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SOME RETARDED NONLINEAR INTEGRAL INEQUALITIES IN TWO VARIABLES AND APPLICATIONS

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ABSTRACT. In this paper, some retarded nonlinear integral inequalities in two variables with more than one distinct nonlinear term are established. Our results are also applied to show the boundedness of the solutions of certain partial differential equations.

Key words and phrases: Integral inequality, Nonlinear, Two variables, Retarded.

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1. Introduction

The Gronwall-Bellman integral inequality plays an important role in the qualitative analysis of the solutions of differential and integral equations. During the past few years, many retarded inequalities have been discovered (see in [1, 2, 4, 5, 6, 10, 11]). Lipovan [4] investigated the following retarded inequality

$$u(t) \leq a + \int_{b(t_0)}^{b(t)} f(s)w(u(s))ds, \qquad t_0 \leq t \leq t_1, \tag{1.1}$$

and Agarwal et al. [6] generalized (1.1) to a more general case as follows

$$u(t) \leq a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} f_i(s)w_i(u(s))ds, \qquad t_0 \leq t \leq t_1. \tag{1.2}$$

Recently, many people such as Wang [10], Cheung [9] and Dragomir [8] established some new integral inequalities involving functions of two independent variables and Zhao et al. [11] also established advanced integral inequalities.

The purpose of this paper, motivated by the works of Agarwal [6], Cheung [9] and Zhao [11], is to discuss more general integral inequalities with n nonlinear terms

$$u(x,y) \leq a(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} f_i(x,y,s,t)w_i(u(s,t))dtds \tag{1.3}$$

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and

$$u(x,y) \le a(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} f_i(x,y,s,t) w_i(u(s,t)) dt ds. \tag{1.4}$$

Our results can be used more effectively to study the boundedness and uniqueness of the solutions of certain partial differential equations. Moreover, at the end of this paper, an example is presented to show the applications of our results.

2. Statement of Main Results

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. $D_1 z(x,y)$ and $D_2 z(x,y)$ denote the first-order partial derivatives of $z(x,y)$ with respect to x and y respectively.

As in [6], define $w_1 \propto w_2$ for $w_1, w_2 : A \subset \mathbb{R} \to \mathbb{R} \backslash \{0\}$ if $\frac{w_2}{w_1}$ is nondecreasing on A. Assume that

- (B_1) $w_i(u)$ $(i = 1, \dots, n)$ is a nonnegative, nondecreasing and continuous function for $u \in \mathbb{R}_+$ with $w_i(u) > 0$ for $u > 0$ such that $w_1 \propto w_2 \propto \cdots \propto w_n$;
- (B_2) $a(x,y)$ is a nonnegative and continuous function for $x, y \in \mathbb{R}_+$;
- (B_3) $f_i(x,y,s,t)$ $(i = 1, \dots, n)$ is a continuous and nonnegative function for $x, y, s, t \in \mathbb{R}_+$.

Take the notation $W_i(u) := \int_{u_i}^{u} \frac{dz}{w_i(z)}$ for $u \ge u_i$, where $u_i > 0$ is a given constant. Clearly, W_i is strictly increasing, so its inverse W_i^{-1} is well defined, continuous and increasing in its corresponding domain.

Theorem 2.1. *Under the assumptions (B_1), (B_2) and (B_3), suppose $a(x,y)$ and $f_i(x,y,s,t)$ are bounded in $y \in \mathbb{R}_+$. Let $\alpha_i(x)$, $\beta_i(y)$ be nonnegative, continuously differentiable and nondecreasing functions with $\alpha_i(x) \le x$ and $\beta_i(y) \ge y$ on $\mathbb{R}_+$ for $i = 1, 2, \dots, n$. If $u(x,y)$ is a continuous and nonnegative function satisfying (1.3), then*

$$u(x,y) \le W_n^{-1}\left[W_n(b_n(x,y)) + \int_{\alpha_n(0)}^{\alpha_n(x)} \int_{\beta_n(y)}^{\infty} \tilde{f}_n(x,y,s,t) dt ds \right] \tag{2.1}$$

for all $0 \le x \le x_1, y_1 \le y < \infty$, where $b_n(x,y)$ is determined recursively by

$$b_1(x,y) = \sup_{0 \le \tau \le x} \sup_{y \le \mu < \infty} a(\tau, \mu),$$

$$b_{i+1}(x,y) = W_i^{-1}\left[W_i(b_i(x,y)) + \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(x,y,s,t) dt ds \right], \tag{2.2}$$

$$\tilde{f}_i(x,y,s,t) = \sup_{0 \le \tau \le x} \sup_{y \le \mu < \infty} f_i(\tau, \mu, s, t),$$

