ON SOME INEQUALITIES IN NORMED ALGEBRAS

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Abstract: Some inequalities in normed algebras that provides lower and upper bounds for the norm of $\sum_{j=1}^{n} a_j x_j$ are obtained. Applications for estimating the quantities $\left\| \left\| x^{-1} \right\| x \pm \left\| y^{-1} \right\| y \right\|$ and $\left\| \left\| y^{-1} \right\| x \pm \left\| x^{-1} \right\| y \right\|$ for invertible elements x, y in unital normed algebras are also given.

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1. Introduction

In [1], in order to provide a generalisation of a norm inequality for n vectors in a normed linear space obtained by Pečarić and Rajić in [2], the author obtained the following result:

$$\max_{k\in\{1,\dots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}|\alpha_j-\alpha_k|\,\|x_j\|\right\} \\ \leq\left\|\sum_{j=1}^{n}\alpha_jx_j\right\|\leq\min_{k\in\{1,\dots,n\}}\left\{|\alpha_k|\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}|\alpha_j-\alpha_k|\,\|x_j\|\right\}, \tag{1.1}$$

where $x_j,\ j\in\{1,\dots,n\}$ are vectors in the normed linear space $(X,\|\cdot\|)$ over $\mathbb{K}$ while $\alpha_j,\ j\in\{1,\dots,n\}$ are scalars in $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$.

For $\alpha_k=\frac{1}{\|x_k\|}$, with $x_k\neq 0,\ k\in\{1,\dots,n\}$ the above inequality produces the following result established by Pečarić and Rajić in [2]:

$$\max_{k\in\{1,\dots,n\}}\left\{\frac{1}{\|x_k\|}\left[\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\big|\|x_j\|-\|x_k\|\big|\right]\right\} \\ \leq\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\leq\min_{k\in\{1,\dots,n\}}\left\{\frac{1}{\|x_k\|}\left[\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}\big|\|x_j\|-\|x_k\|\big|\right]\right\}, \tag{1.2}$$

which implies the following refinement and reverse of the generalised triangle in-

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equality due to M. Kato et al. [3]:

$$\min_{k\in\{1,\dots,n\}}\{\|x_k\|\}\left[n-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right] \le \sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\| \le \max_{k\in\{1,\dots,n\}}\{\|x_k\|\}\left[n-\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|\right]. \tag{1.3}$$

The other natural choice, $\alpha_k=\|x_k\|$, $k\in\{1,\dots,n\}$ in (1.1) produces the result

$$\max_{k\in\{1,\dots,n\}}\left\{\|x_k\|\left\|\sum_{j=1}^{n}x_j\right\|-\sum_{j=1}^{n}\big|\|x_j\|-\|x_k\|\big|\,\|x_j\|\right\} \le \left\|\sum_{j=1}^{n}\|x_j\|\,x_j\right\| \le \min_{k\in\{1,\dots,n\}}\left\{\|x_k\|\left\|\sum_{j=1}^{n}x_j\right\|+\sum_{j=1}^{n}\big|\|x_j\|-\|x_k\|\big|\,\|x_j\|\right\}, \tag{1.4}$$

which in its turn implies another refinement and reverse of the generalised triangle inequality:

$$(0\le)\frac{\sum_{j=1}^{n}\|x_j\|^2-\left\|\sum_{j=1}^{n}\|x_j\|\,x_j\right\|}{\max_{k\in\{1,\dots,n\}}\{\|x_k\|\}} \le \sum_{j=1}^{n}\|x_j\|-\left\|\sum_{j=1}^{n}x_j\right\| \le \frac{\sum_{j=1}^{n}\|x_j\|^2-\left\|\sum_{j=1}^{n}\|x_j\|\,x_j\right\|}{\min_{k\in\{1,\dots,n\}}\{\|x_k\|\}}, \tag{1.5}$$

provided $x_k\neq 0$, $k\in\{1,\dots,n\}$.

