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CONVEXITY OF WEIGHTED STOLARSKY MEANS

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ABSTRACT. We investigate monotonicity and logarithmic convexity properties of one-parameter family of means

$$F_h(r; a, b; x, y) = E(r, r+h; ax, by)/E(r, r+h; a, b)$$

where E is the Stolarsky mean. Some inequalities between classic means are obtained.

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1. INTRODUCTION

Extended mean values of positive numbers x, y introduced by Stolarsky in [6] are defined as

$$E(r,s;x,y) = \begin{cases} \left(\frac{r}{s}\frac{y^s-x^s}{y^r-x^r}\right)^{\frac{1}{s-r}} & sr(s-r)(x-y)\neq 0, \\ \left(\frac{1}{r}\frac{y^r-x^r}{\log y-\log x}\right)^{\frac{1}{r}} & r(x-y)\neq 0,\ s=0, \\ e^{-\frac{1}{r}}\left(\frac{y^{y^r}}{x^{x^r}}\right)^{\frac{1}{y^r-x^r}} & r=s,\ r(x-y)\neq 0, \\ \sqrt{xy} & r=s=0,\ x-y\neq 0, \\ x & x=y. \end{cases} \tag{1.1}$$

This mean is also called the Stolarsky mean.
In [9] the author extended the Stolarsky means to a four-parameter family of means by adding positive weights a, b:

$$F(r,s;a,b;x,y) = \frac{E(r,s;ax,by)}{E(r,s;a,b)}. \tag{1.2}$$

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From the continuity of E it follows that F is continuous in $\mathbb{R}^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$. Our goal in this paper is to investigate the logarithmic convexity of

$$F_h(r;a,b;x,y) = F(r,r+h;a,b;x,y). \tag{1.3}$$

In [1] Horst Alzer investigated the one-parameter mean

$$J(r) = J(r;x,y) = E(r,r+1;x,y) \tag{1.4}$$

and proved that for $x \neq y$, J is strictly log-convex for $r < -1/2$ and strictly log-concave for $r > -1/2$. He also proved that $J(r)J(-r) \le J^2(0)$. In [2] he obtained a similar result for the Lehmer means

$$L(r) = L(r;x,y) = \frac{x^{r+1} + y^{r+1}}{x^r + y^r}. \tag{1.5}$$

With an appropriate choice of parameters in (1.2) one can obtain both the one-parameter mean and the Lehmer mean. Namely,

$$J(r;x,y) = F(r,r+1;1,1;x,y)$$

and

$$L(r,x,y) = F(r,r+1;x,y;x,y).$$

Another example may be the mean created the same way from the Heronian mean

$$N(r;x,y) = F(r,r+1;\sqrt{x},\sqrt{y};x,y) = \frac{x^{r+1} + \sqrt{xy}^{r+1} + y^{r+1}}{x^r + \sqrt{xy}^r + y^r}. \tag{1.6}$$

The following monotonicity properties of weighted Stolarsky means have been established in [9]:

Property 1.1. *F increases in x and y.*

Property 1.2. *F increases in r and s if $(x-y)(a^2x-b^2y) > 0$ and decreases if $(x-y)(a^2x-b^2y) < 0$.*

Property 1.3. *F increases in a if $(x-y)(r+s) > 0$ and decreases if $(x-y)(r+s) < 0$, F decreases in b if $(x-y)(r+s) > 0$ and increases if $(x-y)(r+s) < 0$.*

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be symmetrically convex (concave) with respect to the point r_0 if f is convex (concave) in (r_0,∞) and for every $t > 0$ $f(r_0+t) + f(r_0-t) = 2f(r_0)$.

Definition 1.2. A function $f : \mathbb{R} \to \mathbb{R}_+$ is said to be symmetrically log-convex (log-concave) with respect to the point r_0 if $\log f$ is symmetrically convex (concave) w.r.t. r_0.

For symmetrically log-convex functions the symmetry condition reads $f(r_0+t)f(r_0-t) = f^2(r_0)$. We shall recall now two properties of convex functions.

Property 1.4. *If f is convex (concave) then for $h > 0$ the function $g(t) = f(t+h) - f(t)$ is increasing (decreasing). For $h < 0$ the monotonicity of g reverses.*
For log-convex f the same holds for $g(t) = f(t+h)/f(t)$.

