

ENERGY DECAY OF SOLUTIONS OF A WAVE EQUATION OF p -LAPLACIAN TYPE WITH A WEAKLY NONLINEAR DISSIPATION

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Abstract

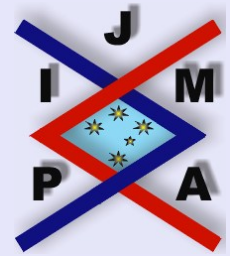
In this paper we study decay properties of the solutions to the wave equation of p -Laplacian type with a weak nonlinear dissipative.

2000 Mathematics Subject Classification: 35B40, 35L70.

Key words: Wave equation of p -Laplacian type, Decay rate.

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1. Introduction

We consider the initial boundary problem for the nonlinear wave equation of p -Laplacian type with a weak nonlinear dissipation of the type

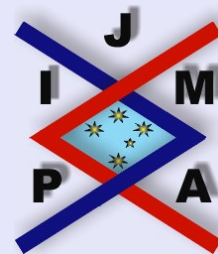
$$(P) \quad \begin{cases} (|u'|^{l-2}u')' - \Delta_p u + \sigma(t)g(u') = 0 \text{ in } \Omega \times [0, +\infty[, \\ u = 0 \text{ on } \partial\Omega \times [0, +\infty[, \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) \text{ in } \Omega. \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla_x u|^{p-2}\nabla_x u)$, $p, l \geq 2$, $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous non-decreasing function and σ is a positive function.

When $p = 2, l = 2$ and $\sigma \equiv 1$, for the case $g(x) = \delta x$ ($\delta > 0$), Ikehata and Suzuki [5] investigated the dynamics, showing that for sufficiently small initial data (u_0, u_1) , the trajectory $(u(t), u'(t))$ tends to $(0, 0)$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \rightarrow +\infty$. When $g(x) = \delta|x|^{m-1}x$ ($m \geq 1$), Nakao [8] investigated the decay property of the problem (P). In [8] the author has proved the existence of global solutions to the problem (P).

For the problem (P) with $\sigma \equiv 1, l = 2$, when $g(x) = \delta|x|^{m-1}x$ ($m \geq 1$), Yao [1] proved that the energy decay rate is $E(t) \leq (1+t)^{-\frac{p}{(mp-m-1)}}$ for $t \geq 0$ by using a general method introduced by Nakao [8]. Unfortunately, this method does not seem to be applicable in the case of more general functions σ and is more complicated.

Our purpose in this paper is to give energy decay estimates of the solutions to the problem (P) for a weak nonlinear dissipation. We extend the results obtained by Yao and prove in some cases an exponential decay when $p > 2$ and the dissipative term is not necessarily superlinear near the origin.



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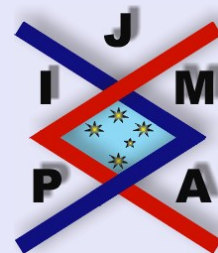
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We use a new method recently introduced by Martinez [7] (see also [2]) to study the decay rate of solutions to the wave equation $u'' - \Delta_x u + g(u') = 0$ in $\Omega \times \mathbb{R}^+$, where Ω is a bounded domain of \mathbb{R}^n . This method is based on a new integral inequality that generalizes a result of Haraux [4].

Throughout this paper the functions considered are all real valued. We omit the space variable x of $u(t, x)$, $u_t(t, x)$ and simply denote $u(t, x)$, $u_t(t, x)$ by $u(t)$, $u'(t)$, respectively, when no confusion arises. Let l be a number with $2 \leq l \leq \infty$. We denote by $\|\cdot\|_l$ the L^l norm over Ω . In particular, the L^2 norm is denoted by $\|\cdot\|_2$. (\cdot) denotes the usual L^2 inner product. We use familiar function spaces $W_0^{1,p}$.



