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ON SOME NEW RETARD INTEGRAL INEQUALITIES IN n INDEPENDENT VARIABLES AND THEIR APPLICATIONS

XUEQIN ZHAO AND FANWEI MENG

DEPARTMENT OF MATHEMATICS
QUFU NORMAL UNIVERSITY
QUFU 273165,
PEOPLE'S REPUBLIC OF CHINA.
xqzhao1972@126.com

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ABSTRACT. In this paper, we established some retard integral inequalities in n independent variables and by means of examples we show the usefulness of our results.

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1. INTRODUCTION

The study of integral inequalities involving functions of one or more independent variables is an important tool in the study of existence, uniqueness, bounds, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations. During the past few years, many new inequalities have been discovered (see [1, 3, 4, 7, 8]). In the qualitative analysis of some classes of partial differential equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new inequalities in order to achieve a diversity of desired goals. Our aim in this paper is to establish some new inequalities in n independent variables, meanwhile, some applications of our results are also given.

2. PRELIMINARIES AND LEMMAS

In this paper, we suppose $\mathbb{R}_+ = [0, \infty)$, is subset of real numbers $\mathbb{R}$, $\widetilde{0} = (0, \dots, 0)$, $\widetilde{\alpha}(t) = (\alpha_1(t_1), \dots, \alpha_n(t_n)) \in \mathbb{R}_+^n$, $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, $s = (s_1, \dots, s_n) \in \mathbb{R}_+^n$, $\widetilde{r} = (r_1, \dots, r_n)$, $\widetilde{r_0} = (r_{10}, \dots, r_{n0})$, $\widetilde{z} = (z_1, \dots, z_n)$, $\widetilde{z_0} = (z_{10}, \dots, z_{n0})$, $T = (T_1, \dots, T_n) \in \mathbb{R}_+^n$.

If $f : \mathbb{R}_+^n \to \mathbb{R}_+$, we suppose

(1) $s \le t \Leftrightarrow s_i \le t_i \;\; (i = 1, 2, \dots, n)$;

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(2) $\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds = \int_0^{\alpha_1(t_1)} \cdots \int_0^{\alpha_n(t_n)} f(s_1, \dots, s_n)ds_n \dots ds_1$;
(3) $D_i = \frac{d}{dt_i}$, $i = 1, 2, \dots, n$.

3. MAIN RESULTS

In this part, we obtain our main results as follows:

Theorem 3.1. *Let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(u) > 0$ on $(0, \infty)$, and let c be a nonnegative constant. Let $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \le t_i$ on $\mathbb{R}_+$ $(i = 1, \dots, n)$. If $u, f \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ and*

$$u(t) \le c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)\psi(u(s))ds, \tag{3.1}$$

for $\widetilde{0} \le t < T$, then

$$u(t) \le G^{-1}\left[G(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right], \tag{3.2}$$

where

$$G(\widetilde{z}) = \int_{\widetilde{z}_0}^{\widetilde{z}} \frac{ds}{\psi(s)}, \qquad \widetilde{z} \ge \widetilde{z}_0 > 0.$$

G^{-1} is the inverse of G, $T \in \mathbb{R}_+^n$ is chosen so that

$$G(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds \in Dom(G^{-1}), \qquad \widetilde{0} \le t < T. \tag{3.3}$$

Define the nondecreasing positive function $z(t)$ and make

$$z(t) = c + \varepsilon + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)\psi(u(s))ds, \qquad \widetilde{0} \le t < T, \tag{3.4}$$

where ε is an arbitrary small positive number. We know that

$$u(t) \le z(t), \qquad D_1D_2\cdots D_n z(t) = f(\widetilde{\alpha})\psi(u(\widetilde{\alpha}))\alpha_1'\alpha_2'\cdots\alpha_n'. \tag{3.5}$$

Using (3.5), we have

$$\frac{D_1D_2\cdots D_n z(t)}{\psi(z(t))} \le f(\widetilde{\alpha})\alpha_1'\alpha_2'\cdots\alpha_n'. \tag{3.6}$$

For

$$D_n\left(\frac{D_1D_2\cdots D_{n-1}z(t)}{\psi(z(t))}\right) = \frac{D_1D_2\cdots D_n z(t)\psi(z(t)) - D_1D_2\cdots D_{n-1}z(t)\psi' D_n z(t)}{\psi^2(z(t))}, \tag{3.7}$$

using $D_1D_2\cdots D_{n-1}z(t) \ge 0$, $\psi' \ge 0$, $D_n z(t) \ge 0$ in (3.7), we get

$$D_n\left(\frac{D_1D_2\cdots D_{n-1}z(t)}{\psi(z(t))}\right) \le \frac{D_1D_2\cdots D_n z(t)}{\psi(z(t))} \le f(\widetilde{\alpha})\alpha_1'\alpha_2'\cdots\alpha_n'. \tag{3.8}$$

