



ON SOME INEQUALITIES IN NORMED ALGEBRAS

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Abstract: Some inequalities in normed algebras that provides lower and upper bounds for the norm of $\sum_{j=1}^n a_j x_j$ are obtained. Applications for estimating the quantities $\| \|x^{-1}\| x \pm \|y^{-1}\| y \|$ and $\| \|y^{-1}\| x \pm \|x^{-1}\| y \|$ for invertible elements x, y in unital normed algebras are also given.

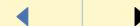
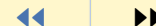
**Inequalities in Normed
Algebras**

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1. Introduction

In [1], in order to provide a generalisation of a norm inequality for n vectors in a normed linear space obtained by Pečarić and Rajić in [2], the author obtained the following result:

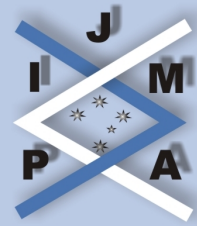
$$(1.1) \quad \max_{k \in \{1, \dots, n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\| \right\} \\ \leq \left\| \sum_{j=1}^n \alpha_j x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ |\alpha_k| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n |\alpha_j - \alpha_k| \|x_j\| \right\},$$

where $x_j, j \in \{1, \dots, n\}$ are vectors in the normed linear space $(X, \|\cdot\|)$ over \mathbb{K} while $\alpha_j, j \in \{1, \dots, n\}$ are scalars in \mathbb{K} ($\mathbb{K} = \mathbb{C}, \mathbb{R}$).

For $\alpha_k = \frac{1}{\|x_k\|}$, with $x_k \neq 0, k \in \{1, \dots, n\}$ the above inequality produces the following result established by Pečarić and Rajić in [2]:

$$(1.2) \quad \max_{k \in \{1, \dots, n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \left| \|x_j\| - \|x_k\| \right| \right] \right\} \\ \leq \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ \frac{1}{\|x_k\|} \left[\left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \left| \|x_j\| - \|x_k\| \right| \right] \right\},$$

which implies the following refinement and reverse of the generalised triangle in-



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equality due to M. Kato et al. [3]:

$$(1.3) \quad \min_{k \in \{1, \dots, n\}} \{ \|x_k\| \} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right] \\ \leq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \leq \max_{k \in \{1, \dots, n\}} \{ \|x_k\| \} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right].$$

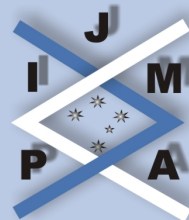
The other natural choice, $\alpha_k = \|x_k\|$, $k \in \{1, \dots, n\}$ in (1.1) produces the result

$$(1.4) \quad \max_{k \in \{1, \dots, n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \| \|x_j\| - \|x_k\| \| \|x_j\| \right\} \\ \leq \left\| \sum_{j=1}^n \|x_j\| x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \left\{ \|x_k\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \| \|x_j\| - \|x_k\| \| \|x_j\| \right\},$$

which in its turn implies another refinement and reverse of the generalised triangle inequality:

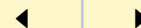
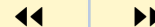
$$(1.5) \quad (0 \leq) \frac{\sum_{j=1}^n \|x_j\|^2 - \left\| \sum_{j=1}^n \|x_j\| x_j \right\|}{\max_{k \in \{1, \dots, n\}} \{ \|x_k\| \}} \\ \leq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \leq \frac{\sum_{j=1}^n \|x_j\|^2 - \left\| \sum_{j=1}^n \|x_j\| x_j \right\|}{\min_{k \in \{1, \dots, n\}} \{ \|x_k\| \}},$$

provided $x_k \neq 0$, $k \in \{1, \dots, n\}$.



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In [2], the authors have shown that the case $n = 2$ in (1.2) produces the *Maligranda-Mercer inequality*:

$$(1.6) \quad \frac{\|x - y\| - \left| \|x\| - \|y\| \right|}{\min \{ \|x\|, \|y\| \}} \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x - y\| + \left| \|x\| - \|y\| \right|}{\max \{ \|x\|, \|y\| \}},$$

for any $x, y \in X \setminus \{0\}$.

