac{K\psi_2(\lambda x)}{|\lambda|} \leq LK\psi_2(x), \quad x \in X.$$

Hence we see that for all constantS $K \in [0,\infty]$ with $d(g,h) \leq K$,

$$d(Tg, Th) \leq LK,$$

$$\text{or} \qquad d(Tg, Th) \leq Ld(g,h),$$

that is, $T$ is a strictly self-mapping of $\Omega$ with the Lipschitz constant $L$.

Substituting $(x, \frac{a}{b}x)$ for $(x, y)$ in the functional inequality (2.6) with the case (2.8), we obtain by (2.8)

$$\|f(2ax) + 2af(-x)\| \leq \varphi_2\left(x, \frac{a}{b}x\right), \tag{2.10}$$

$$\left\|f(x) - \frac{f(\lambda x)}{\lambda}\right\| \leq \frac{1}{|\lambda|}\varphi_2\left(-x, -\frac{a}{b}x\right) = \frac{1}{|\lambda|}\psi_2(\lambda x) \leq L\psi_2(x)$$

for all $x \in X$. Thus $d(f, Tf) \leq L < \infty$.

From (2.10) with the case (2.9), one gets by (2.9)

$$\left\|\lambda f\left(\frac{x}{\lambda}\right) - f(x)\right\| \leq \varphi_2\left(-\frac{x}{\lambda}, -\frac{ax}{b\lambda}\right) = \psi_2(x)$$

for all $x \in X$, and so $d(Tf, f) \leq 1 < \infty$.

Now, it follows from the fixed point alternative in both cases that there exists a unique fixed point $A$ of $T$ in the set $\Delta = \{g \in \Omega | d(f, g) < \infty\}$ such that

$$A(x) := \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n} \quad \left(A(x) := \lim_{n\to\infty} \lambda^n f\left(\frac{x}{\lambda^n}\right), \text{ respectively}\right) \tag{2.11}$$

for all $x \in X$ since $\lim_{n\to\infty} d(T^n f, A) = 0$. According to the fixed point alternative, $A$ is the unique fixed point of $T$ in the set $\Delta$ such that

$$\|f(x) - A(x)\| \leq d(f, A)\psi_2(x) \leq \frac{1}{1-L}d(f, Tf)\psi_2(x) \leq \frac{L}{1-L}\psi_2(x)$$

$$\left(\|f(x) - A(x)\| \leq \frac{1}{1-L}d(f, Tf)\psi_2(x) \leq \frac{1}{1-L}\psi_2(x), \text{ respectively}\right).$$

Now it follows from (2.7) that

$$\begin{aligned}&|\lambda|^{-n}\|f(\lambda^n(ax + by)) + f(\lambda^n(ax - by)) + 2af(-\lambda^n x)\| \\ &\qquad \leq |\lambda|^{-n}\varphi_2(\lambda^n x, \lambda^n y), \\ &\Big(|\lambda|^n\|f(\lambda^{-n}(ax + by)) + f(\lambda^{-n}(ax - by)) + 2af(-\lambda^{-n} x)\| \\ &\qquad \leq |\lambda|^n\varphi_2(\lambda^{-n} x, \lambda^{-n} y), \text{ respectively}\Big)\end{aligned}$$

from which we conclude by $n \to \infty$ that the mapping $A : X \to Y$ satisfies the equation (1.5) and so it is additive.

The proof of the last assertion in our Theorem 2.4 is obvious by [20]. □

**Corollary 2.5.** *If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\|f(ax + by) + f(ax - by) + 2af(-x)\| \leq \delta\|x\|^\alpha\|y\|^\beta,$$

*for all $x, y \in X$ ($X \setminus \{0\}$ if $\alpha, \beta \leq 0$) and for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho := \alpha + \beta \in \mathbb{R}, \rho \neq 1$, $\lambda := -2a \neq 1$ and $\delta \geq 0$, then there exists a unique additive mapping $A : X \to Y$ of Euler–Lagrange which satisfies the equation (1.5) and the inequality*

$$\|f(x) - A(x)\| \leq \begin{cases} \frac{|a|^\beta\delta\|x\|^\rho}{|b|^\beta(|\lambda| - |\lambda|^\rho)}, \ L = \frac{|\lambda|^\rho}{|\lambda|} & \textit{if} \quad |\lambda| > 1, \ \rho < 1, \ (|\lambda| < 1, \ \rho > 1); \\ \frac{|a|^\beta\delta\|x\|^\rho}{|b|^\beta(|\lambda|^\rho - |\lambda|)}, \ L = \frac{|\lambda|}{|\lambda|^\rho} & \textit{if} \quad |\lambda| > 1, \ \rho > 1, \ (|\lambda| < 1, \ \rho < 1) \end{cases}$$

*for all $x \in X$ ($X \setminus \{0\}$ if $\rho \leq 0$).*

We will investigate the conditions under which it is then possible to find a true additive Euler–Lagrange mapping of Eq. (1.6) near an approximate additive Euler–Lagrange mapping of Eq. (1.6) with small error.