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2. Preliminaries and Main Results

First assume that the solution exists in the class

$$(2.1) \quad u \in C(\mathbb{R}_+, W_0^{1,p}(\Omega)) \cap C^1(\mathbb{R}_+, L^l(\Omega)).$$

$\lambda(x), \sigma(t)$ and g satisfy the following hypotheses:

(H1) $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non increasing function of class C^1 on \mathbb{R}_+ satisfying

$$(2.2) \quad \int_0^{+\infty} \sigma(\tau) d\tau = +\infty.$$

(H2) Consider $g : \mathbb{R} \rightarrow \mathbb{R}$ a non increasing C^0 function such that

$$g(v)v > 0 \text{ for all } v \neq 0.$$

and suppose that there exist $c_i > 0; i = 1, 2, 3, 4$ such that

$$(2.3) \quad c_1|v|^m \leq |g(v)| \leq c_2|v|^{\frac{1}{m}} \text{ if } |v| \leq 1,$$

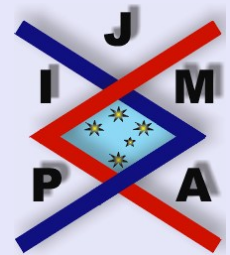
$$(2.4) \quad c_3|v|^s \leq |g(v)| \leq c_4|v|^r \text{ for all } |v| \geq 1,$$

where $m \geq 1, l - 1 \leq s \leq r \leq \frac{n(p-1)+p}{n-p}$.

We define the energy associated to the solution given by (2.1) by the following formula

$$E(t) = \frac{l-1}{l} \|u'\|_l^l + \frac{1}{p} \|\nabla_x u\|_p^p.$$

We first state two well known lemmas, and then state and prove a lemma that will be needed later.



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Lemma 2.1 (Sobolev-Poincaré inequality). Let q be a number with $2 \leq q < +\infty$ ($n = 1, 2, \dots, p$) or $2 \leq q \leq \frac{np}{(n-p)}$ ($n \geq p + 1$), then there is a constant $c_* = c(\Omega, q)$ such that

$$\|u\|_q \leq c_* \|\nabla u\|_p \quad \text{for } u \in W_0^{1,p}(\Omega).$$

Lemma 2.2 ([6]). Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $q \geq 0$ and $A > 0$ such that

$$\int_S^{+\infty} E^{q+1}(t) dt \leq AE(S), \quad 0 \leq S < +\infty,$$

then we have

$$E(t) \leq cE(0)(1+t)^{-\frac{1}{q}} \quad \forall t \geq 0, \quad \text{if } q > 0$$

and

$$E(t) \leq cE(0)e^{-\omega t} \quad \forall t \geq 0, \quad \text{if } q = 0,$$

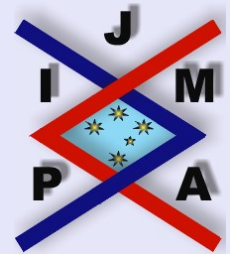
where c and ω are positive constants independent of the initial energy $E(0)$.

Lemma 2.3 ([7]). Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an increasing C^2 function such that

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

Assume that there exist $q \geq 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1}(t)\phi'(t) dt \leq AE(S), \quad 0 \leq S < +\infty,$$



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then we have

$$E(t) \leq cE(0)(1 + \phi(t))^{-\frac{1}{q}} \quad \forall t \geq 0, \quad \text{if } q > 0$$

and

$$E(t) \leq cE(0)e^{-\omega\phi(t)} \quad \forall t \geq 0, \quad \text{if } q = 0,$$

where c and ω are positive constants independent of the initial energy $E(0)$.