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In [2], the authors have shown that the case $n = 2$ in (1.2) produces the *Maligranda-Mercer inequality*:

$$\frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \le \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}, \tag{1.6}$$

for any $x, y \in X \backslash \{0\}$.

We notice that Maligranda proved the right inequality in [5] while Mercer proved the left inequality in [4].

We have shown in [1] that the following dual result for two vectors is also valid:

$$\begin{aligned}(0 \le) \frac{\|x-y\|}{\min\{\|x\|, \|y\|\}} - \frac{|\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} \\ \le \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \le \frac{\|x-y\|}{\max\{\|x\|, \|y\|\}} + \frac{|\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}},\end{aligned} \tag{1.7}$$

for any $x, y \in X \backslash \{0\}$.

Motivated by the above results, the aim of the present paper is to establish lower and upper bounds for the norm of $\sum_{j=1}^{n} a_j x_j$, where $a_j, x_j, j \in \{1, \dots, n\}$ are elements in a normed algebra $(A, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$. In the case where $(A, \|\cdot\|)$ is a unital algebra and x, y are invertible, lower and upper bounds for the quantities

$$\left\| \|x^{-1}\| x \pm \|y^{-1}\| y \right\| \quad \text{and} \quad \left\| \|y^{-1}\| x \pm \|x^{-1}\| y \right\|$$

are provided as well.

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2. Inequalities for n Pairs of Elements

Let $(A, \|\cdot\|)$ be a normed algebra over the real or complex number field $\mathbb{K}$.

Theorem 2.1. *If* $(a_j, x_j) \in A^2$, $j \in \{1, \dots, n\}$, *then*

$$\max_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|a_j - a_k\| \, \|x_j\| \right\} \tag{2.1}$$
$$\leq \max_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|(a_j - a_k) x_j\| \right\}$$
$$\leq \left\| \sum_{j=1}^{n} a_j x_j \right\|$$
$$\leq \min_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|(a_j - a_k) x_j\| \right\}$$
$$\leq \min_{k\in\{1,\dots,n\}} \left\{ \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|a_j - a_k\| \, \|x_j\| \right\}.$$

Proof. Observe that for any $k \in \{1, \dots, n\}$ we have

$$\sum_{j=1}^{n} a_j x_j = a_k \left(\sum_{j=1}^{n} x_j \right) + \sum_{j=1}^{n} (a_j - a_k) x_j.$$

Taking the norm and utilising the triangle inequality and the normed algebra proper-

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ties, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n} a_j x_j\right\| &\leq \left\|a_k\left(\sum_{j=1}^{n} x_j\right)\right\| + \left\|\sum_{j=1}^{n}\left(a_j - a_k\right) x_j\right\| \\ &\leq \left\|a_k\left(\sum_{j=1}^{n} x_j\right)\right\| + \sum_{j=1}^{n}\left\|\left(a_j - a_k\right) x_j\right\| \\ &\leq \left\|a_k\left(\sum_{j=1}^{n} x_j\right)\right\| + \sum_{j=1}^{n}\left\|a_j - a_k\right\|\left\|x_j\right\|,\end{aligned}$$

for any $k \in \{1, \ldots, n\}$, which implies the second part in (2.1).

Observing that

$$\sum_{j=1}^{n} a_j x_j = a_k\left(\sum_{j=1}^{n} x_j\right) - \sum_{j=1}^{n}\left(a_k - a_j\right) x_j$$

and utilising the continuity of the norm, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n} a_j x_j\right\| &\geq \left\|\left\|a_k\left(\sum_{j=1}^{n} x_j\right) - \sum_{j=1}^{n}\left(a_k - a_j\right) x_j\right\|\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n} x_j\right)\right\| - \left\|\sum_{j=1}^{n}\left(a_k - a_j\right) x_j\right\| \\ &\geq \left\|a_k\left(\sum_{j=1}^{n} x_j\right)\right\| - \sum_{j=1}^{n}\left\|\left(a_k - a_j\right) x_j\right\|\end{aligned}$$