**Theorem 2.6.** *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\|f(ax+by) - f(ax-by) + 2bf(-y)\| \le \varphi_3(x,y) \tag{2.12}$$

*and $\varphi_3 : X^2 \to [0,\infty)$ is a mapping satisfying*

$$\lim_{n\to\infty} \frac{\varphi_3(\lambda^n x, \lambda^n y)}{|\lambda|^n} = 0 \quad \left(\lim_{n\to\infty} |\lambda|^n \varphi_3\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0, \ \textit{respectively}\right) \tag{2.13}$$

*for all $x, y \in X$, where $|\lambda := -2b| \neq 1$. If there exists a constant $L < 1$ such that the mapping*

$$x \mapsto \psi_3(x) := \varphi_3\left(-\frac{bx}{a\lambda}, -\frac{x}{\lambda}\right)$$

*has the property*

$$\psi_3(x) \le L|\lambda|\psi_3\left(\frac{x}{\lambda}\right), \tag{2.14}$$

$$\left(\psi_3(x) \le L\frac{\psi_3(\lambda x)}{|\lambda|}, \ \textit{respectively}\right)$$

*for all $x \in X$, then there exists a unique additive mapping $A : X \to Y$ of Euler–Lagrange which satisfies the equation (1.6) and the inequality*

$$\|f(x) - A(x)\| \le \frac{L}{1-L}\psi_3(x)$$

$$\left(\|f(x) - A(x)\| \le \frac{1}{1-L}\psi_3(x), \quad \textit{respectively}\right)$$

*for all $x \in X$. If, moreover, $f$ is measurable or $f(tx)$ is continuous in $t$ for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.*

*Proof.* The proof of this theorem is similar to that of Theorem 2.4. □

**Corollary 2.7.** *If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\|f(ax+by) - f(ax-by) + 2bf(-y)\| \le \delta\|x\|^\alpha\|y\|^\beta,$$

*for all $x, y \in X$ ($X \setminus \{0\}$ if $\alpha, \beta \le 0$) and for some fixed $\alpha, \beta \in \mathbb{R}$, such that $\rho := \alpha + \beta \in \mathbb{R}$, $\rho \neq 1$, $\lambda := -2b \neq 1$ and $\delta \ge 0$, then there exists a unique additive mapping $A : X \to Y$ of Euler–Lagrange which satisfies the equation (1.6) and the inequality*

$$\|f(x) - A(x)\| \le \begin{cases} \frac{|b|^\alpha \delta \|x\|^\rho}{|a|^\alpha(|\lambda| - |\lambda|^\rho)} & \textit{if} \quad |\lambda| > 1, \ \rho < 1, \ (|\lambda| < 1, \ \rho > 1); \\ \frac{|b|^\alpha \delta \|x\|^\rho}{|a|^\alpha(|\lambda|^\rho - |\lambda|)} & \textit{if} \quad |\lambda| > 1, \ \rho > 1 \ (|\lambda| < 1, \ \rho < 1) \end{cases}$$

*for all $x \in X$ ($X \setminus \{0\}$ if $\rho \le 0$).*

**Corollary 2.8.** *If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\|f(ax+by) + f(ax-by) + 2af(-x)\| \le \delta, \quad |\lambda := -2a| \neq 1$$

$$(\|f(ax+by) - f(ax-by) + 2bf(-y)\| \le \delta, \quad |\lambda := -2b| \neq 1, \ \textit{respectively})$$

*for all $x, y \in X$ and for some fixed $\delta \ge 0$, then there exists a unique additive mapping $A : X \to Y$ of Euler–Lagrange which satisfies the equation (1.5) ((1.6), respectively) and the inequality*

$$\|f(x) - A(x)\| \le \begin{cases} \frac{\delta}{|\lambda| - 1} & \textit{if} \quad |\lambda| > 1; \\ \frac{\delta}{1 - |\lambda|} & \textit{if} \quad |\lambda| < 1 \end{cases}$$

*for all $x \in X$.*

## 3. $C^*$-ALGEBRA ISOMORPHISMS BETWEEN UNITAL $C^*$-ALGEBRAS

Throughout this section, assume that $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-algebras. Let $U(\mathcal{A})$ be the unitary group of $\mathcal{A}$, $\mathcal{A}_{in}$ the set of invertible elements in $\mathcal{A}$, $\mathcal{A}_{sa}$ the set of self-adjoint elements in $\mathcal{A}$, $\mathcal{A}_1 := \{a \in \mathcal{A} \mid |a| = 1\}$, $\mathcal{A}^+$ the set of positive elements in $\mathcal{A}$. As an application, we are going to investigate $C^*$-algebra isomorphisms between unital $C^*$-algebras. We denote by $\mathbb{N}_0$ the set of nonnegative integers.