Property 1.5. *If f is convex (concave) then for arbitrary x the function $h_x(t) = f(x-t) + f(x+t)$ is increasing (decreasing) in $(0,\infty)$. For log-convex f the same holds for $h_x(t) = f(x-t)f(x+t)$.*

The property 1.5 holds also for symmetrically convex (concave) functions:

Lemma 1.6. *Let f be symmetrically convex w.r.t. r_0, and let $x > r_0$. Then the function $h_x(t) = f(x-t) + f(x+t)$ is increasing (decreasing) in $(0,\infty)$. If $x < r_0$ then $h_x(t)$ decreases.*
For f symmetrically concave the monotonicity of h_x is reverse.
For the case where f is symmetrically log-convex (log-concave) $h_x(t) = f(x+t)f(x-t)$ is monotone accordingly.

Proof. We shall prove the lemma for f symmetrically convex and $x > r_0$. For $x < r_0$ or f symmetrically convex the proofs are similar.
Consider two cases:

- $0 < t < x - r_0$. In this case $h_x(t)$ is increasing by Property 1.5.
- $t > x - r_0$. Now $h_x(t) = f(x+t) + f(x-t) = 2f(r_0) + f(x+t) - f(t-x+2r_0)$ increases by Property 1.4 because $t - x + 2r_0 > r_0$ and $(x+t) - (t-x+2r_0) > 0$.

□

2. MAIN RESULT

It is obvious that the monotonicity of F_h matches that of F. The main result consists of the following theorem:

Theorem 2.1. *If $(x-y)(a^2x - b^2y) > 0$ (resp. < 0) then $F_h(r)$ is symmetrically log-concave (resp. log-convex) with respect to the point $-h/2$).*

To prove it we need the following

Lemma 2.2. *Let*

$$g(t,A,B) = \frac{A^t \log^2 A}{(A^t-1)^2} - \frac{B^t \log^2 B}{(B^t-1)^2}.$$

Then

(1) $g(t,A,B) = g(\pm t, A^{\pm 1}, B^{\pm 1})$,
(2) g *is increasing in* t *on* $(0,\infty)$ *if* $\log^2 A - \log^2 B > 0$ *and decreasing otherwise.*

Proof. (1) becomes obvious when we write

$$g(t,A,B) = \frac{\log^2 A}{A^t - 2 + A^{-t}} - \frac{\log^2 B}{B^t - 2 + B^{-t}}.$$

From (1) if follows that replacing A, B with A^{-1}, B^{-1} if necessary we may assume that $A, B > 1$. In this case $\operatorname{sgn}(\log^2 A - \log^2 B) = \operatorname{sgn}(A^t - B^t)$.

$$\begin{aligned}\frac{\partial g}{\partial t} &= -\frac{A^t(A^t+1)\log^3 A}{(A^t-1)^3} + \frac{B^t(B^t+1)\log^3 B}{(B^t-1)^3}\\ &= -\frac{1}{t^3}(\phi(A^t) - \phi(B^t)) = -\frac{1}{t^3}(A^t - B^t)\phi'(\xi),\end{aligned}$$

where $\xi > 1$ lies between A^t and B^t and

$$\phi(u) = \frac{u(u+1)\log^3 u}{(u-1)^3}.$$

To complete the proof it is enough to show that $\phi'(u) < 0$ for $u > 1$.

$$\phi'(u) = \frac{(u^2+4u+1)\log^2 u}{(u-1)^4}\left[\frac{3(u^2-1)}{u^2+4u+1} - \log u\right],$$

so the sign of ϕ' is the same as the sign of $\psi(u) = \frac{3(u^2-1)}{u^2+4u+1} - \log u$. But $\psi(1) = 0$ and $\psi'(u) = -(u-1)^4/u(u^2+4u+1)^2 < 0$, so $\phi(u) < 0$. □

Proof of Theorem 2.1. First of all note that

$$\log^2 \frac{ax}{by} - \log^2 \frac{a}{b} = \log \frac{x}{y} \log \frac{a^2x}{b^2y}$$

and because $\operatorname{sgn}(x-y) = \operatorname{sgn} \log \frac{x}{y}$ we see that

$$\operatorname{sgn}(x-y)(a^2x - b^y) = \operatorname{sgn}\left(\log^2 \frac{ax}{by} - \log^2 \frac{a}{b}\right). \tag{2.1}$$