We notice that Maligranda proved the right inequality in [5] while Mercer proved the left inequality in [4].

We have shown in [1] that the following dual result for two vectors is also valid:

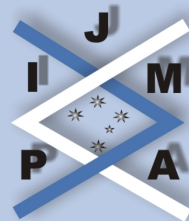
$$(1.7) \quad (0 \leq) \frac{\|x - y\|}{\min \{ \|x\|, \|y\| \}} - \frac{\left| \|x\| - \|y\| \right|}{\max \{ \|x\|, \|y\| \}} \\ \leq \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \frac{\|x - y\|}{\max \{ \|x\|, \|y\| \}} + \frac{\left| \|x\| - \|y\| \right|}{\min \{ \|x\|, \|y\| \}},$$

for any $x, y \in X \setminus \{0\}$.

Motivated by the above results, the aim of the present paper is to establish lower and upper bounds for the norm of $\sum_{j=1}^n a_j x_j$, where $a_j, x_j, j \in \{1, \dots, n\}$ are elements in a normed algebra $(A, \|\cdot\|)$ over the real or complex number field \mathbb{K} . In the case where $(A, \|\cdot\|)$ is a unital algebra and x, y are invertible, lower and upper bounds for the quantities

$$\left\| \|x^{-1}\| x \pm \|y^{-1}\| y \right\| \quad \text{and} \quad \left\| \|y^{-1}\| x \pm \|x^{-1}\| y \right\|$$

are provided as well.



2. Inequalities for n Pairs of Elements

Let $(A, \|\cdot\|)$ be a normed algebra over the real or complex number field \mathbb{K} .

Theorem 2.1. *If $(a_j, x_j) \in A^2$, $j \in \{1, \dots, n\}$, then*

$$\begin{aligned} (2.1) \quad & \max_{k \in \{1, \dots, n\}} \left\{ \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|a_j - a_k\| \|x_j\| \right\} \\ & \leq \max_{k \in \{1, \dots, n\}} \left\{ \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|(a_j - a_k) x_j\| \right\} \\ & \leq \left\| \sum_{j=1}^n a_j x_j \right\| \\ & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|(a_j - a_k) x_j\| \right\} \\ & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|a_j - a_k\| \|x_j\| \right\}. \end{aligned}$$

Proof. Observe that for any $k \in \{1, \dots, n\}$ we have

$$\sum_{j=1}^n a_j x_j = a_k \left(\sum_{j=1}^n x_j \right) + \sum_{j=1}^n (a_j - a_k) x_j.$$

Taking the norm and utilising the triangle inequality and the normed algebra proper-

ties, we have

$$\begin{aligned}
 \left\| \sum_{j=1}^n a_j x_j \right\| &\leq \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| + \left\| \sum_{j=1}^n (a_j - a_k) x_j \right\| \\
 &\leq \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|(a_j - a_k) x_j\| \\
 &\leq \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|a_j - a_k\| \|x_j\|,
 \end{aligned}$$

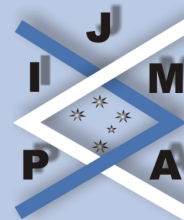
for any $k \in \{1, \dots, n\}$, which implies the second part in (2.1).

Observing that

$$\sum_{j=1}^n a_j x_j = a_k \left(\sum_{j=1}^n x_j \right) - \sum_{j=1}^n (a_k - a_j) x_j$$

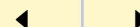
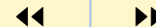
and utilising the continuity of the norm, we have

$$\begin{aligned}
 \left\| \sum_{j=1}^n a_j x_j \right\| &\geq \left\| \left\| a_k \left(\sum_{j=1}^n x_j \right) - \sum_{j=1}^n (a_k - a_j) x_j \right\| \right\| \\
 &\geq \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| - \left\| \sum_{j=1}^n (a_k - a_j) x_j \right\| \\
 &\geq \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|(a_k - a_j) x_j\|
 \end{aligned}$$