**Theorem 3.1.** *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ for which there exist mappings $\varphi : \mathcal{A}^2 \to \mathbb{R}^+ := [0, \infty)$ satisfying*

$$\sum_{i=0}^{\infty} \frac{\varphi(\lambda^i x, \lambda^i y)}{|\lambda|^i} < \infty, \tag{3.1}$$

*$\psi_1 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}^+$, and $\psi : \mathcal{A} \to \mathbb{R}^+$ such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) + 2a\mu h(-x)\| \le \varphi(x, y), \tag{3.2}$$

$$\|h(\lambda^n ux) - h(\lambda^n u)h(x)\| \le \psi_1(\lambda^n u, x), \tag{3.3}$$

$$\|h(\lambda^n u^*) - h(\lambda^n u)^*\| \le \psi(\lambda^n u) \tag{3.4}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, all $u \in U(\mathcal{A})$, all $x, y \in \mathcal{A}$ and all $n \in \mathbb{N}_0$, where $\lambda := -2a \neq 1$. Assume that*

$$\lim_{n\to\infty} \lambda^{-n} \psi_1(\lambda^n u, x) = 0 \quad \text{for all} \quad u \in U(\mathcal{A}), x \in \mathcal{A}, \tag{3.5}$$

$$\lim_{n\to\infty} \lambda^{-n} \psi(\lambda^n u) = 0 \quad \text{for all} \quad u \in U(\mathcal{A}), \tag{3.6}$$

$$\lim_{n\to\infty} \lambda^{-n} h(\lambda^n u_0) \in \mathcal{B}_{in} \quad \text{for some} \quad u_0 \in \mathcal{A}. \tag{3.7}$$

*Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is in fact a $C^*$-algebra isomorphism.*

*Proof.* Substituting $(x, y)$ for $\left(x, \frac{a}{b}x\right)$ in the functional inequality (3.2) with $\mu = 1$, we obtain

$$\|h(2ax) + 2ah(-x)\| \le \varphi\left(x, \frac{a}{b}x\right), \tag{3.8}$$

$$\left\|h(x) - \frac{h(\lambda x)}{\lambda}\right\| \le \frac{1}{|\lambda|}\varphi\left(-x, -\frac{a}{b}x\right),$$

for all $x \in X$. From (3.8), one gets

$$\left\|h(x) - \frac{h(\lambda^n x)}{\lambda^n}\right\| \le \frac{1}{|\lambda|}\sum_{i=0}^{n-1} \frac{\varphi\left(-\lambda^i x, -\frac{a}{b}\lambda^i x\right)}{|\lambda|^i} \tag{3.9}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Thus it follows from (3.1) and (3.9) that a sequence $\left\{\lambda^{-n}h(\lambda^n x)\right\}$ is Cauchy in $Y$ and it thus converges. Therefore we see that there exists a unique mapping $H : \mathcal{A} \to \mathcal{B}$, defined by $H(x) := \lim_{n\to\infty} \lambda^{-n} h(\lambda^n x)$, satisfying $H(0) = 0$, the equation (1.5) and the inequality

$$\|h(x) - H(x)\| \le \frac{1}{|\lambda|}\sum_{i=0}^{\infty} \frac{\varphi\left(-\lambda^i x, -\frac{a}{b}\lambda^i x\right)}{|\lambda|^i} \tag{3.10}$$

for all $x \in \mathcal{A}$. We claim that the mapping $H$ is $\mathbb{C}$-linear. For this, putting $x := 0$ and $y := 0$ separately in (1.5) one gets that $H$ is odd and $H(ax) = aH(x)$ for all $x \in \mathcal{A}$. Now replacing $y$ by $\frac{ay}{b}$ in (1.5) we get $H(ax + ay) + H(ax - ay) = 2aH(x)$ and so $H(x + y) + H(x - y) =$