$W_1(0) := 0$, and $x_1, y_1 \in \mathbb{R}_+$ are chosen such that

$$W_i(b_i(x_1,y_1)) + \int_{\alpha_i(0)}^{\alpha_i(x_1)} \int_{\beta_i(y_1)}^{\infty} \tilde{f}_i(x,y,s,t) dt ds \le \int_{u_i}^{\infty} \frac{dz}{w_i(z)} \tag{2.3}$$

for $i = 1, \dots, n$.

The proof of Theorem 2.1 will be given in the next section.

Remark 1. As in [6], different choices of u_i in W_i do not affect our results. If all w_i $(i = 1, \dots, n)$ satisfy $\int_{u_i}^{\infty} \frac{dz}{w_i(z)} = \infty$, then (2.1) is true for all $x, y \in \mathbb{R}_+$.

Remark 2. As in [10], if $w_i(u)$ $(i = 1, \dots, n)$ are continuous functions on $\mathbb{R}_+$ and positive on $(0, \infty)$ but the sequence of $\{w_i(u)\}$ does not satisfy $w_1 \propto w_2 \propto \cdots \propto w_n$, we can use a technique of monotonization of the sequence of functions $w_i(u)$, calculated by

$$\tilde{w}_1(u) := \max_{\theta \in [0,u]} w_1(\theta),$$

$$\tilde{w}_{i+1}(u) := \max_{\theta \in [0,u]} \left\{ \frac{w_{i+1}(\theta)}{\tilde{w}_i(\theta)} \right\} \tilde{w}_i(u), \qquad i = 1, \dots, n-1. \tag{2.4}$$

Clearly, $\tilde{w}_i(u) \geq w_i(u)$ $(i = 1, \dots, n)$. (1.3) and (1.4) can also become

$$u(x,y) \leq a(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} f_i(x,y,s,t)\tilde{w}_i(u(s,t))dtds \tag{2.5}$$

and

$$u(x,y) \leq a(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} f_i(x,y,s,t)\tilde{w}_i(u(s,t))dtds, \tag{2.6}$$

where the function sequence $\{\tilde{w}_i(u)\}$ satisfies the assumption (B_1).

Theorem 2.2. *Under the assumptions (B_1), (B_2) and (B_3), suppose $a(x,y)$ and $f_i(x,y,s,t)$ are bounded in $x, y \in \mathbb{R}_+$. Let $\alpha_i(x)$, $\beta_i(y)$ be nonnegative, continuously differentiable and nondecreasing functions with $\alpha_i(x) \geq x$ and $\beta_i(y) \geq y$ on $\mathbb{R}_+$ for $i = 1, 2, \dots, n$. If $u(x,y)$ is a continuous and nonnegative function satisfying (1.4), then*

$$u(x,y) \leq W_n^{-1}\left[W_n(b_n(x,y)) + \int_{\alpha_n(x)}^{\infty} \int_{\beta_n(y)}^{\infty} \hat{f}_n(x,y,s,t)dtds\right] \tag{2.7}$$

for all $\hat{x}_1 \leq x < \infty$, $\hat{y}_1 \leq y < \infty$, where $b_n(x,y)$ is determined recursively by

$$b_1(x,y) = \sup_{x \leq \tau < \infty} \sup_{y \leq \mu < \infty} a(\tau, \mu),$$

$$b_{i+1}(x,y) = W_i^{-1}\left[W_i(b_i(x,y)) + \int_{\alpha_i(x)}^{\infty} \int_{\beta_i(y)}^{\infty} \hat{f}_i(x,y,s,t)dtds\right],$$

$$\hat{f}_i(x,y,s,t) = \sup_{x \leq \tau < \infty} \sup_{y \leq \mu < \infty} f_i(\tau, \mu, s, t), \tag{2.8}$$

$W_1(0) := 0$*, and $\hat{x}_1, \hat{y}_1 \in \mathbb{R}_+$ are chosen such that*

$$W_i(b_i(\hat{x}_1, \hat{y}_1)) + \int_{\alpha_i(\hat{x}_1)}^{\infty} \int_{\beta_i(\hat{y}_1)}^{\infty} \hat{f}_i(x,y,s,t)dtds \leq \int_{u_i}^{\infty} \frac{dz}{w_i(z)} \tag{2.9}$$

for $i = 1, \dots, n$.

