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INCLUSION AND NEIGHBORHOOD PROPERTIES OF SOME ANALYTIC AND MULTIVALENT FUNCTIONS

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ABSTRACT. By means of a certain extended derivative operator of Ruscheweyh type, the authors introduce and investigate two new subclasses of p-valently analytic functions of complex order. The various results obtained here for each of these function classes include coefficient inequalities and the consequent inclusion relationships involving the neighborhoods of the p-valently analytic functions.

Key words and phrases: Analytic functions, p-valent functions, Hadamard product (or convolution), Coefficient bounds, Ruscheweyh derivative operator, Neighborhood of analytic functions.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}_p(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geqq 0;\ n, p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

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The Hadamard product (or convolution) of the function $f \in \mathcal{A}_p(n)$ given by (1.1) and the function $g \in \mathcal{A}_p(n)$ given by

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k\, z^k \quad (b_k \geqq 0;\ n, p \in \mathbb{N}) \tag{1.2}$$

is defined (as usual) by

$$(f * g)(z) := z^p + \sum_{k=n+p}^{\infty} a_k\, b_k\, z^k =: (g * f)(z). \tag{1.3}$$

We introduce here an extended linear derivative operator of Ruscheweyh type:

$$\mathcal{D}^{\lambda,p} : \mathcal{A}_p \to \mathcal{A}_p \quad \big(\mathcal{A}_p := \mathcal{A}_p(1)\big),$$

which is defined by the following convolution:

$$\mathcal{D}^{\lambda,p} f(z) = \frac{z^p}{(1-z)^{\lambda+p}} * f(z) \quad (\lambda > -p;\ f \in \mathcal{A}_p)\,. \tag{1.4}$$

In terms of the binomial coefficients, we can rewrite (1.4) as follows:

$$\mathcal{D}^{\lambda,p} f(z) = z^p - \sum_{k=1+p}^{\infty} \binom{\lambda+k-1}{k-p} a_k z^k \quad (\lambda > -p;\ f \in \mathcal{A}_p)\,. \tag{1.5}$$

In particular, when $\lambda = n$ $(n \in \mathbb{N})$, it is easily observed from (1.4) and (1.5) that

$$\mathcal{D}^{n,p} f(z) = \frac{z^p\big(z^{n-p} f(z)\big)^{(n)}}{n!} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ p \in \mathbb{N})\,, \tag{1.6}$$

so that

$$\mathcal{D}^{1,p} f(z) = (1-p) f(z) + z f'(z), \tag{1.7}$$

$$\mathcal{D}^{2,p} f(z) = \frac{(1-p)(2-p)}{2!} f(z) + (2-p) z f'(z) + \frac{z^2}{2!} f''(z), \tag{1.8}$$

and so on.

By using the operator

$$\mathcal{D}^{\lambda,p} f(z) \quad (\lambda > -p;\ p \in \mathbb{N})$$

given by (1.5), we now introduce a new subclass $\mathcal{H}_{n,m}^p(\lambda, b)$ of the p-valently analytic function class $\mathcal{A}_p(n)$, which includes functions $f(z)$ satisfying the following inequality:

$$\left| \frac{1}{b} \left(\frac{z\big(\mathcal{D}^{\lambda,p} f(z)\big)^{(m+1)}}{\big(\mathcal{D}^{\lambda,p} f(z)\big)^{(m)}} - (p-m) \right) \right| < 1 \tag{1.9}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ m \in \mathbb{N}_0;\ \lambda \in \mathbb{R};\ p > \max(m, -\lambda);\ b \in \mathbb{C} \setminus \{0\})\,.$$

Next, following the earlier investigations by Goodman [3], Ruscheweyh [5] and Altintaş *et al.* [2] (see also [1], [4] and [6]), we define the (n, δ)-neighborhood of a function $f(z) \in \mathcal{A}_n(p)$ by (see, for details, [2, p. 1668])

$$\mathcal{N}_{n,\delta}(f) := \left\{ g \in \mathcal{A}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k\, z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqq \delta \right\}. \tag{1.10}$$