$2H(x)$, which means that $H$ is additive. On the other hand, we obtain from (3.1) and (3.2) that $H(a\mu x + b\mu y) + H(a\mu x - b\mu y) - 2a\mu H(x) = 0$ for all $x, y \in \mathcal{A}$ and so

$$H(\mu x) - \mu H(x) = 0 \tag{3.11}$$

for all $x \in \mathcal{A}$ and all $\mu \in S^1 = U(\mathbb{C})$. Now, let $\eta$ be a nonzero element in $\mathbb{C}$ and $K$ a positive integer greater than $4|\eta|$. Then we have $|\frac{\eta}{K}| < \frac{1}{4} < 1 - \frac{2}{3}$. By [15, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in S^1$ such that $3\frac{\eta}{K} = \mu_1 + \mu_2 + \mu_3$. Thus we calculate by (3.11)

$$\begin{aligned}
H(\eta x) &= H\left(\frac{K}{3} \cdot 3\frac{\eta}{K} x\right) \\
&= \left(\frac{K}{3}\right) H(\mu_1 x + \mu_2 x + \mu_3 x) \\
&= \left(\frac{K}{3}\right) \Big(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)\Big) \\
&= \left(\frac{K}{3}\right) (\mu_1 + \mu_2 + \mu_3) H(x) = \left(\frac{K}{3}\right) \cdot 3\frac{\eta}{K} g(x) = \eta H(x)
\end{aligned}$$

for all $\eta \in \mathbb{C}$ $(\eta \neq 0)$ and all $x \in \mathcal{A}$. So the unique mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, as desired.

By (3.4) and (3.6), we have

$$\begin{aligned}
H(u^*) &= \lim_{n\to\infty} \lambda^{-n} h\left(\lambda^n u^*\right) \\
&= \lim_{n\to\infty} \lambda^{-n} h\left(\lambda^n u\right)^* \\
&= \left(\lim_{n\to\infty} \lambda^{-n} h\left(\lambda^n u\right)\right)^* = H(u)^*
\end{aligned} \tag{3.12}$$

for all $u \in U(\mathcal{A})$. Since each $x \in \mathcal{A}$ is a finite linear combination of unitary elements ([16, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m c_j u_j$ $(c_j \in \mathbb{C}, u_j \in U(\mathcal{A}))$, we get by (3.12)

$$\begin{aligned}
H(x^*) &= H\left(\sum_{j=1}^m \bar{c_j} u_j^*\right) = \sum_{j=1}^m \bar{c_j} H(u_j^*) = \sum_{j=1}^m \bar{c_j} H(u_j)^* \\
&= \left(\sum_{j=1}^m c_j H(u_j)\right)^* = H\left(\sum_{j=1}^m c_j u_j\right)^* = H(x)^*
\end{aligned}$$

for all $x \in \mathcal{A}$. So the mapping $H$ is preserved by involution.

Using the relations (3.3) and (3.5), we get

$$\begin{aligned}
H(ux) &= \lim_{n\to\infty} \lambda^{-n} h\left(\lambda^n ux\right) \\
&= \lim_{n\to\infty} \lambda^{-n} h\left(\lambda^n u\right) h(x) = H(u)h(x)
\end{aligned} \tag{3.13}$$

for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. Now, let $z \in \mathcal{A}$ be an arbitrary element. Then $z = \sum_{j=1}^m c_j u_j$ $(c_j \in \mathbb{C}, u_j \in U(\mathcal{A}))$, and it follows from (3.13) that

$$\begin{aligned}
H(zx) &= H\left(\sum_{j=1}^m c_j u_j x\right) = \sum_{j=1}^m c_j H(u_j x) = \sum_{j=1}^m c_j H(u_j) h(x) \\
&= H\left(\sum_{j=1}^m c_j u_j\right) h(x) = H(z)h(x)
\end{aligned} \tag{3.14}$$

for all $z, x \in \mathcal{A}$.

On the other hand, it follows from (3.13) and the linearity of $H$ that the equation

$$\begin{aligned}H(ux) &= \lambda^{-n}H\left(\lambda^n ux\right) = \lambda^{-n}H\left(u\lambda^n x\right)\\ &= \lambda^{-n}H(u)h\left(\lambda^n x\right) = H(u)\lambda^{-n}h\left(\lambda^n x\right)\end{aligned}$$

holds for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. Taking the limit as $n \to \infty$ in the last equation, we obtain

$$H(ux) = H(u)H(x) \tag{3.15}$$

for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. Using the same argument as (3.14), we see from (3.15) that

$$\begin{aligned}H(zx) &= H\left(\sum_{j=1}^{m} c_j u_j x\right) = \sum_{j=1}^{m} c_j H(u_j x) = \sum_{j=1}^{m} c_j H(u_j)H(x)\\ &= H\left(\sum_{j=1}^{m} c_j u_j\right) H(x) = H(z)H(x)\end{aligned} \tag{3.16}$$

for all $z, x \in \mathcal{A}$. Hence the mapping $H$ is multiplicative.