The proof is similar to the argument in the proof of Theorem 2.1 with suitable modifications. In the next section, we omit its proof.

3. PROOF OF THEOREM 2.1

From the assumptions, we know that $b_1(x,y)$ and $\tilde{f}_i(x,y,s,t)$ are well defined. Moreover, $\tilde{a}(x,y)$ and $\tilde{f}_i(x,y,s,t)$ are nonnegative, nondecreasing in x and nonincreasing in y and satisfy $b_1(x,y) \geq a(x,y)$ and $\tilde{f}_i(x,y,s,t) \geq f_i(x,y,s,t)$ for each $i = 1, \dots, n$.

We first discuss the case $a(x,y) > 0$ for all $x, y \in \mathbb{R}_+$. From (1.3), we have

$$u(x,y) \leq b_1(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(x,y,s,t)w_i(u(s,t))dtds. \tag{3.1}$$

Choose arbitrary $\tilde{x}_1, \tilde{y}_1$ such that $0 \le \tilde{x}_1 \le x_1, y_1 \le \tilde{y}_1 < \infty$. From (3.1), we obtain

$$u(x,y) \le b_1(\tilde{x}_1, \tilde{y}_1) + \sum_{i=1}^{n} \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) w_i(u(s,t)) dt ds \tag{3.2}$$

for all $0 \le x \le \tilde{x}_1 \le x_1, y_1 \le \tilde{y}_1 \le y < \infty$.

We claim that

$$u(x,y) \le W_n^{-1} \left[W_n(\tilde{b}_n(\tilde{x}_1, \tilde{y}_1, x, y)) + \int_{\alpha_n(0)}^{\alpha_n(x)} \int_{\beta_n(y)}^{\infty} \tilde{f}_n(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \tag{3.3}$$

for all $0 \le x \le \min\{\tilde{x}_1, x_2\}, \max\{\tilde{y}_1, y_2\} \le y < \infty$, where

$$\tilde{b}_1(\tilde{x}_1, \tilde{y}_1, x, y) = b_1(\tilde{x}_1, \tilde{y}_1),$$

$$\tilde{b}_{i+1}(\tilde{x}_1, \tilde{y}_1, x, y) = W_i^{-1} \left[W_i(\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, x, y)) + \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \tag{3.4}$$

for $i = 1, \dots, n-1$ and $x_2, y_2 \in \mathbb{R}_+$ are chosen such that

$$W_i(\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, x_2, y_2)) + \int_{\alpha_i(0)}^{\alpha_i(x_2)} \int_{\beta_i(y_2)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \le \int_{u_i}^{\infty} \frac{dz}{w_i(z)} \tag{3.5}$$

for $i = 1, \dots, n$.

Note that we may take $x_2 = x_1$ and $y_2 = y_1$. In fact, $\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, x, y)$ and $\tilde{f}_i(\tilde{x}_1, \tilde{y}_1, x, y)$ are nondecreasing in $\tilde{x}_1$ and nonincreasing in $\tilde{y}_1$ for fixed x, y. Furthermore, it is easy to check that $\tilde{b}_i(\tilde{x}_1, \tilde{y}_1, \tilde{x}_1, \tilde{y}_1) = b_i(\tilde{x}_1, \tilde{y}_1)$ for $i = 1, \dots, n$. If x_2 and y_2 are replaced by x_1 and y_1 on the left side of (3.5) respectively, from (2.3) we have

$$\begin{aligned} W_i(\tilde{b}_i(\tilde{x}_1, &\tilde{y}_1, x_1, y_1)) + \int_{\alpha_i(0)}^{\alpha_i(x_1)} \int_{\beta_i(y_1)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \\ &\le W_i(\tilde{b}_i(x_1, y_1, x_1, y_1)) + \int_{\alpha_i(0)}^{\alpha_i(x_1)} \int_{\beta_i(y_1)}^{\infty} \tilde{f}_i(x_1, y_1, s, t) dt ds \\ &= W_i(b_i(x_1, y_1)) + \int_{\alpha_i(0)}^{\alpha_i(x_1)} \int_{\beta_i(y_1)}^{\infty} \tilde{f}_i(x_1, y_1, s, t) dt ds \\ &\le \int_{u_i}^{\infty} \frac{dz}{w_i(z)}. \end{aligned}$$

Thus, we can take $x_2 = x_1, y_2 = y_1$.

