journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 3, Article 88, 14 pp.

UNIQUENESS OF ENTIRE OR MEROMORPHIC FUNCTIONS SHARING ONE VALUE OR A FUNCTION WITH FINITE WEIGHT

HONG-YAN XU AND TING-BIN CAO

DEPARTMENT OF INFORMATICS AND ENGINEERING
JINGDEZHEN CERAMIC INSTITUTE (XIANGHU XIAOQU)
JINGDEZHEN, JIANGXI 333403, CHINA
xhyhhh@126.com

DEPARTMENT OF MATHEMATICS
NANCHANG UNIVERSITY
NANCHANG, JIANGXI 330031, CHINA
tbcao@ncu.edu.cn

Received 13 December, 2008; accepted 27 August, 2009
Communicated by S.S. Dragomir

ABSTRACT. The purpose of this paper is to deal with some uniqueness problems of entire functions or meromorphic functions concerning differential polynomials that share one value or fixed-points with finite weight. We obtain a number of theorems which generalize some results due to M.L. Fang & X.H. Hua, X.Y. Zhang & W.C. Lin, X.Y. Zhang & J.F. Chen and W.C. Lin.

Key words and phrases: Meromorphic function, Entire function, Weighted sharing, Uniqueness.

2000 *Mathematics Subject Classification.* 30D30, 30D35.

1. INTRODUCTION AND MAIN RESULTS

Let f be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of value distribution theory:

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \dots$$

(see Hayman [6],Yang [13] and Yi and Yang [16]). We denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o(T(r,f)),$$

as $r \to +\infty$, possibly outside of a set with finite measure. A meromorphic function a is called a small function with respect to f if $T(r,a) = S(r,f)$. Let $S(f)$ be the set of meromorphic functions in the complex plane $\mathbb{C}$ which are small functions with respect to f. For some $a \in \mathbb{C} \cup \infty$, we define

$$\Theta(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

The research was supported by the NNSF of China (No.10371065), the NSF of Jiangxi of China (2008GQS0075).
335-08

For $a \in \mathbb{C} \cup \infty$ and k a positive integer, we denote by $N(r, a; f| = 1)$ the counting function of simple a-points of f, and denote by $N(r, a; f| \leq k)$ $(N(r, a; f| \geq k))$ the counting functions of those a-points of f whose multiplicities are not greater (less) than k where each a-point is counted according to its multiplicity (see [6]). $\overline{N}(r, a; f| \leq k)(\overline{N}(r, a; f| \geq k))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Set

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2) + \cdots + \overline{N}(r, a; f \geq k).$$

We define

$$\delta_k(a, f) = 1 - \varlimsup_{r\to\infty} \frac{N_k(r, a; f)}{T(r, f)}.$$

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in S(f) \cap S(g)$ the roots of $f - a$ and $g - a$ coincide in locations and multiplicities we say that f and g share the value a CM (counting multiplicities) and if they coincide in locations only we say that f and g share a IM (ignoring multiplicities).

In 1997, Yang and Hua [14] proved the following result.

Theorem A ([14]). *Let f and g be two nonconstant entire functions, $n \geq 6$ a positive integer. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$, where c, c_1, and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = 1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Using the same argument as in [14], Fang [3] proved the following result.

Theorem B ([3]). *Let f and g be two nonconstant entire functions and let n, k be two positive integers with $n > 2k+4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share the value 1 CM, then either $f = c_1e^{cz}, g = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Fang [5] obtained some unicity theorems corresponding to Theorem B.

Theorem C ([5]). *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 8$. If $[f^n(f - 1)]^{(k)}$ and $[g^n(g - 1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

Recently, Zhang and Lin [17], Zhang, Chen and Lin [18] extended Theorem C and obtained the following results.