Finally, it follows from (3.14) and (3.16) that

$$H(u_0)H(x) = H(u_0 x) = H(u_0)h(x)$$

for all $x \in \mathcal{A}$. Since $H(u_0) = \lim_{n\to\infty} \lambda^{-n}h\left(\lambda^n u_0\right)$ is invertible for some $u_0 \in \mathcal{A}$ by (3.7), we see that $H(x) = h(x)$ for all $x \in \mathcal{A}$. Hence the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is in fact a $C^*$-algebra isomorphism, as desired. □

**Theorem 3.2.** *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ for which there exist mappings $\varphi : \mathcal{A}^2 \to \mathbb{R}^+ := [0, \infty)$ satisfying*

$$\sum_{i=1}^{\infty} |\lambda|^i \varphi(\lambda^{-i}x, \lambda^{-i}y) < \infty,$$

*$\psi_1 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}^+$, and $\psi : \mathcal{A} \to \mathbb{R}^+$ such that*

$$\begin{aligned}&\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) + 2a\mu h(-x)\| \le \varphi(x, y),\\ &\|h(\lambda^{-n}ux) - h(\lambda^{-n}u)h(x)\| \le \psi_1(\lambda^{-n}u, x),\\ &\left\|h\left(\lambda^{-n}u^*\right) - h\left(\lambda^{-n}u\right)^*\right\| \le \psi\left(\lambda^{-n}u\right)\end{aligned}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, all $u \in U(\mathcal{A})$, all $x, y \in \mathcal{A}$ and all $n \in \mathbb{N}_0$, where $\lambda := -2a \neq 1$. Assume that*

$$\begin{aligned}&\lim_{n\to\infty} \lambda^n \psi_1\left(\lambda^{-n}u, x\right) = 0 \quad \textit{for all} \quad u \in U(\mathcal{A}), x \in \mathcal{A},\\ &\lim_{n\to\infty} \lambda^n \psi\left(\lambda^{-n}u\right) = 0 \quad \textit{for all} \quad u \in U(\mathcal{A}),\\ &\lim_{n\to\infty} \lambda^n h\left(\lambda^{-n}u_0\right) \in \mathcal{B}_{in} \quad \textit{for some} \quad u_0 \in \mathcal{A}.\end{aligned}$$

*Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is in fact a $C^*$-algebra isomorphism.*

*Proof.* The proof is similar to that of Theorem 3.1. □

As an application we shall derive a stability result for the equation (1.5) which is very connected with the $\beta$-homogeneity of the norm on $F$-spaces.

**Corollary 3.3.** *Suppose that $G$ is an $F$-space and $E$ a $\beta$-homogeneous $F$-space, $0 < \beta \leq 1$. Let $h : G \to E$ be a mapping with $h(0) = 0$ for which there exist constants $p_i$, $\varepsilon_i \geq 0$ and $\delta \geq 0$ such that*

$$\|h(ax+by) + h(ax-by) + 2ah(-x)\| \leq \delta + \varepsilon_1\|x\|^{p_1} + \varepsilon_2\|y\|^{p_2}$$

*for all $x, y \in G$, where $|\lambda := -2a| \neq 1$. Then there exists a unique additive mapping $A : G \to E$ of Euler–Lagrange which satisfies the equation (1.5) and the inequality*

$$\|h(x) - A(x)\| \leq \begin{cases} \frac{\delta}{|\lambda|-1} + \frac{\varepsilon_1\|x\|^{p_1}}{|\lambda|-|\lambda|^{\beta p_1}} + |\frac{a}{b}|^{\beta p_2}\frac{\varepsilon_2\|x\|^{p_2}}{|\lambda|-|\lambda|^{\beta p_2}}, & \text{if} \quad |\lambda| > 1, \beta p_i < 1 \text{ for all } i = 1,2, \\ & \quad (|\lambda| < 1, \beta p_i > 1 \text{ and } \delta = 0); \\ \frac{\delta}{1-|\lambda|} + \frac{\varepsilon_1\|x\|^{p_1}}{|\lambda|^{\beta p_1}-|\lambda|} + |\frac{a}{b}|^{\beta p_2}\frac{\varepsilon_2\|x\|^{p_2}}{|\lambda|^{\beta p_2}-|\lambda|}, & \text{if} \quad |\lambda| < 1, \beta p_i < 1 \text{ for all } i = 1,2, \\ & \quad (|\lambda| > 1, \beta p_i > 1 \text{ and } \delta = 0) \end{cases}$$

*for all $x \in G$.*

*Proof.* Taking $\varphi(x,y) := \delta + \varepsilon_1\|x\|^{p_1} + \varepsilon_2\|y\|^{p_2}$ and applying (3.10) and the corresponding part of Theorem 3.1 and Theorem 3.2, respectively, we obtain the desired results in all cases. □