In the following, we will use mathematical induction to prove (3.3).

For $n = 1$, let

$$z(x,y) = b_1(\tilde{x}_1, \tilde{y}_1) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) w_1(u(s,t)) dt ds.$$

Then $z(x,y)$ is differentiable, nonnegative, nondecreasing for $x \in [0, \tilde{x}_1]$ and nonincreasing for $y \in [\tilde{y}_1, \infty]$ and $z(0,y) = z(x,\infty) = b_1(\tilde{x}_1, \tilde{y}_1)$. From (3.2), we have

$$u(x,y) \le z(x,y). \tag{3.6}$$

Considering $\alpha_1(x) \le x$ and $\alpha_1'(x) \ge 0$ for $x \in \mathbb{R}_+$, we have

$$\begin{aligned} D_1 z(x,y) &= \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) w_1(u(\alpha_1(x), t)) dt \alpha_1'(x) \\ &\le \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) w_1(z(\alpha_1(x), t)) dt \alpha_1'(x) \\ &\le w_1(z(x,y)) \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) dt \alpha_1'(x). \end{aligned} \tag{3.7}$$

Since w_1 is nondecreasing and $z(x,y) > 0$, we get

$$\frac{D_1(z(x,y))}{w_1(z(x,y))} \le \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) dt \alpha_1'(x). \tag{3.8}$$

Integrating both sides of the above inequality from 0 to x, we obtain

$$\begin{aligned} W_1(z(x,y)) &\le W_1(z(0,y)) + \int_0^x \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(s), t) \alpha_1'(s) dt ds \\ &= W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds. \end{aligned} \tag{3.9}$$

Thus the monotonicity of W_1^{-1} and (3.5) imply

$$\begin{aligned} u(x,y) &\le z(x,y) \\ &\le W_1^{-1}\left[W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right], \end{aligned}$$

namely, (3.3) is true for $n = 1$.

Assume that (3.3) is true for $n = m$. Consider

$$u(x,y) \le b_1(\tilde{x}_1, \tilde{y}_1) + \sum_{i=1}^{m+1} \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) w_i(u(s,t)) dt ds$$

for all $0 \le x \le \tilde{x}_1, \tilde{y}_1 \le y < \infty$. Let

$$z(x,y) = b_1(\tilde{x}_1, \tilde{y}_1) + \sum_{i=1}^{m+1} \int_{\alpha_i(0)}^{\alpha_i(x)} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, s, t) w_i(u(s,t)) dt ds.$$

Then $z(x,y)$ is differentiable, nonnegative, nondecreasing for $x \in [0, \tilde{x}_1]$ and nonincreasing for $y \in [\tilde{y}_1, \infty]$. Obviously, $z(0,y) = z(x,0) = b_1(\tilde{x}_1, \tilde{y}_1)$ and $u(x,y) \le z(x,y)$. Since w_1 is nondecreasing and $z(x,y) > 0$, noting that $\alpha_i(x) \le x$ and $\alpha_i'(x) \ge 0$ for $x \in \mathbb{R}_+$, we have

$$\begin{aligned} \frac{D_1(z(x,y))}{w_1(z(x,y))} &\le \frac{\sum_{i=1}^{m+1} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, \alpha_i(x), t) w_i(u(\alpha_i(x), t)) dt \alpha_i'(x)}{w_1(z(x,y))} \\ &\le \frac{\sum_{i=1}^{m+1} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, \alpha_i(x), t) w_i(z(\alpha_i(x), t)) dt \alpha_i'(x)}{w_1(z(x,y))} \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) dt \alpha_1^{'}(x) \\
&\qquad + \sum_{i=2}^{m+1} \int_{\beta_i(y)}^{\infty} \tilde{f}_i(\tilde{x}_1, \tilde{y}_1, \alpha_i(x), t) \phi_i(z(\alpha_i(x), t)) dt \alpha_i^{'}(x) \\
&\leq \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(x), t) dt \alpha_1^{'}(x) \\
&\qquad + \sum_{i=1}^{m} \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, \alpha_{i+1}(x), t) \phi_{i+1}(z(\alpha_{i+1}(x), t)) dt \alpha_{i+1}^{'}(x),
\end{aligned}
$$