Fixing $t_1, \dots, t_{n-1}$, setting $t_n = s_n$, integrating from t_n to ∞, yields

$$\frac{D_1D_2\cdots D_{n-1}z(t)}{\psi(z(t))} \le \int_0^{\alpha_n(t_n)} f(\alpha_1(t_1), \dots, \alpha_{n-1}(t_{n-1}), s_n)\alpha_1'\alpha_2'\cdots\alpha_{n-1}'ds_n. \tag{3.9}$$

Using the same method, we deduce that

$$\frac{D_1 z(t)}{\psi(z(t))} \le \int_0^{\alpha_2(t_2)} \cdots \int_0^{\alpha_n(t_n)} f(t_1, s_2, \dots, s_n)\alpha_1' ds_n \dots ds_2, \tag{3.10}$$

and integration on $[t_1, \infty)$ yields

$$G(z(t)) \le G(c+\varepsilon) + \int_0^{\alpha_1(t_1)} \cdots \int_0^{\alpha_n(t_n)} f(s_1, s_2, \dots, s_n) ds_n \dots ds_1, \qquad t \in \mathbb{R}_+^n. \tag{3.11}$$

From the definition of G and letting $\varepsilon \to 0$, we can obtain inequality (3.2).

Remark 3.2. If we let $G(z) \to \infty$, $z \to \infty$, then condition (3.3) can be omitted.

Corollary 3.3. *If we let $\psi = s^r$, $0 < r \le 1$ in Theorem 3.1, then for $t \in \mathbb{R}_+^n$, we have*

$$u(t) \le \begin{cases} \left[c^{1-r} + (1-r)\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right]^{\frac{1}{1-r}}, & 0 < r < 1; \\ c\exp\left(\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right), & r = 1. \end{cases} \tag{3.12}$$

Remark 3.4. If we let $n = 1$, $r = 1$, $\widetilde{\alpha}(t) = t$, in Corollary 3.3, we obtain the Mate-Nevai inequality.

Theorem 3.5. *Let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$. Let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function and let c be a nonnegative constant. Let $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \le t_i$ on $\mathbb{R}_+ (i = 1, \dots, n)$. If $u, f \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ and*

$$\varphi(u(t)) \le c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)\psi(u(s))ds, \tag{3.13}$$

for $\widetilde{0} \le t < T$, then

$$u(t) \le \varphi^{-1}\left\{G^{-1}[G(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds]\right\}, \tag{3.14}$$

where $G(\widetilde{z}) = \int_{\widetilde{z}_0}^{\widetilde{z}} \frac{ds}{\psi[\varphi^{-1}(s)]}$, $\widetilde{z} \ge \widetilde{z}_0 > 0$, φ^{-1}, G^{-1} are respectively the inverse of φ and G, $T \in \mathbb{R}_+^n$ is chosen so that

$$G(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds \in \mathrm{Dom}(G^{-1}), \qquad \widetilde{0} \le t < T. \tag{3.15}$$

Proof. From the definition of the φ, we know (3.13) can be restated as

$$\varphi(u(t)) \le c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)\psi[\varphi^{-1}(\varphi(u(s)))]ds, \qquad t \in \mathbb{R}_+^n. \tag{3.16}$$

Now an application of Theorem 3.1 gives

$$\varphi(u(t)) \le G^{-1}\left[G(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right], \qquad \widetilde{0} \le t < T. \tag{3.17}$$

So,

$$u(t) \le \varphi^{-1}\left\{G^{-1}[G(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds]\right\}, \qquad \widetilde{0} \le t < T. \tag{3.18}$$

□

Corollary 3.6. *If we let $\varphi = s^p$, $\psi = s^q$, p, q are constants, and $p \geq q > 0$ in Theorem 3.5, for $\widetilde{0} \leq t < T$, then*

$$u(t) \leq \begin{cases} \left[c^{1-\frac{q}{p}} + \left(1 - \frac{q}{p}\right) \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right]^{\frac{1}{p-q}}, & \text{when} \quad p > q; \\ c^{\frac{1}{p}} \exp\left(\frac{1}{p} \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right), & \text{when} \quad p = q. \end{cases} \tag{3.19}$$