Proof of Lemma 2.3. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$. f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned} \int_{\phi(S)}^{\phi(T)} f(x)^{q+1} dx &= \int_{\phi(S)}^{\phi(T)} E(\phi^{-1}(x))^{q+1} dx \\ &= \int_S^T E(t)^{q+1} \phi'(t) dt \\ &\leq AE(S) = Af(\phi(S)) \quad 0 \leq S < T < +\infty. \end{aligned}$$

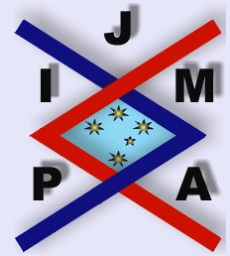
Setting $s := \phi(S)$ and letting $T \rightarrow +\infty$, we deduce that

$$\int_s^{+\infty} f(x)^{q+1} dx \leq Af(s) \quad 0 \leq s < +\infty.$$

By Lemma 2.2, we can deduce the desired results. \square

Our main result is the following

Theorem 2.4. Let $(u_0, u_1) \in W_0^{1,p} \times L^1(\Omega)$ and suppose that **(H1)** and **(H2)** hold. Then the solution $u(x, t)$ of the problem **(P)** satisfies



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(1) If $l \geq m + 1$, we have

$$E(t) \leq C(E(0)) \exp \left(1 - \omega \int_0^t \sigma(\tau) d\tau \right) \quad \forall t > 0.$$

(2) If $l < m + 1$, we have

$$E(t) \leq \left(\frac{C(E(0))}{\int_0^t \sigma(\tau) d\tau} \right)^{\frac{p}{(mp-m-1)}} \quad \forall t > 0.$$

Examples

1) If $\sigma(t) = \frac{1}{t^\theta}$ ($0 \leq \theta \leq 1$), by applying Theorem 2.4 we obtain

$$E(t) \leq C(E(0))e^{1-\omega t^{1-\theta}} \quad \text{if } \theta \in [0, 1[, l \geq m + 1,$$

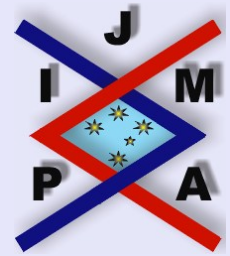
$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{mp-m-1}} \quad \text{if } 0 \leq \theta < 1, l < m + 1$$

and

$$E(t) \leq C(E(0))(\ln t)^{-\frac{p}{(mp-m-1)}} \quad \text{if } \theta = 1, l < m + 1.$$

2) If $\sigma(t) = \frac{1}{t^\theta \ln t \ln_2 t \dots \ln_k t}$, where k is a positive integer and

$$\begin{cases} \ln_1(t) = \ln(t) \\ \ln_{k+1}(t) = \ln(\ln_k(t)), \end{cases}$$



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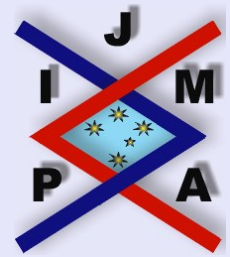


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by applying Theorem 2.4, we obtain

$$E(t) \leq C(E(0))(\ln_{k+1} t)^{-\frac{p}{(mp-m-1)}} \quad \text{if } \theta = 1, l < m + 1,$$

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{mp-m-1}} (\ln t \ln_2 t \dots \ln_k t)^{\frac{p}{mp-m-1}} \quad \text{if } 0 \leq \theta < 1, l < m+1.$$

3) If $\sigma(t) = \frac{1}{t^\theta(\ln t)^\gamma}$, by applying Theorem 2.4, we obtain

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{mp-m-1}} (\ln t)^{\frac{\gamma p}{mp-m-1}} \quad \text{if } 0 \leq \theta < 1, l < m + 1,$$

$$E(t) \leq C(E(0))(\ln t)^{-\frac{(1-\gamma)p}{mp-m-1}} \quad \text{if } \theta = 1, 0 \leq \gamma < 1, l < m + 1,$$

$$E(t) \leq C(E(0))(\ln_2 t)^{-\frac{p}{mp-m-1}} \quad \text{if } \theta = 1, \gamma = 1, l < m + 1.$$

Proof of Theorem 2.4.