where $\phi_{i+1}(u) = \frac{w_{i+1}(u)}{w_1(u)}$, $i = 1, \dots, m$. Integrating the above inequality from 0 to x, we obtain

$$
\begin{aligned}
&W_1(z(x,y)) \\
&\leq W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_0^x \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, \alpha_1(s), t) \alpha_1^{'}(s) dt ds \\
&\qquad + \sum_{i=1}^{m} \int_0^x \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, \alpha_{i+1}(s), t) \phi_{i+1}(z(\alpha_{i+1}(s), t)) \alpha_{i+1}^{'}(s) dt ds \\
&\leq W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \\
&\qquad + \sum_{i=1}^{m} \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x)} \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t) \phi_{i+1}(z(s,t)) dt ds,
\end{aligned}
$$

or

$$
\xi(x,y) \leq c_1(x,y) + \sum_{i=1}^{m} \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x)} \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t) \phi_{i+1}(W_1^{-1}(\xi(s,t))) dt ds
$$

for $0 \leq x \leq \tilde{x}_1$, $\tilde{y}_1 \leq y < \infty$. This is the same as (3.3) for $n = m$, where $\xi(x,y) = W_1(z(x,y))$ and

$$
c_1(x,y) = W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t) dt ds.
$$

From the assumption (B_1), each $\phi_{i+1}(W_1^{-1}(u))$ $(i = 1, \dots, m)$ is continuous and nondecreasing for u. Moreover, $\phi_2(W_1^{-1}) \propto \phi_3(W_1^{-1}) \propto \cdots \propto \phi_{m+1}(W_1^{-1})$. By the inductive assumption, we have

$$
\xi(x,y) \leq \Phi_{m+1}^{-1} \left[\Phi_{m+1}(c_m(x,y)) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right] \tag{3.10}
$$

for all $0 \leq x \leq \min\{\tilde{x}_1, x_3\}, \max\{\tilde{y}_1, y_3\} \leq y < \infty$, where $\Phi_{i+1}(u) = \int_{\tilde{u}_{i+1}}^{u} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))}$, $u > 0$, $\tilde{u}_{i+1} = W_1(u_{i+1})$, Φ_{i+1}^{-1} is the inverse of Φ_{i+1}, $i = 1, \dots, m$,

$$
c_{i+1}(x,y) = \Phi_{i+1}^{-1} \left[\Phi_{i+1}(c_i(x,y)) + \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x)} \int_{\beta_{i+1}(y)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t) dt ds \right], \quad i = 1, \dots, m,
$$

and $x_3, y_3 \in \mathbb{R}_+$ are chosen such that

$$\Phi_{i+1}(c_i(x_3, y_3)) + \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x_3)} \int_{\beta_{i+1}(y_3)}^{\infty} \tilde{f}_{i+1}(\tilde{x}_1, \tilde{y}_1, s, t)dtds \leq \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))} \tag{3.11}$$

for $i = 1, \dots, m$.

Note that

$$\begin{aligned}\Phi_i(u) &= \int_{\tilde{u}_i}^{u} \frac{dz}{\phi_i(W_1^{-1}(z))} \\ &= \int_{W_1(u_i)}^{u} \frac{w_1(W_1^{-1}(z))dz}{w_i(W_1^{-1}(z))} \\ &= \int_{u_i}^{W_1^{-1}(u)} \frac{dz}{w_i(z)} = W_i \circ W_1^{-1}(u), \qquad i = 2, \dots, m+1.\end{aligned}$$