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$$\geq \left\| a_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|a_k - a_j\| \, \|x_j\|$$

for any $k \in \{1, \dots, n\}$, which implies the first part in (2.1). □

Remark 1. If there exists $r > 0$ so that $\|a_j - a_k\| \leq r \|a_k\|$ for any $j, k \in \{1, \dots, n\}$, then, by the second part of (2.1), we have

$$\left\| \sum_{j=1}^{n} a_j x_j \right\| \leq \min_{k \in \{1,\dots,n\}} \{\|a_k\|\} \left[\left\| \sum_{j=1}^{n} x_j \right\| + r \sum_{j=1}^{n} \|x_j\| \right]. \tag{2.2}$$

Corollary 2.2. *If* $x_j \in A$, $j \in \{1, \dots, n\}$, *then*

$$\begin{aligned} \max_{k \in \{1,\dots,n\}} & \left\{ \left\| x_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|x_j - x_k\| \, \|x_j\| \right\} \\ &\leq \max_{k \in \{1,\dots,n\}} \left\{ \left\| x_k \left(\sum_{j=1}^{n} x_j \right) \right\| - \sum_{j=1}^{n} \|(x_j - x_k) x_j\| \right\} \leq \left\| \sum_{j=1}^{n} x_j^2 \right\| \\ &\leq \min_{k \in \{1,\dots,n\}} \left\{ \left\| x_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|(x_j - x_k) x_j\| \right\} \\ &\leq \min_{k \in \{1,\dots,n\}} \left\{ \left\| x_k \left(\sum_{j=1}^{n} x_j \right) \right\| + \sum_{j=1}^{n} \|x_j - x_k\| \, \|x_j\| \right\}. \end{aligned} \tag{2.3}$$

Corollary 2.3. *Assume that* A *is a unital normed algebra. If* $x_j \in A$ *are invertible*

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for any $j \in \{1, \dots, n\}$, *then*

$$\min_{k\in\{1,\dots,n\}} \left\|x_k^{-1}\right\| \left[\sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|\right] \tag{2.4}$$
$$\leq \sum_{j=1}^{n} \left\|x_j^{-1}\right\| \|x_j\| - \left\|\sum_{j=1}^{n} \left\|x_j^{-1}\right\| x_j\right\|$$
$$\leq \max_{k\in\{1,\dots,n\}} \left\|x_k^{-1}\right\| \left[\sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|\right].$$

Proof. If $1 \in A$ is the unity, then on choosing $a_k = \left\|x_k^{-1}\right\| \cdot 1$ in (2.1) we get

$$\max_{k\in\{1,\dots,n\}} \left\{\left\|x_k^{-1}\right\| \left\|\sum_{j=1}^{n} x_j\right\| - \sum_{j=1}^{n} \left|\left\|x_j^{-1}\right\| - \left\|x_k^{-1}\right\|\right| \|x_j\|\right\} \tag{2.5}$$
$$\leq \left\|\sum_{j=1}^{n} \left\|x_j^{-1}\right\| x_j\right\|$$
$$\leq \min_{k\in\{1,\dots,n\}} \left\{\left\|x_k^{-1}\right\| \left\|\sum_{j=1}^{n} x_j\right\| + \sum_{j=1}^{n} \left|\left\|x_j^{-1}\right\| - \left\|x_k^{-1}\right\|\right| \|x_j\|\right\}.$$

Now, assume that $\min_{k\in\{1,\dots,n\}} \left\{\left\|x_k^{-1}\right\|\right\} = \left\|x_{k_0}^{-1}\right\|$. Then

$$\left\|x_{k_0}^{-1}\right\| \left\|\sum_{j=1}^{n} x_j\right\| + \sum_{j=1}^{n} \left|\left\|x_j^{-1}\right\| - \left\|x_{k_0}^{-1}\right\|\right| \|x_j\|$$

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$$= -\left\|x_{k_0}^{-1}\right\|\left(\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\|\right) + \sum_{j=1}^{n}\left\|x_j^{-1}\right\|\|x_j\|.$$

Utilising the second inequality in (2.5), we deduce

$$\left\|x_{k_0}^{-1}\right\|\left(\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\|\right) \le \sum_{j=1}^{n}\left\|x_j^{-1}\right\|\|x_j\| - \left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\|x_j\right\|$$

and the first inequality in (2.4) is proved.