First we have the following energy identity to the problem (P)

Lemma 2.5 (Energy identity). *Let $u(t, x)$ be a local solution to the problem (P) on $[0, \infty)$ as in Theorem 2.4. Then we have*

$$E(t) + \int_{\Omega} \int_0^t \sigma(s)u'(s)g(u'(s)) ds dx = E(0)$$

for all $t \in [0, \infty)$.

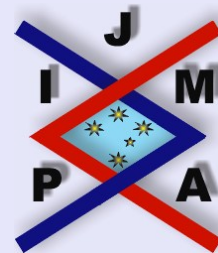
Proof of the energy decay. From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (P) by $E^q \phi' u$, where ϕ is a function satisfying all the hypotheses of Lemma 2.3 to obtain

$$\begin{aligned}
 0 &= \int_S^T E^q \phi' \int_{\Omega} u(|u'|^{l-2} u')_t - \Delta_p u + \sigma(t)g(u') dx dt \\
 &= \left[E^q \phi' \int_{\Omega} uu'|u'|^{l-2} dx \right]_S^T - \int_S^T (qE'E^{q-1}\phi' + E^q\phi'') \int_{\Omega} uu'|u'|^{l-2} dx dt \\
 &\quad - \frac{3l-2}{l} \int_S^T E^q \phi' \int_{\Omega} |u'|^2 dx dt + 2 \int_S^T E^q \phi' \int_{\Omega} \left(\frac{l-1}{l} u'^2 + \frac{1}{p} |\nabla u|^p \right) dx dt \\
 &\quad + \int_S^T E^q \phi' \int_{\Omega} \sigma(t)ug(u') dx dt + \left(1 - \frac{2}{p} \right) \int_S^T E^q \phi' \|\nabla u\|_p^p dx dt.
 \end{aligned}$$

We deduce that

$$\begin{aligned}
 (2.5) \quad 2 \int_S^T E^{q+1} \phi' dt &\leq - \left[E^q \phi' \int_{\Omega} uu'|u'|^{l-2} dx \right]_S^T \\
 &\quad + \int_S^T (qE'E^{q-1}\phi' + E^q\phi'') \int_{\Omega} uu'|u'|^{l-2} dx dt \\
 &\quad + \frac{3l-2}{l} \int_S^T E^q \phi' \int_{\Omega} |u'|^l dx dt - \int_S^T E^q \phi' \int_{\Omega} \sigma(t)ug(u') dx dt.
 \end{aligned}$$

Since E is nonincreasing, ϕ' is a bounded nonnegative function on \mathbb{R}_+ (and we



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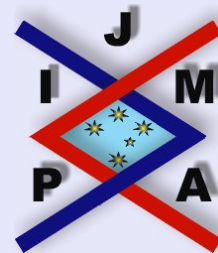
denote by μ its maximum), using the Hölder inequality, we have

$$\left| E(t)^q \phi' \int_{\Omega} uu' |u'|^{l-2} dx \right| \leq c\mu E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} \quad \forall t \geq S.$$

$$\begin{aligned} \int_S^T (qE' E^{q-1} \phi' + E^q \phi'') \int_{\Omega} uu' |u'|^{l-2} dx dt \\ \leq c\mu \int_S^T -E'(t) E(t)^{q-\frac{1}{l}+\frac{1}{p}} dt + c \int_S^T E(t)^{q+\frac{l-1}{l}+\frac{1}{p}} (-\phi''(t)) dt \\ \leq c\mu E(S)^{q+\frac{l-1}{l}+\frac{1}{p}}. \end{aligned}$$

Using these estimates we conclude from the above inequality that

$$\begin{aligned} (2.6) \quad 2 \int_S^T E(t)^{1+q} \phi'(t) dt \\ \leq cE(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + \frac{3l-2}{l} \int_S^T E^q \phi' \int_{\Omega} |u'|^l dx dt \\ - \int_S^T E^q \phi' \int_{\Omega} \sigma(t) u g(u') dx dt \\ \leq cE(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + \frac{3l-2}{l} \int_S^T E^q \phi' \int_{\Omega} |u'|^l dx dt \\ - \int_S^T E^q \phi' \int_{|u'| \leq 1} \sigma(t) u g(u') dx dt \\ - \int_S^T E^q \phi' \int_{|u'| > 1} \sigma(t) u g(u') dx dt. \end{aligned}$$