Theorem 3.7. *Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}_+^n$, let c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_i(u) > 0\,(i = 1, 2)$ on $(0, \infty)$. Let $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on $\mathbb{R}_+$ $(i = 1, \ldots, n)$. If*

$$u(t) \leq c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s) w_1(u(s))ds + \int_{\widetilde{0}}^{t} g(s) w_2(u(s))ds, \tag{3.20}$$

for $\widetilde{0} \leq t < T$, then

(i) *for the case $w_2(u) \leq w_1(u)$,*

$$u(t) \leq G_1^{-1} \left\{ G_1(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds + \int_{\widetilde{0}}^{t} g(s)ds \right\}, \tag{3.21}$$

(ii) *for the case $w_1(u) \leq w_2(u)$,*

$$u(t) \leq G_2^{-1} \left\{ G_2(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds + \int_{\widetilde{0}}^{t} g(s)ds \right\}, \tag{3.22}$$

where

$$G_i(\widetilde{z}) = \int_{\widetilde{z}_0}^{\widetilde{z}} \frac{ds}{w_i(s)}, \quad \widetilde{z} \geq \widetilde{z}_0 > 0, \qquad (i = 1, 2) \tag{3.23}$$

and G_i^{-1} $(i = 1, 2)$ is the inverse of G_i, $T \in \mathbb{R}_+^n$ is chosen so that

$$G_i(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds + \int_{\widetilde{0}}^{t} g(s)ds \in \mathrm{Dom}(G_i^{-1}), \quad (i = 1, 2), \qquad \widetilde{0} \leq t < T. \tag{3.24}$$

Proof. Define the nonincreasing positive function $z(t)$ and make

$$z(t) = c + \varepsilon + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s) w_1(u(s))ds + \int_{\widetilde{0}}^{t} g(s) w_2(u(s))ds, \tag{3.25}$$

where ε is an arbitrary small positive number. From inequality (3.20), we know

$$u(t) \leq z(t) \tag{3.26}$$

and

$$D_1 D_2 \cdots D_n z(t) = [f(\widetilde{\alpha}) w_1(u(\widetilde{\alpha}))\alpha_1' \alpha_2' \cdots \alpha_n' + g(t) w_2(u(t))]. \tag{3.27}$$

The rest of the proof can be completed by following the proof of Theorem 3.1 with suitable modifications. □

Theorem 3.8. *Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}_+^n$, and let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$ and let c be a nonnegative constant.*

Moreover, let $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $w_i(u) > 0\,(i = 1, 2)$ *on* $(0, \infty)$, $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing with* $\alpha_i(t_i) \le t_i$ *on* $\mathbb{R}_+ (i = 1, \dots, n)$. *If*

$$\varphi(u(t)) \le c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s) w_1(u(s)) ds + \int_{\widetilde{0}}^{t} g(s) w_2(u(s)) ds, \tag{3.28}$$

for $\widetilde{0} \le t < T$, *then*

(i) *for the case* $w_2(u) \le w_1(u)$,

$$u(t) \le \varphi^{-1}\left\{G_1^{-1}\left[G_1(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s) ds + \int_{\widetilde{0}}^{t} g(s) ds\right]\right\}; \tag{3.29}$$

(ii) *for the case* $w_1(u) \le w_2(u)$,

$$u(t) \le \varphi^{-1}\left\{G_2^{-1}\left[G_2(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s) ds + \int_{\widetilde{0}}^{t} g(s) ds\right]\right\}, \tag{3.30}$$

where

$$G_i(\widetilde{z}) = \int_{\widetilde{z}_0}^{\widetilde{z}} \frac{ds}{w_i(\varphi^{-1}(s))}, \quad \widetilde{z} > \widetilde{z}_0, \qquad (i = 1, 2),$$

and φ^{-1}, G_i^{-1} $(i = 1, 2)$ *are respectively the inverse of* G_i, φ, $T \in \mathbb{R}_+^n$ *is chosen so that*

$$G_i(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s) ds + \int_{\widetilde{0}}^{t} g(s) ds \in \mathrm{Dom}(G_i^{-1}), \quad (i = 1, 2), \qquad \widetilde{0} \le t < T. \tag{3.31}$$

Proof. From the definition of φ, we know (3.28) can be restated as

$$\begin{aligned} \varphi(u(t)) \le c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} & f(s) w_1[\varphi^{-1}(\varphi(u(s)))] ds \\ & + \int_{\widetilde{0}}^{t} g(s) w_2[\varphi^{-1}(\varphi(u(s)))] ds, \qquad t \in \mathbb{R}_+^n. \end{aligned} \tag{3.32}$$

Now an application of Theorem 3.7 gives

$$\varphi(u(t)) \le G_i^{-1}\left\{G_i(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s) ds + \int_{\widetilde{0}}^{t} g(s) ds\right\}, \qquad \widetilde{0} \le t < T,$$

where T satisfies (3.31). We can obtain the desired inequalities (3.29) and (3.30). □