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$$\geq \left\| a_k \left(\sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|a_k - a_j\| \|x_j\|$$

for any $k \in \{1, \dots, n\}$, which implies the first part in (2.1). \square

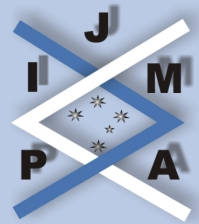
Remark 1. If there exists $r > 0$ so that $\|a_j - a_k\| \leq r \|a_k\|$ for any $j, k \in \{1, \dots, n\}$, then, by the second part of (2.1), we have

$$(2.2) \quad \left\| \sum_{j=1}^n a_j x_j \right\| \leq \min_{k \in \{1, \dots, n\}} \{ \|a_k\| \} \left[\left\| \sum_{j=1}^n x_j \right\| + r \sum_{j=1}^n \|x_j\| \right].$$

Corollary 2.2. If $x_j \in A, j \in \{1, \dots, n\}$, then

$$(2.3) \quad \begin{aligned} & \max_{k \in \{1, \dots, n\}} \left\{ \left\| x_k \left(\sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|x_j - x_k\| \|x_j\| \right\} \\ & \leq \max_{k \in \{1, \dots, n\}} \left\{ \left\| x_k \left(\sum_{j=1}^n x_j \right) \right\| - \sum_{j=1}^n \|(x_j - x_k) x_j\| \right\} \leq \left\| \sum_{j=1}^n x_j^2 \right\| \\ & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| x_k \left(\sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|(x_j - x_k) x_j\| \right\} \\ & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| x_k \left(\sum_{j=1}^n x_j \right) \right\| + \sum_{j=1}^n \|x_j - x_k\| \|x_j\| \right\}. \end{aligned}$$

Corollary 2.3. Assume that A is a unital normed algebra. If $x_j \in A$ are invertible





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for any $j \in \{1, \dots, n\}$, then

$$\begin{aligned}
 (2.4) \quad & \min_{k \in \{1, \dots, n\}} \left\| x_k^{-1} \right\| \left\| \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right\| \\
 & \leq \sum_{j=1}^n \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\| \\
 & \leq \max_{k \in \{1, \dots, n\}} \|x_k^{-1}\| \left\| \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right\|.
 \end{aligned}$$

Proof. If $1 \in A$ is the unity, then on choosing $a_k = \|x_k^{-1}\| \cdot 1$ in (2.1) we get

$$\begin{aligned}
 (2.5) \quad & \max_{k \in \{1, \dots, n\}} \left\{ \left\| x_k^{-1} \right\| \left\| \sum_{j=1}^n x_j \right\| - \sum_{j=1}^n \left| \|x_j^{-1}\| - \|x_k^{-1}\| \right| \|x_j\| \right\} \\
 & \leq \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\| \\
 & \leq \min_{k \in \{1, \dots, n\}} \left\{ \left\| x_k^{-1} \right\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \left| \|x_j^{-1}\| - \|x_k^{-1}\| \right| \|x_j\| \right\}.
 \end{aligned}$$

Now, assume that $\min_{k \in \{1, \dots, n\}} \left\{ \|x_k^{-1}\| \right\} = \|x_{k_0}^{-1}\|$. Then

$$\left\| x_{k_0}^{-1} \right\| \left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \left| \|x_j^{-1}\| - \|x_{k_0}^{-1}\| \right| \|x_j\|$$



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$$= - \|x_{k_0}^{-1}\| \left(\sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right) + \sum_{j=1}^n \|x_j^{-1}\| \|x_j\|.$$

Utilising the second inequality in (2.5), we deduce

$$\|x_{k_0}^{-1}\| \left(\sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right) \leq \sum_{j=1}^n \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\|$$

and the first inequality in (2.4) is proved.