From (3.10), we have

$$\begin{aligned}&u(x, y) \\ &\leq z(x, y) = W_1^{-1}(\xi(x, y)) \\ &\leq W_{m+1}^{-1}\left[W_{m+1}(W_1^{-1}(c_m(x, y))) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty)} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t)dtds\right] \end{aligned}\tag{3.12}$$

for all $0 \leq x \leq \min\{\tilde{x}_1, x_3\}, \max\{\tilde{y}_1, y_3\} \leq y < \infty$. Let $\tilde{c}_i(x, y) = W_1^{-1}(c_i(x, y))$. Then,

$$\begin{aligned}\tilde{c}_1(x, y) &= W_1^{-1}(c_1(x, y)) \\ &= W_1^{-1}\left[W_1(b_1(\tilde{x}_1, \tilde{y}_1)) + \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^{\infty} \tilde{f}_1(\tilde{x}_1, \tilde{y}_1, s, t)dtds\right] \\ &= \tilde{b}_2(\tilde{x}_1, \tilde{y}_1, x, y).\end{aligned}$$

Moreover, with the assumption that $\tilde{c}_m(x, y) = \tilde{b}_{m+1}(\tilde{x}_1, \tilde{y}_1, x, y)$, we have

$$\begin{aligned}&\tilde{c}_{m+1}(x, y) \\ &= W_1^{-1}\left[\Phi_{m+1}^{-1}(\Phi_{m+1}(c_m(x, y)) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t)dtds)\right] \\ &= W_{m+1}^{-1}\left[W_{m+1}(W_1^{-1}(c_m(x, y))) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t)dtds\right] \\ &= W_{m+1}^{-1}\left[W_{m+1}(\tilde{c}_m(x, y)) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t)dtds\right] \\ &= W_{m+1}^{-1}\left[W_{m+1}(\tilde{b}_{m+1}(\tilde{x}_1, \tilde{y}_1, x, y)) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)} \int_{\beta_{m+1}(y)}^{\infty} \tilde{f}_{m+1}(\tilde{x}_1, \tilde{y}_1, s, t)dtds\right] \\ &= \tilde{b}_{m+2}(\tilde{x}_1, \tilde{y}_1, x, y).\end{aligned}$$

This proves that

$$\tilde{c}_i(x, y) = \tilde{b}_{i+1}(\tilde{x}_1, \tilde{y}_1, x, y), \qquad i = 1, \dots, m.$$

Therefore, (3.11) becomes

$$\begin{aligned} W_{i+1}(\tilde{b}_{i+1}(\tilde{x}_1,\tilde{y}_1,x_3,y_3)) &+ \int_{\alpha_{i+1}(0)}^{\alpha_{i+1}(x_3)}\int_{\beta_{i+1}(y_3)}^{\infty}\tilde{f}_{i+1}(\tilde{x}_1,\tilde{y}_1,s,t)dtds \\ &\le \int_{\tilde{u}_{i+1}}^{W_1(\infty)}\frac{dz}{\phi_{i+1}(W_1^{-1}(z))} \\ &= \int_{u_{i+1}}^{\infty}\frac{dz}{w_{i+1}(z)}, \quad i=1,\dots,m. \end{aligned}$$

The above inequalities and (3.5) imply that we may take $x_2 = x_3,\ y_2 = y_3$. From (3.12) we get

$$u(x,y) \le W_{m+1}^{-1}\left[W_{m+1}(\tilde{b}_{m+1}(\tilde{x}_1,\tilde{y}_1,x,y)) + \int_{\alpha_{m+1}(0)}^{\alpha_{m+1}(x)}\int_{\beta_{m+1}(y)}^{\infty}\tilde{f}_{m+1}(\tilde{x}_1,\tilde{y}_1,s,t)dtds\right]$$

for all $0 \le x \le \tilde{x}_1 \le x_2, y_2 \le \tilde{y}_1 \le y < \infty$. This proves (3.3) by mathematical induction.

Taking $x = \tilde{x}_1, y = \tilde{y}_1$, $x_2 = x_1$ and $y_2 = y_1$, we have

$$u(\tilde{x}_1,\tilde{y}_1) \le W_n^{-1}\left[W_n(\tilde{b}_n(\tilde{x}_1,\tilde{y}_1,\tilde{x}_1,\tilde{y}_1)) + \int_{\alpha_n(0)}^{\alpha_n(\tilde{x}_1)}\int_{\beta_n(\tilde{y}_1)}^{\infty}\tilde{f}_n(\tilde{x}_1,\tilde{y}_1,s,t)dtds\right] \tag{3.13}$$