Theorem 3.9. *Let* u, f *and* g *be nonnegative continuous functions defined on* $\mathbb{R}_+^n$ *and let* c *be a nonnegative constant. Moreover, let* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be an increasing function with* $\varphi(\infty) = \infty$, $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be a nondecreasing function with* $\psi(u) > 0$ *on* $(0, \infty)$ *and* $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing with* $\alpha_i(t_i) \le t_i$ *on* $\mathbb{R}_+$ $(i = 1, \dots, n)$. *If*

$$\varphi(u(t)) \le c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} [f(s) u(s) \psi(u(s)) + g(s) u(s)] ds, \tag{3.33}$$

for $\widetilde{0} \le t < T$, *then*

$$u(t) \le \varphi^{-1}\left\{\Omega^{-1}\left[G^{-1}\left(G[\Omega(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} g(s) ds] + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s) ds\right)\right]\right\}, \tag{3.34}$$

where

$$\Omega(\widetilde{r}) = \int_{\widetilde{r}_0}^{\widetilde{r}} \frac{ds}{\varphi^{-1}(s)}, \quad \widetilde{r} \geq \widetilde{r}_0 > 0\,,$$
$$G(\widetilde{z}) = \int_{\widetilde{z}_0}^{\widetilde{z}} \frac{ds}{\psi\{\varphi^{-1}[\Omega^{-1}(s)]\}}, \quad \widetilde{z} \geq \widetilde{z}_0 > 0,$$

$\Omega^{-1}, \varphi^{-1}, G^{-1}$*are respectively the inverse of* Ω, φ, G*. And* $T \in \mathbb{R}_+$ *is chosen so that*

$$G\left[\Omega(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} g(s)ds\right] + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds \in \mathrm{Dom}(G^{-1}), \qquad \widetilde{0} \leq t < T,$$

and

$$G^{-1}\left\{G\left[\Omega(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} g(s)ds\right] + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right\} \in \mathrm{Dom}(\Omega^{-1}), \qquad \widetilde{0} \leq t < T.$$

Proof. Let us first assume that $c > 0$. Defining the nondecreasing positive function $z(t)$ by the right-hand side of (3.33)

$$z(t) = c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} [f(s)u(s)\psi(u(s)) + g(s)u(s)]ds,$$

we know

$$u(t) \leq \varphi^{-1}[z(t)] \tag{3.35}$$

and

$$D_1 D_2 \cdots D_n z(t) = [f(\widetilde{\alpha})u(\widetilde{\alpha})\psi(u(\widetilde{\alpha})) + g(\widetilde{\alpha})u(\widetilde{\alpha})]\alpha_1'\alpha_2' \cdots \alpha_n'. \tag{3.36}$$

Using (3.35), we have

$$\frac{D_1 D_2 \cdots D_n z(t)}{\varphi^{-1}(z(t))} \leq [f(\widetilde{\alpha})\psi(\varphi^{-1}(z(\alpha))) + g(\widetilde{\alpha})]\alpha_1'\alpha_2' \cdots \alpha_n'. \tag{3.37}$$

For

$$\begin{aligned} D_n\left(\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))}\right) & \\ &= \frac{D_1 D_2 \cdots D_n z(t)\varphi^{-1}(z(t)) - D_1 D_2 \cdots D_{n-1} z(t)(\varphi^{-1}(z(t)))' D_n z(t)}{(\varphi^{-1}(z(t)))^2}, \end{aligned} \tag{3.38}$$

using $D_1 D_2 \cdots D_{n-1} z(t) \geq 0$, $D_n z(t) \geq 0$, $(\varphi^{-1}(z(t)))' \geq 0$ in (3.38), we get

$$\begin{aligned} D_n\left(\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))}\right) &\leq \frac{D_1 D_2 \cdots D_n z(t)}{\varphi^{-1}(z(t))} \\ &\leq [f(\widetilde{\alpha})\psi(\varphi^{-1} z(\widetilde{\alpha})) + g(\widetilde{\alpha})]\alpha_1'\alpha_2' \cdots \alpha_n'. \end{aligned} \tag{3.39}$$