The second part of (2.4) can be proved in a similar manner, however, the details are omitted. \square

Remark 2. An equivalent form of (2.4) is:

$$(2.6) \quad \frac{\sum_{j=1}^n \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\|}{\max_{k \in \{1, \dots, n\}} \|x_k^{-1}\|} \leq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \leq \frac{\sum_{j=1}^n \|x_j^{-1}\| \|x_j\| - \left\| \sum_{j=1}^n \|x_j^{-1}\| x_j \right\|}{\min_{k \in \{1, \dots, n\}} \|x_k^{-1}\|},$$

which provides both a refinement and a reverse inequality for the generalised triangle inequality.



3. Inequalities for Two Pairs of Elements

The following particular case of Theorem 2.1 is of interest for applications.

Lemma 3.1. *If $(a, b), (x, y) \in A^2$, then*

$$(3.1) \quad \max \{ \|a(x \pm y)\| - \|(b-a)y\|, \|b(x \pm y)\| - \|(b-a)x\| \} \\ \leq \|ax \pm by\| \leq \min \{ \|a(x \pm y)\| + \|(b-a)y\|, \|b(x \pm y)\| + \|(b-a)x\| \}$$

or, equivalently,

$$(3.2) \quad \frac{1}{2} \{ \|a(x \pm y)\| + \|b(x \pm y)\| - [\|(b-a)y\| + \|(b-a)x\|] \} \\ + \frac{1}{2} \| \|a(x \pm y)\| - \|b(x \pm y)\| + \|(b-a)y\| - \|(b-a)x\| \| \\ \leq \|ax \pm by\| \\ \leq \frac{1}{2} \{ \|a(x \pm y)\| + \|b(x \pm y)\| + [\|(b-a)y\| + \|(b-a)x\|] \} \\ - \frac{1}{2} \| \|a(x \pm y)\| + \|b(x \pm y)\| - \|(b-a)y\| - \|(b-a)x\| \|.$$

Proof. The inequality (3.1) follows from Theorem 2.1 for $n = 2$, $a_1 = a$, $a_2 = b$, $x_1 = x$ and $x_2 = \pm y$.

Utilising the properties of real numbers,

$$\min \{ \alpha, \beta \} = \frac{1}{2} [\alpha + \beta - |\alpha - \beta|], \quad \max \{ \alpha, \beta \} = \frac{1}{2} [\alpha + \beta + |\alpha - \beta|]; \quad \alpha, \beta \in \mathbb{R};$$

the inequality (3.1) is clearly equivalent with (3.2). \square

The following result contains some upper bounds for $\|ax \pm by\|$ that are perhaps more useful for applications.



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Theorem 3.2. *If $(a, b), (x, y) \in A^2$, then*

$$(3.3) \quad \|ax \pm by\| \leq \min \{ \|a(x \pm y)\|, \|b(x \pm y)\| \} + \|b - a\| \max \{ \|x\|, \|y\| \} \\ \leq \|x \pm y\| \min \{ \|a\|, \|b\| \} + \|b - a\| \max \{ \|x\|, \|y\| \}$$

and

$$(3.4) \quad \|ax \pm by\| \leq \|x \pm y\| \max \{ \|a\|, \|b\| \} + \min \{ \|(b - a)x\|, \|(b - a)y\| \} \\ \leq \|x \pm y\| \max \{ \|a\|, \|b\| \} + \|b - a\| \min \{ \|x\|, \|y\| \}.$$

Proof. Observe that $\|(b - a)x\| \leq \|b - a\| \|x\|$ and $\|(b - a)y\| \leq \|b - a\| \|y\|$, and then

$$(3.5) \quad \|(b - a)x\|, \|(b - a)y\| \leq \|b - a\| \max \{ \|x\|, \|y\| \},$$

which implies that

$$\min \{ \|a(x \pm y)\| + \|(b - a)y\|, \|b(x \pm y)\| + \|(b - a)x\| \} \\ \leq \min \{ \|a(x \pm y)\|, \|b(x \pm y)\| \} + \|b - a\| \max \{ \|x\|, \|y\| \} \\ \leq \|x \pm y\| \min \{ \|a\|, \|b\| \} + \|b - a\| \max \{ \|x\|, \|y\| \}.$$

Utilising the second inequality in (3.1), we deduce (3.3).