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Define

$$\phi(t) = \int_0^t \sigma(s) ds.$$

It is clear that ϕ is a non decreasing function of class C^2 on \mathbb{R}_+ . The hypothesis (2.2) ensures that

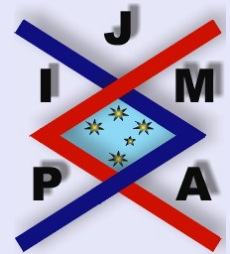
$$(2.7) \quad \phi(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty.$$

Now, we estimate the terms of the right-hand side of (2.6) in order to apply the results of Lemma 2.3:

Using the Hölder inequality, we get for $l < m + 1$

$$\begin{aligned} & \int_S^T E^q \phi' \int_{\Omega} |u'|^l dx dt \\ & \leq C \int_S^T E^q \phi' \int_{\Omega} \frac{1}{\sigma(t)} u' \rho(t, u') dx dt \\ & \quad + C' \int_S^T E^q \phi' \int_{\Omega} \left(\frac{1}{\sigma(t)} u' \rho(t, u') \right)^{\frac{l}{m+1}} dx dt \\ & \leq C \int_S^T E^q \frac{\phi'}{\sigma(t)} (-E') dt + C'(\Omega) \int_S^T E^q \frac{\phi'}{\sigma^{\frac{l}{m+1}}(t)} (-E')^{\frac{l}{m+1}} dt \\ & \leq CE^{q+1}(S) + C'(\Omega) \int_S^T E^q \phi' \frac{m+1-l}{m+1} \left(\frac{\phi'}{\sigma(t)} \right)^{\frac{l}{m+1}} (-E')^{\frac{l}{m+1}} dt. \end{aligned}$$

Now, fix an arbitrarily small $\varepsilon > 0$ (to be chosen later). By applying Young's



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inequality, we obtain

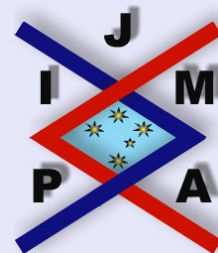
$$\begin{aligned}
 (2.8) \quad & \int_S^T E^q \phi' \int_{\Omega} |u'|^l dx dt \\
 & \leq C E^{q+1}(S) + C'(\Omega) \frac{m+l}{m+1} \varepsilon^{\frac{(m+1)}{(m+1-l)}} \int_S^T E^{q \frac{m+1}{m+1-l}} \phi' dt \\
 & \quad + C'(\Omega) \frac{l}{m+1} \frac{1}{\varepsilon^{\frac{(m+1)}{l}}} E(S).
 \end{aligned}$$

If $l \geq m+1$, we easily obtain from (2.3) and (2.4)

$$(2.9) \quad \int_S^T E^q \phi' \int_{\Omega} |u'|^l dx dt \leq C E^{q+1}(S).$$

Next, we estimate the third term of the right-hand of (2.6). We get for $l < m+1$

$$\begin{aligned}
 (2.10) \quad & \int_S^T E^q \phi' \int_{|u'| \leq 1} \sigma(t) u g(u') dx dt \\
 & \leq \varepsilon_1 \int_S^T E^q \phi' \int_{|u'| \leq 1} \|u\|_p^p dt + C(\varepsilon_1) \int_S^T E^q \phi' \int_{|u'| \leq 1} (\sigma g(u'))^{\frac{p}{p-1}} dx \\
 & \leq c \varepsilon_1 \int_S^T E^{q+1} \phi' dt + C(\varepsilon_1) \int_S^T E^q \phi' \int_{|u'| \leq 1} (\sigma g(u'))^{\frac{p}{p-1}} dx.
 \end{aligned}$$