Fixing $t_1, \ldots, t_{n-1}$, setting $t_n = s_n$, integrating from 0 to t_n with respect to s_n yields

$$\begin{aligned} &\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))} \\ &\leq \int_0^{\alpha_n(t_n)} [f(\alpha_1(t_1), \ldots, \alpha_{n-1}(t_{n-1}), s_n)\psi(\varphi^{-1}(z(\alpha_1(t_1), \ldots, \alpha_{n-1}(t_{n-1}), s_n))) \\ &\qquad + g(\alpha_1(t_1), \ldots, \alpha_{n-1}(t_{n-1}), s_n)]\alpha_1'\alpha_2' \cdots \alpha_{n-1}' ds_n. \end{aligned} \tag{3.40}$$

Using the same method, we deduce that

$$\frac{D_1 z(t)}{\varphi^{-1}(z(t))} \leq \int_0^{\alpha_2(t_2)} \cdots \int_0^{\alpha_n(t_n)} [f(\alpha_1(t_1), s_2, \dots, s_n)\psi(\varphi(z(\alpha_1(t_1), s_2, \dots, s_n))) + g(\alpha_1(t_1), s_2, \dots, s_n)]\alpha_1' ds_n \dots ds_2. \tag{3.41}$$

Setting $t_1 = s_1$, and integrating it from 0 to t_1 with respect to s_1 yields

$$\Omega(z(t)) \leq \Omega(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)\psi(\varphi^{-1}(z(s))ds + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} g(s)ds, \tag{3.42}$$

Let $T_1 \leq T$ be arbitrary, we denote $p(T_1) = \Omega(c) + \int_0^{\widetilde{\alpha}(T_1)} g(s)ds$, from (3.42), we deduce that

$$\Omega(z(t)) \leq p(T_1) + \int_0^{\widetilde{\alpha}(t)} f(s)\psi[\varphi^{-1}z(s)]ds, \qquad \widetilde{0} \leq t \leq T_1 \leq T.$$

Now an application of Theorem 3.5 gives

$$z(t) \leq \Omega^{-1}\left\{G^{-1}\left[G(p(T_1)) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right]\right\}, \qquad \widetilde{0} \leq t \leq T_1 \leq T,$$

so

$$u(t) \leq \varphi^{-1}\left\{\Omega^{-1}\left[G^{-1}\left(G(p(T_1)) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right)\right]\right\}, \qquad \widetilde{0} \leq t \leq T_1 \leq T.$$

Taking $t = T_1$ in the above inequality, since T_1 is arbitrary, we can prove the desired inequality (3.34). □

If $c = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \to 0$.

Setting $f(t) = 0$, $n = 1$, we can obtain a retarded Ou-Iang inequality.

Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}_+^n$ and let c be a nonnegative constant. Moreover, let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(u) > 0$ on $(0, \infty)$ and $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on $\mathbb{R}_+$ $(i = 1, \dots, n)$. If

$$u^2(t) \leq c^2 + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} [f(s)u(s)\psi(u(s)) + g(s)u(s)]ds,$$

for $\widetilde{0} \leq t < T$, then

$$u(t) \leq \Omega^{-1}\left[\Omega\left(c + \frac{1}{2}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} g(s)ds\right) + \frac{1}{2}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right],$$

where

$$\Omega(\widetilde{z}) = \int_{\widetilde{z}_0}^{\widetilde{z}} \frac{ds}{\psi(s)} \qquad \widetilde{z} > \widetilde{z}_0,$$

Ω^{-1} is the inverse of Ω, and $T \in \mathbb{R}_+^n$ is chosen so that

$$\Omega\left(c + \frac{1}{2}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} g(s)ds\right) + \frac{1}{2}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds \in \mathrm{Dom}(\Omega^{-1}), \qquad \widetilde{0} \leq t < T.$$

Corollary 3.10. *Let u, f and g be nonnegative continuous functions defined on $\mathbb{R}_+^n$ and let c be a nonnegative constant. Moreover, let p, q be positive constants with $p \geq q$, $p \neq 1$. Let $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on $\mathbb{R}_+$ $(i = 1, \dots, n)$. If*

$$u^p(t) \leq c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} [f(s)u^q(s) + g(s)u(s)]ds, \qquad t \geq 0$$

for $\widetilde{0} \le t < T$ *then*

$$u(t) \le \begin{cases} \left(c^{(1-\frac{1}{p})} + \frac{p-1}{p}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} g(s)ds\right)^{\frac{p}{p-1}} \exp\left[\frac{1}{p}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right], & \text{when} \quad p = q; \\ \left[\left(c^{(1-\frac{1}{p})} + \frac{p-1}{p}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} g(s)ds\right)^{\frac{p-q}{p-1}} + \frac{p-q}{p}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds\right]^{\frac{1}{p-q}}, & \text{when} \quad p > q. \end{cases}$$