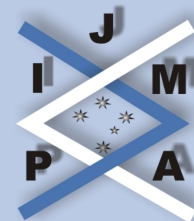
Also, since $\|a(x \pm y)\| \leq \|a\| \|x \pm y\|$ and $\|b(x \pm y)\| \leq \|b\| \|x \pm y\|$, hence

$$\|a(x \pm y)\|, \|b(x \pm y)\| \leq \|x \pm y\| \max \{ \|a\|, \|b\| \},$$

which implies that

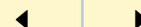
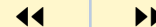
$$\min \{ \|a(x \pm y)\| + \|(b - a)y\|, \|b(x \pm y)\| + \|(b - a)x\| \} \\ \leq \|x \pm y\| \max \{ \|a\|, \|b\| \} + \min \{ \|(b - a)x\|, \|(b - a)y\| \} \\ \leq \|x \pm y\| \max \{ \|a\|, \|b\| \} + \|b - a\| \min \{ \|x\|, \|y\| \},$$

and the inequality (3.4) is also proved. □



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The following corollary may be more useful for applications.

Corollary 3.3. *If $(a, b), (x, y) \in A^2$, then*

$$(3.6) \quad \|ax \pm by\| \leq \|x \pm y\| \cdot \frac{\|a\| + \|b\|}{2} + \|b - a\| \cdot \frac{\|x\| + \|y\|}{2}.$$

Proof. Follows from Theorem 3.2 by adding the last inequality in (3.3) to the last inequality (3.4) and utilising the property that $\min\{\alpha, \beta\} + \max\{\alpha, \beta\} = \alpha + \beta$, $\alpha, \beta \in \mathbb{R}$. \square

The following lower bounds for $\|ax \pm by\|$ can be stated as well:

Theorem 3.4. *For any (a, b) and $(x, y) \in A^2$, we have:*

$$(3.7) \quad \begin{aligned} & \max\{\| \|ax\| - \|ay\| \|, \| \|bx\| - \|by\| \| \} - \|b - a\| \max\{\|x\|, \|y\|\} \\ & \leq \max\{\|a(x \pm y)\|, \|b(x \pm y)\|\} - \|b - a\| \max\{\|x\|, \|y\|\} \\ & \leq \|ax \pm by\| \end{aligned}$$

and

$$(3.8) \quad \begin{aligned} & \min\{\| \|ax\| - \|ay\| \|, \| \|bx\| - \|by\| \| \} - \|b - a\| \min\{\|x\|, \|y\|\} \\ & \leq \min\{\|a(x \pm y)\|, \|b(x \pm y)\|\} - \min\{\|(b - a)x\|, \|(b - a)y\|\} \\ & \leq \|ax \pm by\|. \end{aligned}$$

Proof. Observe that, by (3.5) we have that

$$\begin{aligned} & \max\{\|a(x \pm y)\| - \|(b - a)y\|, \|b(x \pm y)\| - \|(b - a)x\|\} \\ & \geq \max\{\|ax \pm ay\|, \|bx \pm by\|\} - \|b - a\| \max\{\|x\|, \|y\|\} \\ & \geq \max\{\| \|ax\| - \|ay\| \|, \| \|bx\| - \|by\| \| \} - \|b - a\| \max\{\|x\|, \|y\|\} \end{aligned}$$



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and on utilising the first inequality in (3.1), the inequality (3.7) is proved.