The second part of (2.4) can be proved in a similar manner, however, the details are omitted. □

Remark 2. An equivalent form of (2.4) is:

$$\frac{\sum_{j=1}^{n}\left\|x_j^{-1}\right\|\|x_j\| - \left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\|x_j\right\|}{\max_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|} \tag{2.6}$$
$$\le \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \le \frac{\sum_{j=1}^{n}\left\|x_j^{-1}\right\|\|x_j\| - \left\|\sum_{j=1}^{n}\left\|x_j^{-1}\right\|x_j\right\|}{\min_{k\in\{1,\dots,n\}}\left\|x_k^{-1}\right\|},$$

which provides both a refinement and a reverse inequality for the generalised triangle inequality.

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3. Inequalities for Two Pairs of Elements

The following particular case of Theorem 2.1 is of interest for applications.

Lemma 3.1. *If* $(a,b),(x,y)\in A^2$, *then*

$$\max\{\|a(x\pm y)\|-\|(b-a)y\|,\|b(x\pm y)\|-\|(b-a)x\|\} \le \|ax\pm by\| \le \min\{\|a(x\pm y)\|+\|(b-a)y\|,\|b(x\pm y)\|+\|(b-a)x\|\} \tag{3.1}$$

or, equivalently,

$$\begin{aligned}&\frac{1}{2}\{\|a(x\pm y)\|+\|b(x\pm y)\|-[\|(b-a)y\|+\|(b-a)x\|]\}\\ &\qquad+\frac{1}{2}\left|\|a(x\pm y)\|-\|b(x\pm y)\|+\|(b-a)y\|-\|(b-a)x\|\right|\\ &\le\|ax\pm by\|\\ &\le\frac{1}{2}\{\|a(x\pm y)\|+\|b(x\pm y)\|+[\|(b-a)y\|+\|(b-a)x\|]\}\\ &\qquad-\frac{1}{2}\left|\|a(x\pm y)\|+\|b(x\pm y)\|-\|(b-a)y\|-\|(b-a)x\|\right|.\end{aligned} \tag{3.2}$$

Proof. The inequality (3.1) follows from Theorem 2.1 for $n=2$, $a_1=a$, $a_2=b$, $x_1=x$ and $x_2=\pm y$.

Utilising the properties of real numbers,

$$\min\{\alpha,\beta\}=\frac{1}{2}[\alpha+\beta-|\alpha-\beta|],\quad \max\{\alpha,\beta\}=\frac{1}{2}[\alpha+\beta+|\alpha-\beta|];\quad \alpha,\beta\in\mathbb{R};$$

the inequality (3.1) is clearly equivalent with (3.2). □

The following result contains some upper bounds for $\|ax\pm by\|$ that are perhaps more useful for applications.

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Theorem 3.2. *If* $(a,b),(x,y)\in A^2$, *then*

$$\|ax\pm by\| \le \min\{\|a(x\pm y)\|,\|b(x\pm y)\|\}+\|b-a\|\max\{\|x\|,\|y\|\} \le \|x\pm y\|\min\{\|a\|,\|b\|\}+\|b-a\|\max\{\|x\|,\|y\|\} \tag{3.3}$$

and

$$\|ax\pm by\| \le \|x\pm y\|\max\{\|a\|,\|b\|\}+\min\{\|(b-a)x\|,\|(b-a)y\|\} \le \|x\pm y\|\max\{\|a\|,\|b\|\}+\|b-a\|\min\{\|x\|,\|y\|\}. \tag{3.4}$$