Let $A = \frac{ax}{by}$ and $B = \frac{a}{b}$. Suppose that $A, B \neq 1$ (in other cases we use a standard continuity argument). $F_h(r)$ can be written as

$$F_h(r) = y\left(\frac{A^{r+h}-1}{B^{r+h}-1}\Big/\frac{A^r-1}{B^r-1}\right)^{\frac{1}{h}},$$

We show symmetry by performing simple calculations:

$$\begin{aligned} & F_h^h(-h/2-r)F_h^h(-h/2+r) \\ & = y^{2h}\frac{A^{h/2-r}-1}{B^{h/2-r}-1}\cdot\frac{B^{-h/2-r}-1}{A^{-h/2-r}-1}\cdot\frac{A^{h/2+r}-1}{B^{h/2+r}-1}\cdot\frac{B^{-h/2+r}-1}{A^{-h/2+r}-1} \\ & = y^{2h}\frac{B^{-h}}{A^{-h}}\cdot\frac{A^{h/2-r}-1}{B^{h/2-r}-1}\cdot\frac{1-B^{h/2+r}}{1-A^{h/2+r}}\cdot\frac{A^{h/2+r}-1}{B^{h/2+r}-1}\cdot\frac{1-B^{h/2-r}}{1-A^{h/2-r}} \\ & = y^{2h}\left(\frac{x}{y}\right)^h = (xy)^h = F_h^{2h}(-h/2). \end{aligned} \tag{2.2}$$

Differentiating twice we obtain

$$\begin{aligned} \frac{d^2}{dr^2}\log F_h(r) & = \frac{g(r,A,B)-g(r+h,A,B)}{h} \\ & = \frac{g(|r|,A,B)-g(|r+h|,A,B)}{h} \quad \text{(by Lemma 2.2 (1))}, \end{aligned}$$

hence by Lemma 2.2 (2)

$$\begin{aligned} \operatorname{sgn}\frac{d^2}{dr^2}\log F_h(r) & = \operatorname{sgn} h(|r|-|r+h|)(\log^2 A - \log^2 B) \\ & = \operatorname{sgn}(r+h/2)(x-y)(a^2x-b^2y). \end{aligned}$$

The last equation follows from (2.1) and from the fact that the inequality $|r| < |r+h|$ is valid if and only if $r > -h/2$ and $h > 0$ or $r < -h/2$ and $h < 0$. □

The following theorem is an immediate consequence of Theorem 2.1 and Lemma 1.6.

Theorem 2.3. *If* $(x-y)(a^2x-b^2y)(r_0+h/2) > 0$ *then the function*

$$\Phi(t) = F_h(r_0-t)F_h(r_0+t)$$

is decreasing in $(0,\infty)$. *In particular for every real* t

$$F_h(r_0-t)F_h(r_0+t) \le F_h^2(r_0). \tag{2.3}$$

If $(x-y)(a^2x-b^2y)(r_0+h/2) < 0$ *then* $\Phi(t)$ *is increasing in* $(0,\infty)$. *In particular for every real* t

$$F_h(r_0-t)F_h(r_0+t) \ge F_h^2(r_0). \tag{2.4}$$

The following corollaries are immediate consequences of Theorems 2.1 and 2.3:

Corollary 2.4. *For $x \neq y$ the one-parameter mean $J(r)$ defined by* (1.4) *is log-convex for $r < -1/2$ and log-concave for $r > -1/2$. If $r_0 > -1/2$ then for all real t, $J(r_0 - t)J(r_0 + t) \leq J^2(r_0)$. For $r_0 < -1/2$ the inequality reverses.*

Proof. $J(r; x, y) = F_1(r; 1, 1; x, y)$. □

Corollary 2.5. *For $x \neq y$ the Lehmer mean $L(r)$ defined by* (1.5) *is log-convex for $r < -1/2$ and log-concave for $r > -1/2$. If $r_0 > -1/2$ then for all real t, $L(r_0 - t)L(r_0 + t) \leq L^2(r_0)$. For $r_0 < -1/2$ the inequality reverses.*