Observe also that, since

$$\|a(x \pm y)\|, \|b(x \pm y)\| \geq \min \{ \left| \|ax\| - \|ay\| \right|, \left| \|bx\| - \|by\| \right| \},$$

then

$$\begin{aligned} & \max \{ \|a(x \pm y)\| - \|(b-a)y\|, \|b(x \pm y)\| - \|(b-a)x\| \} \\ & \geq \min \{ \left| \|ax\| - \|ay\| \right|, \left| \|bx\| - \|by\| \right| \} - \min \{ \|(b-a)x\|, \|(b-a)y\| \} \\ & \geq \min \{ \left| \|ax\| - \|ay\| \right|, \left| \|bx\| - \|by\| \right| \} - \|b-a\| \min \{ \|x\|, \|y\| \}. \end{aligned}$$

Then, by the first inequality in (3.1), we deduce (3.8). □

Corollary 3.5. For any $(a, b), (x, y) \in A^2$, we have

$$(3.9) \quad \frac{1}{2} \cdot \left[\left| \|ax\| - \|ay\| \right| + \left| \|bx\| - \|by\| \right| \right] - \|b-a\| \cdot \frac{\|x\| + \|y\|}{2} \leq \|ax \pm by\|.$$

The proof follows from Theorem 3.4 by adding (3.7) to (3.8). The details are omitted.



4. Applications for Two Invertible Elements

In this section we assume that A is a unital algebra with the unity 1. The following results provide some upper bounds for the quantity $|||x^{-1}|| x \pm ||y^{-1}|| y||$, where x and y are invertible in A .

Proposition 4.1. *If $(x, y) \in A^2$ are invertible, then*

$$(4.1) \quad |||x^{-1}|| x \pm ||y^{-1}|| y|| \leq ||x \pm y|| \min \{ ||x^{-1}||, ||y^{-1}|| \} + |||x^{-1}|| - ||y^{-1}||| \max \{ ||x||, ||y|| \}$$

and

$$(4.2) \quad |||x^{-1}|| x \pm ||y^{-1}|| y|| \leq ||x \pm y|| \max \{ ||x^{-1}||, ||y^{-1}|| \} + |||x^{-1}|| - ||y^{-1}||| \min \{ ||x||, ||y|| \}.$$

Proof. Follows by Theorem 3.2 on choosing $a = ||x^{-1}|| \cdot 1$ and $b = ||y^{-1}|| \cdot 1$. \square

Corollary 4.2. *With the above assumption for x and y , we have*

$$(4.3) \quad |||x^{-1}|| x \pm ||y^{-1}|| y|| \leq ||x \pm y|| \cdot \frac{||x^{-1}|| + ||y^{-1}||}{2} + |||x^{-1}|| - ||y^{-1}||| \cdot \frac{||x|| + ||y||}{2}.$$

Lower bounds for $|||x^{-1}|| x \pm ||y^{-1}|| y||$ are provided below:

Proposition 4.3. *If $(x, y) \in A^2$ are invertible, then*

$$(4.4) \quad ||x \pm y|| \max \{ ||x^{-1}||, ||y^{-1}|| \} - |||x^{-1}|| - ||y^{-1}||| \max \{ ||x||, ||y|| \} \leq |||x^{-1}|| x \pm ||y^{-1}|| y||$$



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and

$$(4.5) \quad \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} - \| \|x^{-1}\| - \|y^{-1}\| \| \min \{ \|x\|, \|y\| \} \\ \leq \| \|x^{-1}\| x \pm \|y^{-1}\| y \| .$$

Proof. The first inequality in (4.4) follows from the second inequality in (3.7) on choosing $a = \|x^{-1}\| \cdot 1$ and $b = \|y^{-1}\| \cdot 1$.