for $0 \le \tilde{x}_1 \le x_1, y_1 \le \tilde{y}_1 < \infty$. It is easy to verify that $\tilde{b}_n(\tilde{x}_1,\tilde{y}_1,\tilde{x}_1,\tilde{y}_1) = b_n(\tilde{x}_1,\tilde{y}_1)$. Thus, (3.13) can be written as

$$u(\tilde{x}_1,\tilde{y}_1) \le W_n^{-1}\left[W_n(b_n(\tilde{x}_1,\tilde{y}_1)) + \int_{\alpha_n(0)}^{\alpha_n(\tilde{x}_1)}\int_{\beta_n(\tilde{y}_1)}^{\infty}\tilde{f}_n(\tilde{x}_1,\tilde{y}_1,s,t)dtds\right].$$

Since $\tilde{x}_1,\tilde{y}_1$ are arbitrary, replace $\tilde{x}_1$ and $\tilde{y}_1$ by x and y respectively and we have

$$u(x,y) \le W_n^{-1}\left[W_n(b_n(x,y)) + \int_{\alpha_n(0)}^{\alpha_n(x)}\int_{\beta_n(y)}^{\infty}\tilde{f}_n(x,y,s,t)dtds\right]$$

for all $0 \le x \le x_1, y_1 \le y < \infty$.

In case $a(x,y) = 0$ for some $x,y \in \mathbb{R}_+$. Let $b_{1,\epsilon}(x,y) := b_1(x,y) + \epsilon$ for all $x,y \in \mathbb{R}_+$, where $\epsilon > 0$ is arbitrary, and then $b_{1,\epsilon}(x,y) > 0$. Using the same arguments as above, where $b_1(x,y)$ is replaced with $b_{1,\epsilon}(x,y) > 0$, we get

$$u(x,y) \le W_n^{-1}\left[W_n(b_{n,\epsilon}(x,y)) + \int_{\alpha_n(0)}^{\alpha_n(x)}\int_{\beta_n(y)}^{\infty}\tilde{d}_n(x,y,s,t)dtds\right].$$

Letting $\epsilon \to 0^+$, we obtain (2.1) by the continuity of $b_{1,\epsilon}$ in ϵ and the continuity of W_i and W_i^{-1} under the notation $W_1(0) := 0$. □

4. APPLICATIONS

Consider the retarded partial differential equation

$$\begin{aligned} D_1D_2v(x,y) = \frac{1}{(x+1)^2(y+1)^2} &+ \exp(-x)\exp(-y)\sqrt{|v(x,y)|} \\ &+ \frac{3}{4}x\exp\left(-\frac{x}{2}\right)\exp(-3y)v\left(\frac{x}{2},3y\right), \end{aligned} \tag{4.1}$$

$$v(x,\infty) = \sigma(x), v(0,y) = \tau(y), v(0,\infty) = k, \tag{4.2}$$

for $x, y \in \mathbb{R}_+$, where $\sigma, \tau \in C(\mathbb{R}_+, \mathbb{R})$, $\sigma(x)$ is nondecreasing in x, $\tau(y)$ is nonincreasing in y, and k is a real constant. Integrating (4.1) with respect to x and y and using the initial conditions (4.2), we get

$$\begin{aligned} v(x,y) &= \sigma(x) + \tau(y) - k - \frac{x}{(x+1)(y+1)} \\ &\qquad - \int_0^x \int_y^\infty \exp(-s)\exp(-t)\sqrt{|v(s,t)|}dtds \\ &\qquad - \frac{3}{4}\int_0^x \int_y^\infty s\exp\left(-\frac{s}{2}\right)\exp(-3t)v(\frac{s}{2}, 3t)dtds \\ &= \sigma(x) + \tau(y) - k - \frac{x}{(x+1)(y+1)} \\ &\qquad - \int_0^x \int_y^\infty \exp(-s)\exp(-t)\sqrt{|v(s,t)|}dtds \\ &\qquad - \int_0^{\frac{x}{2}} \int_{3y}^\infty s\exp(-s)\exp(-t)v(s,t)dtds. \end{aligned}$$

Thus,

$$\begin{aligned} |v(x,y)| \le |\sigma(x) + \tau(y) - k| &+ \frac{x}{(x+1)(y+1)} \\ &+ \int_0^x \int_y^\infty \exp(-s)\exp(-t)\sqrt{|v(s,t)|}dtds \\ &\qquad + \int_0^{\frac{x}{2}} \int_{3y}^\infty s\exp(-s)\exp(-t)|v(s,t)|dtds. \end{aligned}$$