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We now estimate the last term of the above inequality to get

$$\begin{aligned}
 (2.11) \quad & \int_S^T E^q \phi' \int_{|u'| \leq 1} (\sigma g(u'))^{\frac{p}{p-1}} dx dt \\
 & \leq \int_S^T E^q \phi' \int_{|u'| \leq 1} (u' g(u'))^{\frac{p}{(m+1)(p-1)}} dx dt \\
 & \leq \int_S^T E^q \phi' \frac{1}{\sigma^{\frac{p}{(m+1)(p-1)}}} \int_{|u'| \leq 1} (\sigma u' g(u'))^{\frac{p}{(m+1)(p-1)}} dx dt \\
 & \leq C(\Omega) \int_S^T E^q \phi' \frac{1}{\sigma^{\frac{p}{(m+1)(p-1)}}} (-E')^{\frac{p}{(m+1)(p-1)}} dt.
 \end{aligned}$$

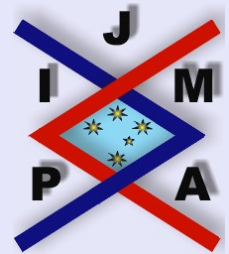
Set $\varepsilon_2 > 0$; due to Young's inequality, we obtain

$$\begin{aligned}
 (2.12) \quad & \int_S^T E^q \phi' \int_{|u'| \leq 1} (\sigma g(u'))^{\frac{p}{p-1}} dx dt \\
 & \leq C(\Omega) \frac{(m+1)(p-1) - p}{(m+1)(p-1)} \varepsilon_2^{\frac{(m+1)(p-1)}{(m+1)(p-1) - p}} \int_S^T E^q \frac{(m+1)(p-1)}{(m+1)(p-1) - p} \phi' dt \\
 & \quad + \frac{C(\Omega) p}{(m+1)(p-1)} \frac{1}{\varepsilon_2^{\frac{(m+1)(p-1)}{p}}} E(S),
 \end{aligned}$$

we chose q such that

$$q \frac{(m+1)(p-1)}{(m+1)(p-1) - p} = q + 1.$$

thus we find $q = \frac{mp - m - 1}{p}$ and thus $q \frac{m+1}{m+1-l} = q + 1 + \alpha$ with $\alpha = \frac{(m+1)(p-l-p-l)}{p(m+1-l)}$.



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Using the Hölder inequality, the Sobolev imbedding and the condition (2.4), we obtain

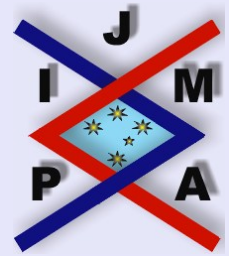
$$\begin{aligned}
 & \int_S^T E^q \phi' \int_{|u'| \geq 1} \sigma(t) u g(u') \, dx dt \\
 & \leq \int_S^T E^q \phi' \sigma(t) \left(\int_{\Omega} |u|^{r+1} \, dx \right)^{\frac{1}{(r+1)}} \left(\int_{|u'| > 1} |g(u')|^{\frac{r+1}{r}} \, dx \right)^{\frac{r}{r+1}} dt \\
 & \leq c \int_S^T E^{q+\frac{1}{p}} \phi' \sigma^{\frac{1}{(r+1)}}(t) \left(\int_{|u'| > 1} \sigma u' g(u') \, dx \right)^{\frac{r}{r+1}} dt \\
 & \leq c \int_S^T E^{q+\frac{1}{p}} \phi' \sigma^{\frac{1}{(r+1)}}(t) (-E')^{\frac{r}{r+1}} dt.
 \end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}
 (2.13) \quad & \int_S^T E^q \phi' \int_{|u'| \geq 1} \sigma(t) u g(u') \, dx dt \\
 & \leq \varepsilon_3 \int_S^T (E^{q+\frac{1}{p}} \phi' \sigma^{\frac{1}{(r+1)}}(t))^{r+1} dt + c(\varepsilon_3) \int_S^T (-E') dt \\
 & \leq \varepsilon_3 \mu^{r+1} E^{\frac{(p-1)(mr-1)}{p}}(0) \int_S^T E^{q+1} \phi' dt + c(\varepsilon_3) E(S).
 \end{aligned}$$