Proof. $L(r; x, y) = F_1(r; x, y; x, y)$. □

Corollary 2.6. *For $x \neq y$ the mean $N(r)$ defined by* (1.6) *is log-convex for $r < -1/2$ and log-concave for $r > -1/2$. If $r_0 > -1/2$ then for all real t, $N(r_0 - t)N(r_0 + t) \leq N^2(r_0)$. For $r_0 < -1/2$ the inequality reverses.*

Proof. $N(r; x, y) = F_1(r; \sqrt{x}, \sqrt{y}; x, y)$. □

3. APPLICATION

In this section we show some inequalities between classic means:

Power means	$A_r = A_r(x, y) = \left(\frac{x^r + y^r}{2}\right)^{\frac{1}{r}}$,
Harmonic mean	$H = A_{-1}(x, y) = \frac{2xy}{x + y}$,
Geometric mean	$G = A_0(x, y) = \sqrt{xy}$,
Logarithmic mean	$L = L(x, y) = \frac{x - y}{\log x - \log y}$,
Heronian mean	$N = N(x, y) = \frac{x + \sqrt{xy} + y}{3}$,
Arithmetic mean	$A = A_1(x, y) = \frac{x + y}{2}$,
Centroidal mean	$T = T(x, y) = \frac{2}{3}\frac{x^2 + xy + y^2}{x + y}$,
Root-mean-square	$R = A_2(x, y) = \sqrt{\frac{x^2 + y^2}{2}}$,
Contrharmonic mean	$C = C(x, y) = \frac{x^2 + y^2}{x + y}$.

Corollary 3.1 (Tung-Po Lin inequality [4])**.**

$$L \leq A_{1/3}.$$

Proof. By Theorem 2.3

$$F_{1/3}(0; 1, 1, ; x, y)F_{1/3}(2/3; 1, 1; x, y) \leq F^2_{1/3}(1/3; 1, 1; x, y)$$

or

$$\left(3\frac{\sqrt[3]{x} - \sqrt[3]{y}}{\log x - \log y}\right)^3 \left(\frac{2}{3}\frac{x - y}{\sqrt[3]{x^2} - \sqrt[3]{y^2}}\right)^3 \leq \left(\frac{1}{2}\frac{\sqrt[3]{x^2} - \sqrt[3]{y^2}}{\sqrt[3]{x} - \sqrt[3]{y}}\right)^6.$$

Simplifying we obtain

$$L^3(x, y) \leq A^3_{1/3}(x, y).$$

□

Inequalities in the table below can be shown the same way as above by an appropriate choice of parameters in (2.3) and (2.4).

No	Inequality	h	r_0	t	a	b
1	$L^2 \geq GN$	$1/2$	0	1	1	1
2	$L^2 \geq HT$	1	0	2	1	1
3	$A_{1/2}^2 \geq AG$	$1/2$	0	$1/2$	x	y
4	$A_{1/2}^2 \geq LN$	$1/2$	$1/2$	$1/2$	1	1
5	$N^2 \geq AL$	1	$1/2$	$1/2$	1	1
6	$A^2 \geq LT$	1	1	1	1	1
7	$A^2 \geq CH$	1	0	1	x	y
8	$LN \geq AG$	$1/2$	$1/2$	1	1	1
9	$GN \geq HT$	1	-1	$1/2$	x	y
10	$AN \geq TG$	$1/2$	0	1	x	y
11	$LT \geq HC$	1	1	2	1	1
12	$TA \geq NR$	1	$1/2$	$1/2$	x	y
13	$L^3 \geq AG^2$	1	0	1	1	1
14	$L^3 \geq GA_{1/2}^2$	$1/2$	$-1/2$	$1/2$	1	1
15	$N^3 \geq AA_{1/2}^2$	$1/2$	1	$1/2$	1	1
16	$T^3 \geq AR^2$	1	2	1	1	1
17	$LN^2 \geq G^2T$	1	$1/2$	$3/2$	1	1

Note that 4 is stronger than 3 (due to inequality 8), 14 is stronger than 13 (due to 3). Also, 1 is stronger than 2 because of 9.

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