**Theorem 3.4.** *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and (3.7) for which there exists a mapping $\varphi : \mathcal{A}^2 \to \mathbb{R}^+$ satisfying (3.1), and mappings $\psi_1, \psi$ such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) + 2a\mu h(-x)\| \leq \varphi(x,y),$$

$$\|h(\lambda^n ux) - h(\lambda^n u)h(x)\| \leq \psi_1(\lambda^n u, x), \tag{3.17}$$

$$\|h(\lambda^n u^*) - h(\lambda^n u)^*\| \leq \psi(\lambda^n u) \tag{3.18}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, all $u \in {\mathcal{A}_1}^+$ and all $x, y \in \mathcal{A}$ and all $n \in \mathbb{N}_0$, where $\lambda := -2a \neq 1$. Assume that*

$$\lim_{n\to\infty} \lambda^{-n}\psi_1(\lambda^n u, x) = 0 \quad \text{for all} \quad u \in {\mathcal{A}_1}^+, \ \text{all } x \in \mathcal{A}, \tag{3.19}$$

$$\lim_{n\to\infty} \lambda^{-n}\psi(\lambda^n u) = 0 \quad \text{for all} \quad u \in {\mathcal{A}_1}^+. \tag{3.20}$$

*Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is in fact a $C^*$-algebra isomorphism.*

*Proof.* By the same reasoning as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$, defined by $H(x) := \lim_{n\to\infty} \lambda^{-n}h(\lambda^n x)$, satisfying $H(0) = 0$, the equation (1.5) and the functional inequality (3.10).

By (3.18) and (3.20), we have $H(u^*) = H(u)^*$ for all $u \in {\mathcal{A}_1}^+$, and so

$$H(v^*) = H\left(|v| \cdot \frac{v^*}{|v|}\right) = |v|H\left(\frac{v^*}{|v|}\right) = \left[|v|H\left(\frac{v}{|v|}\right)\right]^* = H(v)^* \tag{3.21}$$

for all nonzero $v \in \mathcal{A}^+$. Now, for any element $v \in A$, $v = v_1 + iv_2$, where $v_1, v_2 \in \mathcal{A}_{sa}$; furthermore, $v = v_1^+ - v_1^- + iv_2^+ - iv_2^-$, where $v_1^+, v_1^-, v_2^+$ and $v_2^-$ are all positive elements (see [2, Lemma 38.8]). Since $H$ is $\mathbb{C}$-linear, we figure out by (3.21)

$$\begin{aligned} H(v^*) &= H\left((v_1^+ - v_1^- + iv_2^+ - iv_2^-)^*\right) \\ &= H({v_1^+}^*) - H({v_1^-}^*) + H((iv_2^+)^*) - H((iv_2^-)^*) \\ &= H(v_1^+)^* - H(v_1^-)^* - iH(v_2^+)^* + iH(v_2^-)^* \\ &= \left[H(v_1^+ - v_1^- + iv_2^+ - iv_2^-)\right]^* = H(v)^* \end{aligned}$$

for all $v \in \mathcal{A}$.

Using (3.17) and (3.19) we get $H(ux) = H(u)h(x)$ for all $u \in {\mathcal{A}_1}^+$ and all $x \in \mathcal{A}$, and so $H(vx) = H(v)h(x)$ for all $v \in \mathcal{A}^+$ and all $x \in \mathcal{A}$ because

$$H(vx) = H\left(|v|\frac{v}{|v|} \cdot x\right) = |v|H\left(\frac{v}{|v|} \cdot x\right) = |v|H\left(\frac{v}{|v|}\right) \cdot h(x) = H(v)h(x), \quad \forall v \in \mathcal{A}^+. \tag{3.22}$$

Now, for any element $v \in A$, $v = v_1^+ - v_1^- + iv_2^+ - iv_2^-$, where $v_1^+, v_1^-, v_2^+$ and $v_2^-$ are positive elements (see [2, Lemma 38.8]). Thus we calculate by (3.22) and the linearity of $H$

$$\begin{aligned} H(vx) &= H\Big(v_1^+x - v_1^-x + iv_2^+x - iv_2^-x\Big) \\ &= H(v_1^+x) - H(v_1^-x) + iH(v_2^+x) - iH(v_2^-x) \\ &= \Big(H(v_1^+) - H(v_1^-) + iH(v_2^+) - iH(v_2^-)\Big)h(x) \\ &= H(v)h(x) \end{aligned} \tag{3.23}$$

for all $v, x \in \mathcal{A}$. By (3.23) and the linearity of $H$, one has

$$\begin{aligned} H(vx) &= \lambda^{-n}H(\lambda^n vx) = \lambda^{-n}H(v\lambda^n x) \\ &= \lambda^{-n}H(v)h(\lambda^n x) = H(v)\lambda^{-n}h(\lambda^n x), \end{aligned}$$

which yields by taking the limit as $n \to \infty$

$$H(vx) = H(v)H(x) \tag{3.24}$$

for all $v, x \in \mathcal{A}$.