If $l \geq m + 1$, the last inequality is also valid in the domain $\{|u'| < 1\}$ and with m instead of r .

Choosing $\varepsilon, \varepsilon_1, \varepsilon_2$ and ε_3 small enough, we deduce from (2.6), (2.8), (2.10),



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(2.12) and (2.13) for $l < m + 1$

$$\int_S^T E(t)^{1+q} \phi'(t) dt \leq CE(S)^{q+1} + C'E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + C''E(S) \\ + C'''E(0)^{\frac{(p-l-p-l)(m+1)}{p^l}} E(S) + C''''E(0)^{\frac{(m-r-1)(p-1)}{p^r}} E(S),$$

where C, C', C'', C''', C'''' are different positive constants independent of $E(0)$.

Choosing ε_3 small enough, we deduce from (2.6), (2.9) and (2.13) for $l \geq m + 1$

$$\int_S^T E(t)^{1+q} \phi'(t) dt \leq CE(S)^{q+1} + C'E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + C''E(0)^{\frac{(m^2-1)(p-1)}{p^m}} E(S),$$

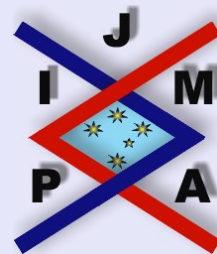
where C, C', C'' are different positive constants independent of $E(0)$, we may thus complete the proof by applying Lemma 2.3. \square

Remark 1. We obtain the same results for the following problem

$$\begin{cases} (|u'|^{l-2}u')' - e^{-\Phi(x)} \operatorname{div}(e^{\Phi(x)} |\nabla_x u|^{p-2} \nabla_x u) + \sigma(t)g(u') = 0 \text{ in } \Omega \times [0, +\infty[, \\ u = 0 \text{ on } \partial\Omega \times [0, +\infty[, \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) \text{ in } \Omega, \end{cases}$$

where Φ is a positive function such that $\Phi \in L^\infty(\Omega)$, in this case $(u_0, u_1) \in W_{0,\Phi}^{1,p} \times L_\Phi^l$, where

$$W_{0,\Phi}^{1,p}(\Omega) = \left\{ u \in W_0^{1,p}(\Omega), \int_\Omega e^{\Phi(x)} |\nabla_x u|^p dx < \infty \right\}, \\ L_\Phi^l(\Omega) = \left\{ u \in L^l(\Omega), \int_\Omega e^{\Phi(x)} |u|^l dx < \infty \right\}.$$



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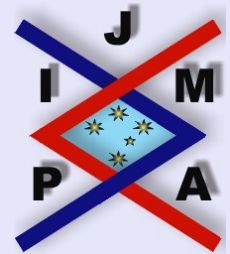
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Thus the energy associated to the solution is given by the following formula

$$E(t) = \frac{l-1}{l} \|e^{\Phi(x)/l} u'\|_l^l + \frac{1}{p} \|e^{\Phi(x)/p} \nabla_x u\|_p^p.$$



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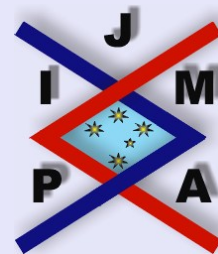
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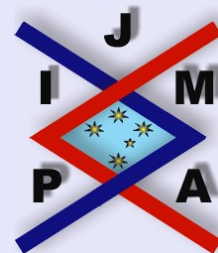
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