journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 1, Article 7, 3 pp.

A NEW PROOF OF SOME INEQUALITY CONNECTED WITH QUADRATURES

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Received 04 November, 2007; accepted 02 January, 2008
Communicated by I. Gavrea

ABSTRACT. Another proof of some inequality from [5] is given. It is based on the spline approximation of convex functions of higher order.

Key words and phrases: Convex functions of higher order, Inequalities, Quadrature rules, Splines, Spline approximation.

2000 *Mathematics Subject Classification.* Primary: 26A51, 26D15; Secondary: 41A55, 65D07, 65D32.

1. INTRODUCTION

Recall that a real function f defined on a real interval I is n–convex ($n \in \mathbb{N}$), if its divided differences involving $n+2$ points are nonnegative, i.e.

$$[x_1, \dots, x_{n+2}; f] \geq 0$$

for any $n+2$ distinct points $x_1, \dots, x_{n+2} \in I$ (cf. [3]). Then 1–convexity reduces to an ordinary convexity.

In the recent paper [5] we established the order structure of a set of some six quadrature operators in the class of 3–convex functions mapping $[-1,1]$ into $\mathbb{R}$. In this setting we have proved among others the inequality

$$\frac{1}{3}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right] \leq \frac{1}{12}[f(-1)+f(1)] + \frac{5}{12}\left[f\left(-\frac{\sqrt{5}}{5}\right) + f\left(\frac{\sqrt{5}}{5}\right)\right] \tag{1.1}$$

between a three–point Chebyshev quadrature operator (on the left–hand side) and a four–point Lobatto quadrature operator (on the right–hand side). The proof was rather complicated. In its main part we needed to determine the inverse of a 4×4 matrix with entries all of the form $a + b\sqrt{2} + c\sqrt{5} + d\sqrt{10}$. This was done using computer software. The only thing done precisely was the computation of the determinant to be sure the inverse does exist.

Now we would like to propose an easy to verify proof of (1.1), without the use of any computer software, based on the spline approximation of convex functions of higher order. In the paper [4] one can find the following result (as a quotation from [1]).

Theorem 1.1. *Every n–convex function mapping $[a, b]$ into $\mathbb{R}$ can be uniformly approximated on $[a, b]$ by spline functions of the form*

$$x \mapsto p(x) + \sum_{i=1}^{m} a_i (x - c_i)_+^n,$$

where p is a polynomial of degree at most n, $a_i > 0$, $c_i \in [a, b]$ $(i = 1, \dots, m)$ and $y_+ := \max\{y, 0\}$, $y \in \mathbb{R}$.

Observe that all spline functions of the above form are also n–convex. Indeed, if $x_1, \dots, x_{n+2} \in [a, b]$ are distinct, then by the properties of divided differences we have $[x_1, \dots, x_{n+2}; p] = 0$. If $c \in \mathbb{R}$, then

$$B(c) := [x_1, \dots, x_{n+2}; (\cdot - c)_+^n]$$

is a value at the point c of the B–spline on the knots $x_1, \dots, x_{n+2}$. Hence $B(c) \geq 0$ (cf. [2]) and the function $(\cdot - c)_+^n$ is n–convex. Obviously, the conical combination of n–convex functions is n–convex by linearity of the divided differences.

2. A NEW PROOF OF THE INEQUALITY (1.1)

Theorem 2.1. *If $f : [-1, 1] \to \mathbb{R}$ is 3–convex then* (1.1) *holds.*

Proof. By Theorem 1.1 it is enough to prove (1.1) only for polynomials of degree at most 3 and for any function of the form $x \mapsto (x - c)_+^3$, $c \in \mathbb{R}$.

To show that (1.1) holds for polynomials of degree at most 3 (even with the equality), we check it for the monomials 1, x, x^2, x^3 and use linearity.

Now let $c \in \mathbb{R}$. Rearranging (1.1) we have to prove that

$$\begin{aligned}\varphi(c) := (-1 - c)_+^3 + (1 - c)_+^3 + 5\left[\left(-\frac{\sqrt{5}}{5} - c\right)_+^3 + \left(\frac{\sqrt{5}}{5} - c\right)_+^3\right] \\ - 4\left[\left(-\frac{\sqrt{2}}{2} - c\right)_+^3 + (-c)_+^3 + \left(\frac{\sqrt{2}}{2} - c\right)_+^3\right] \geq 0.\end{aligned}$$

Obviously $\varphi(c) = 0$ for $c \notin [-1, 1]$. We compute

$$\varphi(c) = \begin{cases} (c+1)^3 & \text{for } -1 \leq c < -\frac{\sqrt{2}}{2}, \\ -3c^3 + 3(1 - 2\sqrt{2})c^2 - 3c + 1 - \sqrt{2} & \text{for } -\frac{\sqrt{2}}{2} \leq c < -\frac{\sqrt{5}}{5}, \\ 2c^3 + 3(1 + \sqrt{5} - 2\sqrt{2})c^2 + 1 + \frac{\sqrt{5}}{5} - \sqrt{2} & \text{for } -\frac{\sqrt{5}}{5} \leq c < 0, \\ -2c^3 + 3(1 + \sqrt{5} - 2\sqrt{2})c^2 + 1 + \frac{\sqrt{5}}{5} - \sqrt{2} & \text{for } 0 \leq c < \frac{\sqrt{5}}{5}, \\ 3c^3 + 3(1 - 2\sqrt{2})c^2 + 3c + 1 - \sqrt{2} & \text{for } \frac{\sqrt{5}}{5} \leq c < \frac{\sqrt{2}}{2}, \\ (1-c)^3 & \text{for } \frac{\sqrt{2}}{2} \leq c \leq 1. \end{cases}$$

Then we can see that φ is an even function and it is enough to check that $\varphi(c) \geq 0$ only for $0 \leq c \leq 1$. We have

$$\varphi'(c) = \begin{cases} -6c^2 + 6(1+\sqrt{5}-2\sqrt{2})c & \text{for } 0 \leq c < \frac{\sqrt{5}}{5}, \\ 9c^2 + 6(1-2\sqrt{2})c + 3 & \text{for } \frac{\sqrt{5}}{5} \leq c < \frac{\sqrt{2}}{2}, \\ -3(1-c)^2 & \text{for } \frac{\sqrt{2}}{2} \leq c \leq 1. \end{cases}$$

Then

$$\begin{aligned} \varphi'(c) = 0 \quad & \text{for } c \in \{0, 1+\sqrt{5}-2\sqrt{2}, 1\}, \\ \varphi'(c) > 0 \quad & \text{for } c \in (0, 1+\sqrt{5}-2\sqrt{2}), \\ \varphi'(c) < 0 \quad & \text{for } c \in (1+\sqrt{5}-2\sqrt{2}, 1). \end{aligned}$$

Hence φ is increasing on $[0, 1+\sqrt{5}-2\sqrt{2})$ and decreasing on $[1+\sqrt{5}-2\sqrt{2}, 1]$. Finally we compute

$$\begin{aligned} \varphi(0) &= 1 + \frac{\sqrt{5}}{5} - \sqrt{2} > 0, \\ \varphi(1+\sqrt{5}-2\sqrt{2}) &= (1+\sqrt{5}-2\sqrt{2})^3 + 1 + \frac{\sqrt{5}}{5} - \sqrt{2} > 0, \\ \varphi(1) &= 0, \end{aligned}$$

which proves that $\varphi \geq 0$ on $[0,1]$ and finishes the proof. $\square$

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