We know from the proof of Theorem 3.4 that

$$(4.6) \quad \max \{ \|a(x \pm y)\| - \|(b-a)y\|, \|b(x \pm y)\| - \|(b-a)x\| \} \\ \leq \|ax \pm by\| .$$

If in this inequality we choose $a = \|x^{-1}\| \cdot 1$ and $b = \|y^{-1}\| \cdot 1$, then we get

$$\| \|x^{-1}\| x \pm \|y^{-1}\| y \| \\ \geq \max \left\{ \|x^{-1}\| \|x \pm y\| - \| \|x^{-1}\| - \|y^{-1}\| \| \|y\|, \right. \\ \left. \|y^{-1}\| \|x \pm y\| - \| \|x^{-1}\| - \|y^{-1}\| \| \|x\| \right\} \\ \geq \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} - \| \|x^{-1}\| - \|y^{-1}\| \| \min \{ \|x\|, \|y\| \}$$

and the inequality (4.5) is obtained. □

Corollary 4.4. *If $(x, y) \in A^2$ are invertible, then*

$$(4.7) \quad \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} - \| \|x^{-1}\| - \|y^{-1}\| \| \cdot \frac{\|x\| + \|y\|}{2} \\ \leq \| \|x^{-1}\| x \pm \|y^{-1}\| y \| .$$



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Remark 3. We observe that the inequalities (4.3) and (4.7) are in fact equivalent with:

$$(4.8) \quad \left| \left| \|x^{-1}\| x \pm \|y^{-1}\| y \right\| - \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} \right| \\ \leq \left| \|x^{-1}\| - \|y^{-1}\| \right| \cdot \frac{\|x\| + \|y\|}{2}.$$

Now we consider the dual expansion $\left| \|y^{-1}\| x \pm \|x^{-1}\| y \right\|$, for which the following upper bounds can be stated.

Proposition 4.5. *If (x, y) are invertible in A , then*

$$(4.9) \quad \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right\| \\ \leq \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \max \{ \|x\|, \|y\| \}$$

and

$$(4.10) \quad \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right\| \\ \leq \|x \pm y\| \max \{ \|x^{-1}\|, \|y^{-1}\| \} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \min \{ \|x\|, \|y\| \}.$$

In particular,

$$(4.11) \quad \left| \|y^{-1}\| x \pm \|x^{-1}\| y \right\| \\ \leq \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} + \left| \|x^{-1}\| - \|y^{-1}\| \right| \cdot \frac{\|x\| + \|y\|}{2}.$$

The proof follows from Theorem 3.2 on choosing $a = \|y^{-1}\| \cdot 1$ and $b = \|x^{-1}\| \cdot 1$. The lower bounds for the quantity $\left| \|y^{-1}\| x \pm \|x^{-1}\| y \right\|$ are incorporated in:



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Proposition 4.6. *If (x, y) are invertible in A , then*

$$(4.12) \quad \|x \pm y\| \max \{ \|x^{-1}\|, \|y^{-1}\| \} - \| \|x^{-1}\| - \|y^{-1}\| \| \max \{ \|x\|, \|y\| \} \\ \leq \| \|y^{-1}\| x \pm \|x^{-1}\| y \|$$

and

$$(4.13) \quad \|x \pm y\| \min \{ \|x^{-1}\|, \|y^{-1}\| \} - \| \|x^{-1}\| - \|y^{-1}\| \| \min \{ \|x\|, \|y\| \} \\ \leq \| \|y^{-1}\| x \pm \|x^{-1}\| y \|.$$

In particular,

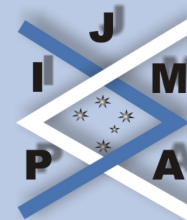
$$(4.14) \quad \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} - \| \|x^{-1}\| - \|y^{-1}\| \| \cdot \frac{\|x\| + \|y\|}{2} \\ \leq \| \|y^{-1}\| x \pm \|x^{-1}\| y \|.$$

Remark 4. We observe that the inequalities (4.11) and (4.14) are equivalent with

$$(4.15) \quad \left| \| \|y^{-1}\| x \pm \|x^{-1}\| y \| - \|x \pm y\| \cdot \frac{\|x^{-1}\| + \|y^{-1}\|}{2} \right| \\ \leq \| \|x^{-1}\| - \|y^{-1}\| \| \cdot \frac{\|x\| + \|y\|}{2}.$$

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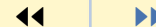
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