Proof. Observe that $\|(b-a)x\|\le\|b-a\|\,\|x\|$ and $\|(b-a)y\|\le\|b-a\|\,\|y\|$, and then

$$\|(b-a)x\|,\|(b-a)y\|\le\|b-a\|\max\{\|x\|,\|y\|\}, \tag{3.5}$$

which implies that

$$\begin{aligned}&\min\{\|a(x\pm y)\|+\|(b-a)y\|,\|b(x\pm y)\|+\|(b-a)x\|\}\\ &\le \min\{\|a(x\pm y)\|,\|b(x\pm y)\|\}+\|b-a\|\max\{\|x\|,\|y\|\}\\ &\le \|x\pm y\|\min\{\|a\|,\|b\|\}+\|b-a\|\max\{\|x\|,\|y\|\}.\end{aligned}$$

Utilising the second inequality in (3.1), we deduce (3.3).

Also, since $\|a(x\pm y)\|\le\|a\|\,\|x\pm y\|$ and $\|b(x\pm y)\|\le\|b\|\,\|x\pm y\|$, hence

$$\|a(x\pm y)\|,\|b(x\pm y)\|\le\|x\pm y\|\max\{\|a\|,\|b\|\},$$

which implies that

$$\begin{aligned}&\min\{\|a(x\pm y)\|+\|(b-a)y\|,\|b(x\pm y)\|+\|(b-a)x\|\}\\ &\le \|x\pm y\|\max\{\|a\|,\|b\|\}+\min\{\|(b-a)x\|,\|(b-a)y\|\}\\ &\le \|x\pm y\|\max\{\|a\|,\|b\|\}+\|b-a\|\min\{\|x\|,\|y\|\},\end{aligned}$$

and the inequality (3.4) is also proved. □

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The following corollary may be more useful for applications.

Corollary 3.3. *If* $(a,b),(x,y)\in A^2$, *then*

$$\|ax\pm by\|\le\|x\pm y\|\cdot\frac{\|a\|+\|b\|}{2}+\|b-a\|\cdot\frac{\|x\|+\|y\|}{2}. \tag{3.6}$$

Proof. Follows from Theorem 3.2 by adding the last inequality in (3.3) to the last inequality (3.4) and utilising the property that $\min\{\alpha,\beta\}+\max\{\alpha,\beta\}=\alpha+\beta$, $\alpha,\beta\in\mathbb{R}$. □

The following lower bounds for $\|ax\pm by\|$ can be stated as well:

Theorem 3.4. *For any* (a,b) *and* $(x,y)\in A^2$, *we have:*

$$\begin{aligned}&\max\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\le\max\{\|a(x\pm y)\|,\|b(x\pm y)\|\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\le\|ax\pm by\|\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\min\{\|x\|,\|y\|\}\\&\le\min\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\min\{\|(b-a)x\|,\|(b-a)y\|\}\\&\le\|ax\pm by\|.\end{aligned} \tag{3.8}$$

Proof. Observe that, by (3.5) we have that

$$\begin{aligned}&\max\{\|a(x\pm y)\|-\|(b-a)y\|,\|b(x\pm y)\|-\|(b-a)x\|\}\\&\ge\max\{\|ax\pm ay\|,\|bx\pm by\|\}-\|b-a\|\max\{\|x\|,\|y\|\}\\&\ge\max\{|\|ax\|-\|ay\||,|\|bx\|-\|by\||\}-\|b-a\|\max\{\|x\|,\|y\|\}\end{aligned}$$

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and on utilising the first inequality in (3.1), the inequality (3.7) is proved.