It follows from (3.23) and (3.24) that for a given $u_0$ subject to (3.7)

$$H(u_0)H(x) = H(u_0x) = H(u_0)h(x)$$

for all $x \in \mathcal{A}$. Since $H(u_0) = \lim_{n\to\infty} \lambda^{-n}h(\lambda^n u_0) \in \mathcal{B}_{in}$, we see that $H(x) = h(x)$ for all $x \in \mathcal{A}$. Hence the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a $C^*$-algebra isomorphism, as desired. □

**Theorem 3.5.** *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ satisfying (3.1), (3.3) and (3.4) such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) + 2a\mu h(-x)\| \le \varphi(x, y) \tag{3.25}$$

*holds for $\mu = 1, i$. Assume that the conditions (3.5), (3.6) and (3.7) are satisfied, and that $h$ is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a $C^*$-algebra isomorphism.*

*Proof.* Fix $\mu = 1$ in (3.25). By the same reasoning as in the proof of Theorem 3.1, there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying $H(0) = 0$, the equation (1.5) and the inequality (3.10).

By the assumption that $h$ is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear, that is, $H(tx) = tH(x)$ for all $t \in \mathbb{R}$ and all $x \in \mathcal{A}$ [20, 31]. Put $\mu = i$ in (3.25). Then applying the same argument to (3.11) as in the proof of Theorem 3.1, we obtain that

$$H(ix) = iH(x),$$

and so for any $\mu = s + it \in \mathbb{C}$

$$\begin{aligned} H(\mu x) &= H(sx + itx) \\ &= H(sx) + H(itx) \\ &= sH(x) + itH(x) \\ &= (s+it)H(x) = \mu H(x) \end{aligned}$$

for all $x \in \mathcal{A}$. Hence the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the corresponding part of Theorem 3.1. □

**Theorem 3.6.** *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ satisfying (3.1), (3.7), (3.17) and (3.18) such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) + 2a\mu h(-x)\| \le \varphi(x,y) \tag{3.26}$$

*holds for $\mu = 1, i$. Assume that the equations (3.19), (3.20) are satisfied, and that $h$ is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a $C^*$-algebra isomorphism.*

*Proof.* The proof is similar to that of Theorem 3.5. □

## 4. DERIVATIONS MAPPING INTO THE RADICALS OF BANACH ALGEBRAS

Throughout this section, assume that $\mathcal{A}$ is a complex Banach algebra with norm $\|\cdot\|$. As an application, we are going to investigate the stability of derivations on Banach algebras and consider the range of derivations on Banach algebras.

**Lemma 4.1.** *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $h(0) = 0$ for which there exists a mapping $\varphi : \mathcal{A}^2 \to \mathbb{R}^+$ satisfying (3.1) and a mapping $\psi : \mathcal{A}^2 \to \mathbb{R}^+$ satisfying*

$$\lim_{n\to\infty} \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0 \tag{4.1}$$

*for all $x, y \in X$, where $\lambda := -2a \neq 1$, such that*

$$\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) + 2a\mu h(-x)\| \le \varphi(x,y), \tag{4.2}$$

$$\|h(xy) - h(x)y - xh(y)\| \le \psi(x,y) \tag{4.3}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$ and all $x, y \in \mathcal{A}$. Then there exists a unique $\mathbb{C}$-linear derivation $H : \mathcal{A} \to \mathcal{A}$ which satisfies the inequality*

$$\|h(x) - H(x)\| \le \frac{1}{|\lambda|} \sum_{i=0}^{\infty} \frac{\varphi\left(-\lambda^i x, -\frac{a}{b}\lambda^i x\right)}{|\lambda|^i} \tag{4.4}$$

*for all $x \in \mathcal{A}$.*

*Proof.* By the same reasoning as in the proof of Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{A}$, defined by $H(x) := \lim_{n\to\infty} \lambda^{-n} h(\lambda^n x)$, satisfying $H(0) = 0$, the equation (1.5) and the functional inequality (4.4).

Replacing $x$ and $y$ in (4.2) by $\lambda^n x$ and $\lambda^n y$, respectively, and dividing the result by $|\lambda|^{2n}$, we obtain

$$\left\| \frac{h(\lambda^{2n} xy)}{\lambda^{2n}} - \frac{h(\lambda^n x)}{\lambda^n} y - x \frac{h(\lambda^n y)}{\lambda^n} \right\| \le \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}}$$

for all $x, y \in \mathcal{A}$. Taking the limit in the last inequality, one obtains that

$$H(xy) - H(x)y - xH(y) = 0$$

for all $x, y \in \mathcal{A}$ because $\lim_{n\to\infty} \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0$ and $\lim_{n\to\infty} \frac{h(\lambda^{2n} xy)}{\lambda^{2n}} = H(xy)$. Thus the mapping $H : \mathcal{A} \to \mathcal{A}$ is a unique $\mathbb{C}$-linear derivation satisfying the functional inequality (4.4). □