It follows from (1.10) that, if

$$h(z) = z^p \quad (p \in \mathbb{N}), \tag{1.11}$$

then

$$\mathcal{N}_{n,\delta}(h) = \left\{ g \in \mathcal{A}_p(n) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k\, z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k\,|b_k| \leqq \delta \right\}. \tag{1.12}$$

Finally, we denote by $\mathcal{L}^p_{n,m}(\lambda, b; \mu)$ the subclass of $\mathcal{A}_p(n)$ consisting of functions $f(z)$ which satisfy the inequality (1.13) below:

$$\left| \frac{1}{b} \left(p(1-\mu) \left(\frac{\mathcal{D}^{\lambda,p} f(z)}{z} \right)^{(m)} + \mu \left(\mathcal{D}^{\lambda,p} f(z) \right)^{(m+1)} - (p-m) \right) \right| < p - m \tag{1.13}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ m \in \mathbb{N}_0;\ \lambda \in \mathbb{R};\ p > \max(m, -\lambda);\ \mu \geqq 0;\ b \in \mathbb{C} \setminus \{0\}).$$

The object of the present paper is to investigate the various properties and characteristics of analytic p-valent functions belonging to the subclasses

$$\mathcal{H}^p_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^p_{n,m}(\lambda, b; \mu),$$

which we have introduced here. Apart from deriving a set of coefficient bounds for each of these function classes, we establish several inclusion relationships involving the (n, δ)-neighborhoods of analytic p-valent functions (with negative *and* missing coefficients) belonging to these subclasses.

Our definitions of the function classes

$$\mathcal{H}^p_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^p_{n,m}(\lambda, b; \mu)$$

are motivated essentially by two earlier investigations [1] and [4], in each of which further details and references to other closely-related subclasses can be found. In particular, in our definition of the function class $\mathcal{L}^p_{n,m}(\lambda, b; \mu)$ involving the inequality (1.13), we have relaxed the parametric constraint $0 \leqq \mu \leqq 1$, which was imposed earlier by Murugusundaramoorthy and Srivastava [4, p. 3, Equation (1.14)] (see also Remark 3 below).

2. A SET OF COEFFICIENT BOUNDS

In this section, we prove the following results which yield the coefficient inequalities for functions in the subclasses

$$\mathcal{H}^p_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^p_{n,m}(\lambda, b; \mu).$$

Theorem 1. *Let* $f(z) \in \mathcal{A}_p(n)$ *be given by* (1.1). *Then* $f(z) \in \mathcal{H}^p_{n,m}(\lambda, b)$ *if and only if*

$$\sum_{k=n+p}^{\infty} \binom{\lambda + k - 1}{k - p} \binom{k}{m} (k + |b| - p)\, a_k \leqq |b| \binom{p}{m}. \tag{2.1}$$

Proof. Let a function $f(z)$ of the form (1.1) belong to the class $\mathcal{H}^p_{n,m}(\lambda, b)$. Then, in view of (1.5), (1.9) yields the following inequality:

$$\Re \left(\frac{\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} (p-k) z^{k-p}}{\binom{p}{m} - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} z^{k-p}} \right) > -|b| \quad (z \in \mathbb{U}). \tag{2.2}$$

Putting $z = r$ $(0 \leqq r < 1)$ in (2.2), we observe that the expression in the denominator on the left-hand side of (2.2) is positive for $r = 0$ and also for all r $(0 < r < 1)$. Thus, by letting $r \to 1-$ through *real* values, (2.2) leads us to the desired assertion (2.1) of Theorem 1.

Conversely, by applying (2.1) and setting $|z| = 1$, we find by using (1.5) that

$$\begin{aligned}&\left|\frac{z\left(\mathcal{D}^{\lambda,p}f(z)\right)^{(m+1)}}{\left(\mathcal{D}^{\lambda,p}f(z)\right)^{(m)}}-(p-m)\right|\\&\qquad=\left|\frac{\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}(p-k)z^{k-m}}{\binom{p}{m}z^{p-m}-\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}z^{k-m}}\right|\\&\qquad\leqq\frac{|b|\left[\binom{p}{m}-\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}a_k\right]}{\binom{p}{m}-\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}a_k}=|b|\,.\end{aligned}$$

Hence, by the maximum modulus principle, we infer that $f(z) \in \mathcal{H}^p_{n,m}(\lambda, b)$, which completes the proof of Theorem 1. □

Remark 1. In the special case when

(2.3) $$m = 0,\ p = 1, \quad \text{and} \quad b = \beta\gamma \quad (0 < \beta \leqq 1;\ \gamma \in \mathbb{C} \setminus \{0\})\,,$$

Theorem 1 corresponds to a result given earlier by Murugusundaramoorthy and Srivastava [4, p. 3, Lemma 1].