Theorem 3.11. *Let* u, f *and* g *be nonnegative continuous functions defined on* $\mathbb{R}_+^n$, *and let* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be an increasing function with* $\varphi(\infty) = \infty$ *and let* c *be a nonnegative constant. Moreover, let* $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $w_i(u) > 0$ $(i = 1, 2)$ *on* $(0, \infty)$, *and* $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing with* $\alpha_i(t_i) \le t_i$ $(i = 1, \dots, n)$. *If*

$$\varphi(u(t)) \le c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)u(s)w_1(u(s))ds + \int_{\widetilde{0}}^{t} g(s)u(s)w_2(u(s))ds, \tag{3.43}$$

for $\widetilde{0} \le t < T$, *then*

(i) *for the case* $w_2(u) \le w_1(u)$,

$$u(t) \le \varphi^{-1}\left\{\Omega^{-1}\left[G_1^{-1}\left(G_1(\Omega(c)) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds + \int_{\widetilde{0}}^{t} g(s)ds\right)\right]\right\}, \tag{3.44}$$

(ii) *for the case* $w_1(u) \le w_2(u)$,

$$u(t) \le \varphi^{-1}\left\{\Omega^{-1}\left[G_2^{-1}\left(G_2(\Omega(c)) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds + \int_{\widetilde{0}}^{t} g(s)ds\right)\right]\right\}, \tag{3.45}$$

where

$$\Omega(\widetilde{r}) = \int_{\widetilde{r_0}}^{\widetilde{r}} \frac{ds}{\varphi^{-1}(s)}, \quad \widetilde{r} \ge \widetilde{r_0} > 0,$$

$$G_i(\widetilde{z}) = \int_{\widetilde{z_0}}^{\widetilde{z}} \frac{ds}{w_i\{\varphi^{-1}[\Omega^{-1}(s)]\}}, \quad \widetilde{z} \ge \widetilde{z_0} > 0 \quad (i = 1, 2)$$

$\Omega^{-1}, \varphi^{-1}, G^{-1}$ *are respectively the inverse of* Ω, φ, G, *and* $T \in \mathbb{R}_+$ *is chosen so that*

$$G_i\left(\Omega(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds + \int_{\widetilde{0}}^{t} g(s)ds\right) \in \mathrm{Dom}(G_i^{-1}), \quad \widetilde{0} \le t \le T,$$

and

$$G_i^{-1}\left[G_i\left(\Omega(c) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds + \int_{\widetilde{0}}^{t} g(s)ds\right)\right] \in \mathrm{Dom}(\Omega^{-1}), \quad \widetilde{0} \le t \le T.$$

Proof. Let $c > 0$ and define the nonincreasing positive function $z(t)$ and make

$$z(t) = c + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)u(s)w_1(u(s))ds + \int_{\widetilde{0}}^{t} g(s)u(s)w_2(u(s))ds. \tag{3.46}$$

From inequality (3.43), we know

$$u(t) \le \varphi^{-1}[z(t)], \tag{3.47}$$

and

$$D_1 D_2 \cdots D_n z(t) = [f(\widetilde{\alpha})u(\widetilde{\alpha})w_1(u(\widetilde{\alpha}))\alpha_1'\alpha_2' \cdots \alpha_n' + g(t)u(t)w_2(u(t))]. \tag{3.48}$$

Using (3.47), we have

$$\frac{D_1D_2\cdots D_n z(t)}{\varphi^{-1}(z(t))} \le f(\widetilde{\alpha})w_1(u(\widetilde{\alpha}))\alpha_1'\alpha_2'\cdots\alpha_n' + g(t)w_2(u(t)). \tag{3.49}$$

For

$$D_n\left(\frac{D_1D_2\cdots D_{n-1}z(t)}{\varphi^{-1}(z(t))}\right) = \frac{D_1D_2\cdots D_n z(t)\varphi^{-1}(z(t)) - D_1D_2\cdots D_{n-1}z(t)(\varphi^{-1}(z(t)))'D_n z(t)}{(\varphi^{-1}(z(t)))^2}, \tag{3.50}$$

using $D_1D_2\cdots D_{n-1}z(t)\ge 0$, $(\varphi^{-1}(z(t)))'\ge 0$, $D_nz(t)\ge 0$ in (3.50), we get

$$D_n\left(\frac{D_1D_2\cdots D_{n-1}z(t)}{\varphi^{-1}(z(t))}\right) \le \frac{D_1D_2\cdots D_n z(t)}{\varphi^{-1}(z(t))} \le f(\widetilde{\alpha})\alpha_1'\alpha_2'\cdots\alpha_n' w_1(\varphi^{-1}(\widetilde{\alpha}(t))) + g(t)w_2(\varphi^{-1}(t)).$$