Observe also that, since

$$\|a(x \pm y)\|, \|b(x \pm y)\| \geq \min\{|\|ax\| - \|ay\||, |\|bx\| - \|by\||\},$$

then

$$\begin{aligned}
&\max\{\|a(x \pm y)\| - \|(b-a)y\|, \|b(x \pm y)\| - \|(b-a)x\|\} \\
&\geq \min\{|\|ax\| - \|ay\||, |\|bx\| - \|by\||\} - \min\{\|(b-a)x\|, \|(b-a)y\|\} \\
&\geq \min\{|\|ax\| - \|ay\||, |\|bx\| - \|by\||\} - \|b-a\| \min\{\|x\|, \|y\|\}.
\end{aligned}$$

Then, by the first inequality in (3.1), we deduce (3.8). □

Corollary 3.5. *For any* $(a, b), (x, y) \in A^2$*, we have*

$$\frac{1}{2} \cdot [|\|ax\| - \|ay\|| + |\|bx\| - \|by\||] - \|b-a\| \cdot \frac{\|x\| + \|y\|}{2} \leq \|ax \pm by\|. \tag{3.9}$$

The proof follows from Theorem 3.4 by adding (3.7) to (3.8). The details are omitted.

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4. Applications for Two Invertible Elements

In this section we assume that A is a unital algebra with the unity 1. The following results provide some upper bounds for the quantity $\left\| \left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y \right\|$, where x and y are invertible in A.

Proposition 4.1. *If $(x,y) \in A^2$ are invertible, then*

$$\begin{aligned} \left\| \left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y \right\| & \\ \leq \|x \pm y\| \min\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} &+ \left| \left\|x^{-1}\right\| - \left\|y^{-1}\right\| \right| \max\left\{\|x\|, \|y\|\right\} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \left\| \left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y \right\| & \\ \leq \|x \pm y\| \max\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} &+ \left| \left\|x^{-1}\right\| - \left\|y^{-1}\right\| \right| \min\left\{\|x\|, \|y\|\right\}. \end{aligned} \tag{4.2}$$

Proof. Follows by Theorem 3.2 on choosing $a = \left\|x^{-1}\right\| \cdot 1$ and $b = \left\|y^{-1}\right\| \cdot 1$. □

Corollary 4.2. *With the above assumption for x and y, we have*

$$\begin{aligned} \left\| \left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y \right\| & \\ \leq \|x \pm y\| \cdot \frac{\left\|x^{-1}\right\| + \left\|y^{-1}\right\|}{2} &+ \left| \left\|x^{-1}\right\| - \left\|y^{-1}\right\| \right| \cdot \frac{\|x\| + \|y\|}{2}. \end{aligned} \tag{4.3}$$

Lower bounds for $\left\| \left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y \right\|$ are provided below:

Proposition 4.3. *If $(x,y) \in A^2$ are invertible, then*

$$\begin{aligned} \|x \pm y\| \max\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} - \left| \left\|x^{-1}\right\| - \left\|y^{-1}\right\| \right| \max\left\{\|x\|, \|y\|\right\} & \\ \leq \left\| \left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y \right\| & \end{aligned} \tag{4.4}$$

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and

$$\|x \pm y\| \min\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \min\left\{\|x\|, \|y\|\right\} \leq \left\|\left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y\right\|. \tag{4.5}$$

Proof. The first inequality in (4.4) follows from the second inequality in (3.7) on choosing $a = \|x^{-1}\| \cdot 1$ and $b = \|y^{-1}\| \cdot 1$.

We know from the proof of Theorem 3.4 that

$$\max\left\{\|a(x \pm y)\| - \|(b-a)y\|, \|b(x \pm y)\| - \|(b-a)x\|\right\} \leq \|ax \pm by\|. \tag{4.6}$$