Theorem D ([17]). *Let f and g be two nonconstant entire functions, n, m and k be three positive integers with $n > 2k+m+4$, and λ, μ be constants such that $|\lambda|+|\mu| \neq 0$. If $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 CM, then*

(i) *when $\lambda\mu \neq 0$, $f \equiv g$;*
(ii) *when $\lambda\mu = 0$, either $f \equiv tg$, where t is a constant satisfying $t^{n+m} = 1$, or $f = c_1e^{cz}, g = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying*

$$(-1)^k\lambda^2(c_1c_2)^{n+m}[(n + m)c]^{2k} = 1 \qquad \text{or} \qquad (-1)^k\mu^2(c_1c_2)^{n+m}[(n + m)c]^{2k} = 1.$$

Theorem E ([18]). *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n \geq 3m + 2k + 5$, and let $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$ or $P(z) \equiv c_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0, c_0 \neq 0$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 CM, then*

(i) *when $P(z) = a_mz^m+a_{m-1}z^{m-1}+\cdots+a_1z+a_0$, either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_1\omega_1 + a_0) \\ - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_1\omega_2 + a_0);$$

(ii) *when* $P(z) \equiv c_0$, *either* $f = c_1/\sqrt[n]{c_0}e^{cz}$, $g = c_2/\sqrt[n]{c_0}e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying* $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, *or* $f \equiv tg$ *for a constant* t *such that* $t^n = 1$.

Regarding Theorems D and E, it is natural to ask the following question.

Problem 1.1. In Theorems D and E, can the nature of sharing 1 CM be further relaxed?

For meromorphic functions, Yang and Hua [14] proved the following result corresponding to Theorem A.

Theorem F ([14])**.** *Let* f *and* g *be two nonconstant meromorphic functions,* $n \geq 11$ *an integer, and* $a \in C - \{0\}$. *If* f^nf' *and* g^ng' *share the value* a CM, *then either* $f = dg$ *for some* $(n+1)th$ *root of unity* d *or* $g = c_1e^{cz}$ *and* $f = c_2e^{-cz}$, *where* c, c_1, *and* c_2 *are constants satisfying* $(c_1c_2)^{n+1}c^2 = -a^2$.

Lin and Yi [7] obtained some unicity theorems corresponding to Theorem F.

Theorem G ([7])**.** *Let* f *and* g *be two nonconstant meromorphic functions satisfying* $\Theta(\infty, f) > \frac{2}{n+1}$, $n \geq 12$. *If* $[f^n(f-1)]f'$ *and* $[g^n(g-1)]g'$ *share 1* CM, *then* $f \equiv g$.

Lin and Yi [8] extended Theorem G by replacing the value 1 with the function z and obtained the following result.

Theorem H ([8])**.** *Let* f *and* g *be two transcendental meromorphic functions,* $n \geq 12$ *an integer. If* $f^n(f-1)f'$ *and* $g^n(g-1)g'$ *share* z CM, *then either* $f \equiv g$ *or* $g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$ *and* $f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, *where* h *is a nonconstant meromorphic function.*

Recently, Zhang, Chen and Lin [18] extended Theorems F and G and obtained the following result.

Theorem I ([18])**.** *Let* f *and* g *be two nonconstant meromorphic functions, and let* n *and* m *be two positive integers with* $n > \max\{m+10, 3m+3\}$, *and let* $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$, *where* $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ *are complex constants. If* $f^nP(f)f'$ *and* $g^nP(g)g'$ *share 1* CM, *then either* $f \equiv tg$ *for a constant* t *such that* $t^d = 1$, *where* $d = (n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ *for some* $i = 0, 1, \ldots, m$, *or* f *and* g *satisfy the algebraic equation* $R(f, g) \equiv 0$, *where*

$$R(\omega_1, \omega_2) = \omega_1^n \left(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1} \right) - \omega_2^n \left(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1} \right).$$

Regarding Theorem I, it is natural to ask the following questions.

Problem 1.2. Is it possible that the value 1 can be replaced by a function z in Theorem I?

Problem 1.3. Is it possible to relax the nature of sharing z in Theorem I and if possible, how far?