**Lemma 4.2.** *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $h(0) = 0$ for which there exists a mapping $\varphi : \mathcal{A}^2 \to \mathbb{R}^+$ satisfying*

$$\sum_{i=1}^{\infty} |\lambda|^i \varphi\left(\frac{x}{\lambda^i}, \frac{y}{\lambda^i}\right) < \infty \tag{4.5}$$

*and a mapping $\psi : \mathcal{A}^2 \to \mathbb{R}^+$ satisfying*

$$\lim_{n\to\infty} |\lambda|^{2n} \psi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0 \tag{4.6}$$

*for all $x, y \in X$, where $\lambda := -2a \neq 1$, such that*

$$\begin{aligned} &\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) + 2a\mu h(-x)\| \leq \varphi(x, y), \\ &\|h(xy) - h(x)y - xh(y)\| \leq \psi(x, y) \end{aligned} \tag{4.7}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$ and all $x, y \in \mathcal{A}$. Then there exists a unique $\mathbb{C}$-linear derivation $H : \mathcal{A} \to \mathcal{A}$ which satisfies the inequality*

$$\|h(x) - H(x)\| \leq \frac{1}{|\lambda|} \sum_{i=1}^{\infty} |\lambda|^i \varphi\left(-\frac{x}{\lambda^i}, -\frac{a}{b}\frac{x}{\lambda^i}\right) \tag{4.8}$$

*for all $x \in \mathcal{A}$.*

**Corollary 4.3.** *Let $|\lambda := -2a| \neq 1$. Assume that $h : \mathcal{A} \to \mathcal{A}$ is a mapping satisfying $h(0) = 0$ for which there exist nonnegative constants $\varepsilon_1, \varepsilon_2$, such that*

$$\begin{aligned} &\|h(a\mu x + b\mu y) + h(a\mu x - b\mu y) + 2a\mu h(-x)\| \leq \varepsilon_1, \\ &\|h(xy) - h(x)y - xh(y)\| \leq \varepsilon_2 \end{aligned}$$

*for all $\mu \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$ and all $x, y \in \mathcal{A}$. Then there exists a unique $\mathbb{C}$-linear derivation $H : \mathcal{A} \to \mathcal{A}$ which satisfies the inequality*

$$\|h(x) - H(x)\| \leq \frac{\varepsilon_1}{||\lambda| - 1|}$$

*for all $x \in \mathcal{A}$.*

**Lemma 4.4.** *Let $h : \mathcal{A} \to \mathcal{A}$ be a linear mapping for which there exists a mapping $\psi : \mathcal{A}^2 \to \mathbb{R}^+$ satisfying either*

$$\lim_{n\to\infty} \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0 \quad \textit{or}, \quad \lim_{n\to\infty} |\lambda|^{2n} \psi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0 \tag{4.9}$$

*for all $x, y \in X$, where $\lambda := -2a$ is a nonzero real number with $\lambda \neq 1$, such that*

$$\|h(xy) - h(x)y - xh(y)\| \leq \psi(x, y) \tag{4.10}$$

*for all $x, y \in \mathcal{A}$. Then the mapping $h$ is in fact a derivation on $\mathcal{A}$.*

*Proof.* Taking $\varphi(x, y) := 0$ in the previous two lemmas, we have the desired result. □

**Theorem 4.5.** *Let $\mathcal{A}$ be a commutative Banach algebra. Let $h : \mathcal{A} \to \mathcal{A}$ be a given linear mapping and an approximate derivation with difference $Dh$ bounded by $\psi$, that is, there exists a mapping $\psi : \mathcal{A} \times \mathcal{A} \to \mathbb{R}^+$ such that*

$$\|Dh(x, y) := h(xy) - h(x)y - xh(y)\| \leq \psi(x, y) \tag{4.11}$$

*for all $x, y \in \mathcal{A}$. Assume that there exists a nonzero real number $\lambda$ with $\lambda \neq 1$ such that the limit*

$$\lim_{n\to\infty} \frac{\psi(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0 \qquad \left(\lim_{n\to\infty} |\lambda|^{2n}\psi\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0, \ \textit{respectively}\right) \tag{4.12}$$

*for all $x, y \in \mathcal{A}$. Then the mapping $h$ is in fact a linear derivation which maps the algebra into its radical.*

*Proof.* By Lemma 4.4, the mapping $h$ is in fact a linear derivation which maps the algebra into its radical by Thomas' result [33]. $\square$

It is well-known that all linear derivations on commutative semi-simple Banach algebras are zero [33]. We remark that every linear mapping $h$ on a commutative semi-simple Banach algebra, which is an approximate derivation satisfying (4.11) and (4.12), is also zero.

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