Fixing $t_1,\dots,t_{n-1}$, setting $t_n = s_n$, integrating from t_n to ∞, yields

$$\begin{aligned}\frac{D_1D_2\cdots D_{n-1}z(t)}{\varphi^{-1}(z(t))} &\le \int_0^{\alpha_n(t_n)} f(\alpha_1(t_1),\dots,\alpha_{n-1}(t_{n-1}),s_n)w_1(\varphi^{-1}(\alpha_1(t_1),\dots,\alpha_{n-1}(t_{n-1}),s_n))\alpha_1'\alpha_2'\cdots\alpha_{n-1}'ds_n \\ &\quad + \int_0^{t_n} g(t_1,\dots,t_{n-1},s_n)w_2(\varphi^{-1}(t_1,\dots,t_{n-1},s_n))ds_n.\end{aligned}$$

Deductively

$$\begin{aligned}\frac{D_1z(t)}{\varphi^{-1}(z(t))} &\le \int_0^{\alpha_2(t_2)}\cdots\int_0^{\alpha_n(t_n)} f(\alpha_1(t_1),s_2,\dots,s_n)w_1(\varphi^{-1}(\alpha_1(t_1),s_2,\dots,s_n))\alpha_1'ds_n\dots ds_2 \\ &\quad + \int_0^{t_2}\cdots\int_0^{t_n} g(t_1,s_2,\dots,s_n)w_2(\varphi^{-1}(t_1,s_2,\dots,s_n))ds_n\dots ds_2.\end{aligned} \tag{3.51}$$

Fixing $t_2,\dots,t_n$, setting $t_1 = s_1$, integrating from 0 to t_1 with respect to s_1 yields

$$\begin{aligned}\Omega(z(t)) \le \Omega(c) &+ \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s_1,\dots,s_n)w_1(\varphi^{-1}(z(s)) \\ &+ \int_{\widetilde{0}}^{t} g(s)w_2(\varphi^{-1}(z(s))ds, \qquad t\in\mathbb{R}_+^n.\end{aligned} \tag{3.52}$$

From Theorem 3.8, we obtain

$$z(t)\le \Omega^{-1}\left[G_1^{-1}\left(G_1(\Omega(c)) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} f(s)ds + \int_{\widetilde{0}}^{t} g(s)ds\right)\right],$$

using (3.47), we get the inequality (3.44).

If $c = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon\to 0$.
(ii) when $w_1(u)\le w_2(u)$.

The proof can be completed with suitable changes. □

4. SOME APPLICATIONS

Example 4.1. Consider the integral equation:

$$(4.1)\quad u^p(t_1,\dots,t_n) = f(t_1,\dots,t_n) + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} K(s_1,\dots,s_n)g(s_1,\dots,s_n,u(s_1,\dots,s_n))ds_1\dots ds_n,$$

where $f, K : \mathbb{R}_+^n \to \mathbb{R}$, $g : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $p > 0$ and $p \neq 1$is constant, $\widetilde{\alpha}_i(t) \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ is nondecreasing with $\alpha_i(t) \le t_i$ on $\mathbb{R}_+$ $(i = 1,\dots,n)$. In [8] B.G. Pachpatte studied the problem when $\alpha(t) = t$, $n = 1$. Here we assume that every solution under discussion exists on an interval $\mathbb{R}_+^n$. We suppose that the functions f, K, g in (4.1) satisfy the following conditions

$$(4.2)\quad \begin{gathered} |f(t_1,\dots,t_n)| \le c_1, \qquad |K(t_1,\dots,t_n)| \le c_2, \\ |g(t_1,\dots,t_n,u)| \le r(t_1,\dots,t_n)|u|^q + h(t_1,\dots,t_n)|u|, \end{gathered}$$

where $c_1,\ c_2$, are nonnegative constants, and $p \ge q > 0$, and $r : \mathbb{R}_+^n \to \mathbb{R}_+$, $h : \mathbb{R}_+^n \to \mathbb{R}_+$ are continuous functions. From (4.1) and using (4.2), we get

$$(4.3)\quad |u^p(t_1,\dots,t_n)| \le c_1 + \int_{\widetilde{0}}^{\widetilde{\alpha}(t)} [c_2 r(s_1,\dots,s_n)|u|^q + c_2 h(s_1,\dots,s_n)|u|ds_1\dots ds_n.$$