In 2001, Lahiri [9, 10] first employed the idea of weighted sharing of values which measures how close a shared value is to being shared IM or to being shared CM. Recently, many mathematicians (such as H. X. Yi, I. Lahiri, M. L. Fang, A. Banerjee, W. C. Lin, X. Yan) have been interested in investigating meromorphic functions sharing values with finite weight in the field of complex analysis.

We first introduced the notion of weighted sharing of values as follows.

Definition 1.1 ([9, 10]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$, then we say that f, g share the value a CM.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$; and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k, clearly if f, g share (a, k), then f, g share (a, p) for all integers p $(0 \leq p \leq k)$.Also, we note that f, g share a value a IM or CM if and only if they share $(a, 0)$ or (a, ∞), respectively.

With the notion of weighted sharing of values, we investigate the solution of the above question and obtain the following results.

Theorem 1.1. *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n \geq 5m + 5k + 8$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, 0)$, then the conclusion of Theorem E still holds.*

Theorem 1.2. *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n > \frac{9}{2}m + 4k + \frac{9}{2}$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, 1)$, then the conclusion of Theorem E still holds.*

Theorem 1.3. *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n \geq 3m + 3k + 5$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, 2)$, then the conclusion of Theorem E still holds.*

Remark 1. From Theorems 1.1 – 1.3, we obtain a positive answer to Question 1.1.

Theorem 1.4. *Let f and g be two transcendental meromorphic functions, and let n and m be two positive integers with $n > m + 10$, and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^n P(f) f'$ and $g^n P(g) g'$ share z CM, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^n \left(\frac{a_m \omega_1^m}{n + m + 1} + \frac{a_{m-1} \omega_1^{m-1}}{n + m} + \cdots + \frac{a_0}{n + 1} \right) - \omega_2^n \left(\frac{a_m \omega_2^m}{n + m + 1} + \frac{a_{m-1} \omega_2^{m-1}}{n + m} + \cdots + \frac{a_0}{n + 1} \right).$$

Theorem 1.5. *Let f and g be two transcendental meromorphic functions, and let n and m be two positive integers with $n > 4m + 22$, and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^n P(f) f'$ and $g^n P(g) g'$ share z IM, then the conclusion of Theorem 1.4 still holds.*

Theorem 1.6. *Let f and g be two transcendental meromorphic functions, let n, l and m be three positive integers, and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $E_{l)}(z, f^n P(f) f') = E_{l)}(z, g^n P(g) g')$,*

(i) *If $l = 1$ and $n > 3m + 18$, then the conclusion of Theorem 1.4 still holds.*

(ii) *If* $l = 2$ *and* $n > \frac{3}{2}m + 12$*, then the conclusion of Theorem 1.4 still holds.*

Remark 2. Theorem 1.4 is an improvement of Theorem H. Theorem 1.5 and 1.6 are complements to Theorem H.

Though the standard definitions and notations of value distribution theory are available in [6, 13], we explain the ones which are used in the paper.

Definition 1.2 ([1, 16])**.** When f and g share $1\ IM$, We denote by $\overline{N}_L(r, 1; f)$ the counting function of the 1-points of f whose multiplicities are greater than 1-points of g, where each zero is counted only once; Similarly, we have $\overline{N}_L(r, 1; g)$. Let z_0 be a zero of $f - 1$ of multiplicity p and a zero of $g - 1$ of multiplicity q, we also denote by $N_{11}(r, 1; f)$ the counting function of those 1-points of f where $p = q = 1$; $\overline{N}_E^{(2}(r, 1; f)$ denotes the counting function of those 1-points of f where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way, one can define $N_{11}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.

Definition 1.3 ([9, 10])**.** Let f, g share a value $1\ IM$. We denote by $\overline{N}_*(r, 1; f, g)$ the reduced counting function of those 1-points of f whose multiplicities differ from the multiplicities of the corresponding 1-points of g. Clearly $\overline{N}_*(r, 1; f, g) \equiv \overline{N}_*(r, 1; g, f)$ and $\overline{N}_*(r, 1; f, g) = \overline{N}_L(r, 1; f) + \overline{N}_L(r, 1; g)$.