If in this inequality we choose $a = \|x^{-1}\| \cdot 1$ and $b = \|y^{-1}\| \cdot 1$, then we get

$$\begin{aligned} &\left\|\left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y\right\| \\ &\quad\geq \max\left\{ \left\|x^{-1}\right\| \|x \pm y\| - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \|y\|, \right. \\ &\qquad\qquad \left. \left\|y^{-1}\right\| \|x \pm y\| - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \|x\| \right\} \\ &\quad\geq \|x \pm y\| \min\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \min\left\{\|x\|, \|y\|\right\} \end{aligned}$$

and the inequality (4.5) is obtained. □

Corollary 4.4. *If $(x, y) \in A^2$ are invertible, then*

$$\|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \cdot \frac{\|x\| + \|y\|}{2} \leq \left\|\left\|x^{-1}\right\| x \pm \left\|y^{-1}\right\| y\right\|. \tag{4.7}$$

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Remark 3. We observe that the inequalities (4.3) and (4.7) are in fact equivalent with:

$$\left|\left\|\left\|x^{-1}\right\|x\pm\left\|y^{-1}\right\|y\right\|-\left\|x\pm y\right\|\cdot\frac{\left\|x^{-1}\right\|+\left\|y^{-1}\right\|}{2}\right| \le\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\cdot\frac{\|x\|+\|y\|}{2}. \tag{4.8}$$

Now we consider the dual expansion $\left\|\left\|y^{-1}\right\|x\pm\left\|x^{-1}\right\|y\right\|$, for which the following upper bounds can be stated.

Proposition 4.5. *If* (x,y) *are invertible in* A, *then*

$$\left\|\left\|y^{-1}\right\|x\pm\left\|x^{-1}\right\|y\right\| \le\|x\pm y\|\min\left\{\left\|x^{-1}\right\|,\left\|y^{-1}\right\|\right\}+\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\max\left\{\|x\|,\|y\|\right\} \tag{4.9}$$

and

$$\left\|\left\|y^{-1}\right\|x\pm\left\|x^{-1}\right\|y\right\| \le\|x\pm y\|\max\left\{\left\|x^{-1}\right\|,\left\|y^{-1}\right\|\right\}+\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\min\left\{\|x\|,\|y\|\right\}. \tag{4.10}$$

In particular,

$$\left\|\left\|y^{-1}\right\|x\pm\left\|x^{-1}\right\|y\right\| \le\|x\pm y\|\cdot\frac{\left\|x^{-1}\right\|+\left\|y^{-1}\right\|}{2}+\left|\left\|x^{-1}\right\|-\left\|y^{-1}\right\|\right|\cdot\frac{\|x\|+\|y\|}{2}. \tag{4.11}$$

The proof follows from Theorem 3.2 on choosing $a=\left\|y^{-1}\right\|\cdot 1$ and $b=\left\|x^{-1}\right\|\cdot 1$.
The lower bounds for the quantity $\left\|\left\|y^{-1}\right\|x\pm\left\|x^{-1}\right\|y\right\|$ are incorporated in:

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Proposition 4.6. *If* (x, y) *are invertible in* A, *then*

$$\|x \pm y\| \max\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \max\{\|x\|, \|y\|\} \leq \left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\| \tag{4.12}$$

and

$$\|x \pm y\| \min\left\{\left\|x^{-1}\right\|, \left\|y^{-1}\right\|\right\} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \min\{\|x\|, \|y\|\} \leq \left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\|. \tag{4.13}$$

In particular,

$$\|x \pm y\| \cdot \frac{\left\|x^{-1}\right\| + \left\|y^{-1}\right\|}{2} - \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \cdot \frac{\|x\| + \|y\|}{2} \leq \left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\|. \tag{4.14}$$

Remark 4. We observe that the inequalities (4.11) and (4.14) are equivalent with

$$\left|\left\|\left\|y^{-1}\right\| x \pm \left\|x^{-1}\right\| y\right\| - \|x \pm y\| \cdot \frac{\left\|x^{-1}\right\| + \left\|y^{-1}\right\|}{2}\right| \leq \left|\left\|x^{-1}\right\| - \left\|y^{-1}\right\|\right| \cdot \frac{\|x\| + \|y\|}{2}. \tag{4.15}$$

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