Now an application of Corollary 3.10 yields

$$|u(t)| \le \begin{cases} \left(c_1^{(1-\frac{1}{p})} + \frac{c_2(p-1)}{p}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} h(s)ds_1\dots ds_n\right)^{\frac{p-1}{p}} \exp\left[\frac{c_2}{p}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} r(s)ds_1\dots ds_n\right] & \text{when} \quad p = q, \\ \left[\left(c_1^{(1-\frac{1}{p})} + \frac{c_2(p-1)}{p}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} h(s)ds_1\dots ds_n\right)^{\frac{p-q}{p-1}} + \frac{c_2(p-q)}{p}\int_{\widetilde{0}}^{\widetilde{\alpha}(t)} r(s)ds_1\dots ds_n\right]^{\frac{1}{p-q}} & \text{when} \quad p > q. \end{cases}$$

If the integrals of $r(s), h(s)$ are bounded, then we can have the bound of the solution $u(t)$ of (4.1). Similarly, we can obtain many other kinds of estimates.

Example 4.2. Consider the partial delay differential equation:

$$(4.4)\quad \frac{\partial^2 u^p(x,y)}{\partial x_1 \partial x_2} = f(x,y,u(x,y),u(x-h_1(x),y-h_2(y))),$$

$$(4.5)\quad u^p(x,0) = a_1(x) \qquad u^p(0,y) = a_2(y)$$

$$(4.6)\quad a_1(0) = a_2(0) = 0, \qquad |a_1(x) + a_2(y)| \le c,$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R})$, $a_1 \in C^1(\mathbb{R}_+,\mathbb{R})$, $a_2 \in C^1(\mathbb{R}_+,\mathbb{R})$, c and p are nonnegative constants. $h_1 \in C^1(\mathbb{R}_+,\mathbb{R}_+)$, $h_2 \in C^1(\mathbb{R}_+,\mathbb{R}_+)$, such that

$$\begin{gathered} x - h_1(x) \ge 0, \qquad y - h_2(y) \ge 0, \\ h_1'(x) < 1, \qquad h_2'(y) < 1. \end{gathered}$$

Suppose that

$$(4.7)\quad |f(x,y,u,v)| \le a(x,y)|v|^q + b(x,y)|v|,$$

where $a, b \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ and let

$$M_1 = \max_{x \in R_+} \frac{1}{1 - h_1'(x)}, \qquad M_2 = \max_{y \in R_+} \frac{1}{1 - h_2'(y)}. \tag{4.8}$$

If $u(x, y)$ is any solution of (4.4) – (4.7), then

(i) if $p = q$, we have

$$|u(x,y)| \leq \left(c^{(1-\frac{1}{p})} + \frac{M_1 M_2 (p-1)}{p} \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \widetilde{b}(\sigma, \tau) d\sigma d\tau \right)^{\frac{p}{p-1}} \times \exp \left[\frac{M_1 M_2}{p} \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \widetilde{a}(\sigma, \tau) d\sigma d\tau \right] \tag{4.9}$$

(ii) if $p > q$, we have

$$|u(x,y)| \leq \left[\left(c^{(1-\frac{1}{p})} + \frac{M_1 M_2 (p-1)}{p} \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \widetilde{b}(\sigma, \tau) d\sigma d\tau \right)^{\frac{p-q}{p-1}} + \frac{M_1 M_2 (p-q)}{p} \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \widetilde{a}(\sigma, \tau) d\sigma d\tau \right]^{\frac{1}{p-q}}. \tag{4.10}$$

In which $\phi_1(x) = x - h_1(x),\ x \in \mathbb{R}_+^n,\ \phi_2(y) = y - h_2(y),\ y \in \mathbb{R}_+^n$ and

$$\widetilde{b}(\sigma, \tau) = b(\sigma + h_1(s), \tau + h_2(t)), \widetilde{a}(\sigma, \tau) = a(\sigma + h_1(s), \tau + h_2(t)),$$

for $\sigma, s, \tau, t \in \mathbb{R}_+^n$.

In fact, if $u(x, y)$ is a solution of (4.4) – (4.7), then it satisfies the equivalent integral equation:

$$[u(x,y)]^p = a_1(x) + a_2(y) + \int_0^x \int_0^y f(s, t, u(s,t), u(s - h_1(s), t - h_2(t))) dt ds. \tag{4.11}$$

for $x, y \in (\mathbb{R}_+^n \times \mathbb{R}_+^n, \mathbb{R})$.

Using (4.5), (4.7) in (4.11) and making the change of variables, we have

$$|u(x,y)|^p \leq c + M_1 M_2 \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \widetilde{a}(\sigma, \tau) |u(\sigma, \tau)|^q + \widetilde{b}(\sigma, \tau) |u(\sigma, \tau)| d\sigma d\tau. \tag{4.12}$$

Now a suitable application of the inequality in Corollary 3.10 to (4.12) yields (4.9) and (4.10).

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