By using the same arguments as in the proof of Theorem 1, we can establish Theorem 2 below.

Theorem 2. *Let* $f(z) \in \mathcal{A}_p(n)$ *be given by* (1.1)*. Then* $f(z) \in \mathcal{L}^p_{n,m}(\lambda, b; \mu)$ *if and only if*

(2.4) $$\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k-1}{m}\left[\mu\,(k-1)+1\right]a_k \leqq (p-m)\left[\frac{|b|-1}{m!}+\binom{p}{m}\right].$$

Remark 2. Making use of the same parametric substitutions as mentioned above in (2.3), Theorem 2 yields another known result due to Murugusundaramoorthy and Srivastava [4, p. 4, Lemma 2].

3. Inclusion Relationships Involving (n, δ)-Neighborhoods

In this section, we establish several inclusion relationships for the function classes

$$\mathcal{H}^p_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^p_{n,m}(\lambda, b; \mu)$$

involving the (n, δ)-neighborhood defined by (1.12).

Theorem 3. *If*

(3.1) $$\delta = \frac{(n+p)\,|b|\,\binom{p}{m}}{(n+|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}} \qquad (p > |b|)\,,$$

then

(3.2) $$\mathcal{H}^p_{n,m}(\lambda, b) \subset \mathcal{N}_{n,\delta}(h).$$

Proof. Let $f(z) \in \mathcal{H}^p_{n,m}(\lambda, b)$. Then, in view of the assertion (2.1) of Theorem 1, we have

(3.3) $$(n+|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}\sum_{k=n+p}^{\infty}a_k \leqq |b|\binom{p}{m}.$$

This yields

$$\sum_{k=n+p}^{\infty} a_k \leqq \frac{|b| \binom{p}{m}}{(n+|b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m}}. \tag{3.4}$$

Applying the assertion (2.1) of Theorem 1 again, in conjunction with (3.4), we obtain

$$\begin{aligned}
\binom{\lambda+n+p-1}{n} \binom{n+p}{m} \sum_{k=n+p}^{\infty} k a_k \\
&\leqq |b| \binom{p}{m} + (p-|b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m} \sum_{k=n+p}^{\infty} a_k \\
&\leqq |b| \binom{p}{m} + (p-|b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m} \\
&\qquad \cdot \frac{|b| \binom{p}{m}}{(n+|b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m}} \\
&= |b| \binom{p}{m} \left(\frac{n+p}{n+|b|} \right).
\end{aligned}$$

Hence

$$\sum_{k=n+p}^{\infty} k a_k \leqq \frac{|b| (n+p) \binom{p}{m}}{(n+|b|) \binom{\lambda+n+p-1}{n} \binom{n+p}{m}} =: \delta \quad (p > |b|), \tag{3.5}$$

which, by virtue of (1.12), establishes the inclusion relation (3.2) of Theorem 3. □

In an analogous manner, by applying the assertion (2.4) of Theorem 2 instead of the assertion (2.1) of Theorem 1 to functions in the class $\mathcal{L}_{n,m}^{p}(\lambda, b; \mu)$, we can prove the following inclusion relationship.