Letting $u(x,y) = |v(x,y)|$, we have

$$\begin{aligned} u(x,y) \le a(x,y) &+ \int_{\alpha_1(0)}^{\alpha_1(x)} \int_{\beta_1(y)}^\infty f_1(x,y,s,t)w_1(u)dtds \\ &\qquad + \int_{\alpha_2(0)}^{\alpha_2(x)} \int_{\beta_2(y)}^\infty f_2(x,y,s,t)w_2(u)dtds, \end{aligned}$$

where

$$a(x,y) = |\sigma(x) + \tau(y) - k| + \frac{x}{(x+1)(y+1)},$$

$$\alpha_1(x) = x, \quad \beta_1(y) = y, \quad \alpha_2(x) = \frac{x}{2}, \quad \beta_2(y) = 3y, \quad w_1(u) = \sqrt{u}, \quad w_2(u) = u,$$
$$f_1(x,y,s,t) = \exp(-s)\exp(-t), \quad f_2(x,y,s,t) = s\exp(-s)\exp(-t).$$

Clearly, $\frac{w_2(u)}{w_1(u)} = \frac{u}{\sqrt{u}} = \sqrt{u}$ is nondecreasing for $u > 0$, that is, $w_1 \propto w_2$. Then for $u_1, u_2 > 0$

$$\begin{aligned} &b_1(x,y) = a(x,y), \qquad \tilde{f}_1(x,y,s,t) = f_1(x,y,s,t), \qquad \tilde{f}_2(x,y,s,t) = f_2(x,y,s,t), \\ &W_1(u) = \int_{u_1}^u \frac{dz}{\sqrt{z}} = 2\left(\sqrt{u} - \sqrt{u_1}\right), \qquad W_1^{-1}(u) = \left(\frac{u}{2} + \sqrt{u_1}\right)^2, \\ &W_2(u) = \int_{u_2}^u \frac{dz}{z} = \ln\frac{u}{u_2}, \qquad W_2^{-1}(u) = u_2\exp(u), \end{aligned}$$

$$
\begin{aligned}
b_2(x,y) &= W_1^{-1}\left[W_1(b_1(x,y)) + \int_0^x \int_y^\infty \tilde{f}_1(x,y,s,t)dtds\right] \\
&= W_1^{-1}\left[2\left(\sqrt{b_1(x,y)} - \sqrt{u_1}\right) + (1-\exp(-x))\exp(-y)\right] \\
&= \left[\sqrt{b_1(x,y)} + \frac{1}{2}(1-\exp(-x))\exp(-y)\right]^2.
\end{aligned}
$$

By Theorem 2.1, we have

$$
\begin{aligned}
|v(x,y)| &\le W_2^{-1}\left[W_2(b_2(x,y)) + \int_0^{\frac{x}{2}} \int_{3y}^\infty \tilde{d}_2(x,y,s,t)dtds\right] \\
&= W_2^{-1}\left[\ln\frac{b_2(x,y)}{u_2} + \left(1 - \left(\frac{x}{2}+1\right)\exp\left(-\frac{x}{2}\right)\right)\exp(-3y)\right] \\
&= u_2 \exp\left[\ln\frac{b_2(x,y)}{u_2} + \left(1 - \left(\frac{x}{2}+1\right)\exp\left(-\frac{x}{2}\right)\right)\exp(-3y)\right] \\
&= b_2(x,y)\exp\left[\left(1 - \left(\frac{x}{2}+1\right)\exp\left(-\frac{x}{2}\right)\right)\exp(-3y)\right] \\
&= \left(\sqrt{|\sigma(x)+\tau(y)-k| + \frac{x}{(x+1)(y+1)}} + \frac{1}{2}(1-\exp(-x))\exp(-y)\right)^2 \\
&\qquad \times \exp\left[\left(1 - \left(\frac{x}{2}+1\right)\exp\left(-\frac{x}{2}\right)\right)\exp(-3y)\right].
\end{aligned}
$$

This implies that the solution of (4.1) is bounded for $x, y \in \mathbb{R}_+$ provided that $\sigma(x) + \tau(y) - k$ is bounded for all $x, y \in \mathbb{R}_+$.

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