Theorem 4. *If*

$$\delta = \frac{(p-m)(n+p) \left[\frac{|b|-1}{m!} + \binom{p}{m} \right]}{[\mu (n+p-1) + 1] \binom{\lambda+n+p-1}{n} \binom{n+p}{m}} \quad (\mu > 1), \tag{3.6}$$

then

$$\mathcal{L}_{n,m}^{p}(\lambda, b; \mu) \subset \mathcal{N}_{n,\delta}(h).$$

Remark 3. Applying the parametric substitutions listed in (2.3), Theorems 3 and 4 would yield the known results due to Murugusundaramoorthy and Srivastava [4, p. 4, Theorem 1; p. 5, Theorem 2]. Incidentally, just as we indicated in Section 2 above, the condition $\mu > 1$ is needed in the proof of one of these known results [4, p. 5, Theorem 2]. This implies that the constraint $0 \leqq \mu \leqq 1$ in [4, p. 3, Equation (1.14)] should be replaced by the less stringent constraint $\mu \geqq 0$.

4. FURTHER NEIGHBORHOOD PROPERTIES

In this last section, we determine the neighborhood properties for each of the following (slightly modified) function classes:

$$\mathcal{H}_{n,m}^{p,\alpha}(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p,\alpha}(\lambda, b; \mu).$$

Here the class $\mathcal{H}^{p,\alpha}_{n,m}(\lambda, b)$ consists of functions $f(z) \in \mathcal{A}_p(n)$ for which there exists another function $g(z) \in \mathcal{H}^{p}_{n,m}(\lambda, b)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < p)\,. \tag{4.1}$$

Analogously, the class $\mathcal{L}^{p,\alpha}_{n,m}(\lambda, b; \mu)$ consists of functions $f(z) \in \mathcal{A}_p(n)$ for which there exists another function $g(z) \in \mathcal{L}^{p}_{n,m}(\lambda, b; \mu)$ satisfying the inequality (4.1).

The proofs of the following results involving the neighborhood properties for the classes

$$\mathcal{H}^{p,\alpha}_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^{p,\alpha}_{n,m}(\lambda, b; \mu)$$

are similar to those given in [1] and [4]. We, therefore, skip their proofs here.

Theorem 5. *Let* $g(z) \in \mathcal{H}^{p}_{n,m}(\lambda, b)$. *Suppose also that*

$$\alpha = p - \frac{\delta\,(n + |b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}}{(n+p)\left[(n+|b|)\binom{\lambda+n+p-1}{n+p}\binom{n+p}{m} - |b|\binom{p}{m}\right]}. \tag{4.2}$$

Then

$$\mathcal{N}_{n,\delta}(g) \subset \mathcal{H}^{p,\alpha}_{n,m}(\lambda, b). \tag{4.3}$$

Theorem 6. *Let* $g(z) \in \mathcal{L}^{p}_{n,m}(\lambda, b; \mu)$. *Suppose also that*

$$\alpha = p - \frac{\delta\,[\mu\,(n+p-1)+1]\binom{\lambda+n+p-1}{n}\binom{n+p-1}{m}}{(n+p)\left[[\mu\,(n+p-1)+1]\binom{\lambda+n+p-1}{n}\binom{n+p-1}{m} - (p-m)\left\{\frac{|b|-1}{m!} + \binom{p}{m}\right\}\right]}. \tag{4.4}$$

Then

$$\mathcal{N}_{n,\delta}(g) \subset \mathcal{L}^{p,\alpha}_{n,m}(\lambda, b; \mu). \tag{4.5}$$

REFERENCES

[1] O. ALTINTAŞ, Ö. ÖZKAN AND H.M. SRIVASTAVA, Neighborhoods of a class of analytic functions with negative coefficients, *Appl. Math. Lett.*, **13**(3) (2000), 63–67.

[2] O. ALTINTAŞ, Ö. ÖZKAN AND H.M. SRIVASTAVA, Neighborhoods of a certain family of multivalent functions with negative coefficients, *Comput. Math. Appl.*, **47** (2004), 1667–1672.

[3] A.W. GOODMAN, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, **8** (1957), 598–601.

[4] G. MURUGUSUNDARAMOORTHY AND H.M. SRIVASTAVA, Neighborhoods of certain classes of analytic functions of complex order, *J. Inequal. Pure Appl. Math.*, **5**(2) (2004), Art. 24. 8 pp. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=374`].

[5] S. RUSCHEWEYH, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81** (1981), 521–527.

[6] H.M. SRIVASTAVA AND S. OWA (Eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.