2. SOME LEMMAS

For the proof of our results we need the following lemmas.

Lemma 2.1 ([15, p. 27, Theorem 1.12])**.** *Let* f *be a nonconstant meromorphic function and* $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$*, where* $a_0, a_1, a_2, \ldots, a_n$ *are constants and* $a_n \neq 0$*. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([18])**.** *Let* f *be a transcendental entire function, let* n, k, m *be positive integers with* $n \geq k+2$*, and* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$*, where* $a_0, a_1, a_2, \ldots, a_m$ *are complex constants. Then* $[f^n P(f)]^{(k)} = 1$ *has infinitely many solutions.*

Lemma 2.3 ([18])**.** *Let* f *and* g *be two nonconstant entire functions, and let* n, k *be two positive integers with* $n > k$*, and let* $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ *be a nonzero polynomial, where* $a_0, a_1, \ldots, a_{m-1}, a_m$ *are complex constants. If* $[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv 1$*, then* $P(z)$ *is reduced to a nonzero monomial, that is,* $P(z) = a_i z^i \not\equiv 0$ *for some* $i = 0, 1, \ldots, m$*; further,*$f = c_1 / \sqrt[n+i]{a_i} e^{cz}$, $g = c_2 / \sqrt[n+i]{a_i} e^{-cz}$*, where* c_1, c_2 *and* c *are three constants satisfying* $(-1)^k (c_1 c_2)^n [(n+1)c)]^{2k} = 1$.

Let f be an entire function; we have $\Theta(\infty, f) = 1$. Using the same argument as [12, Lemma 2.12], we can easily obtain the following lemma.

Lemma 2.4. *Let* f *and* g *be two entire functions, and let* k *be a positive integer.If* $f^{(k)}$ *and* $g^{(k)}$ *share* $(1, l)$ $(l = 0, 1, 2)$*. Then*

(i) *If* $l = 0$*,*

$$\Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \delta_{k+2}(0, f) + \delta_{k+2}(0, g) > 5, \tag{2.1}$$

then either $f^{(k)} g^{(k)} \equiv 1$ *or* $f \equiv g$*;*

(ii) *If* $l = 1$*,*

$$\frac{1}{2}(\Theta(0, f) + \delta_k(0, f) + \delta_{k+2}(0, f)) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \Theta(0, g) + \delta_k(0, g) > \frac{9}{2}, \tag{2.2}$$

then either $f^{(k)} g^{(k)} \equiv 1$ *or* $f \equiv g$*;*

(iii) If $l = 2$,

$$\Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f)\} + \delta_{k+2}(0, g) > 3, \tag{2.3}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Lemma 2.5. *Let f and g be two transcendental meromorphic functions, let n and m be three positive integers with $n > 7$, and let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^n P(f) f'$ and $g^n P(g) g'$ share z IM, then $S(r, f) = S(r, g)$.*

Proof. Using the same arguments as in [8] and [18], we easily obtain Lemma 2.5. □

Lemma 2.6. *Let f and g be two transcendental meromorphic functions, and let n and m be three positive integers with $n \geq m + 3$, $F_1 = \frac{f^n P(f) f'}{z}$ and $G_1 = \frac{g^n P(g) g'}{z}$, where $n(\geq 4)$ is a positive integer. If $F_1 \equiv G_1$, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2)$ is as stated in Theorem 1.4.*

Proof. Using the same arguments as those in [11] and [18], we can easily get Lemma 2.6. □

Lemma 2.7. *Let f and g be two transcendental meromorphic functions. Then*

$$f^n P(f) f' g^n P(g) g' \not\equiv z^2,$$

where $n \geq m + 4$ is a positive integer.

Proof. Using the same argument as in [11] and [18], we easily obtain Lemma 2.7. □

Lemma 2.8 ([3])**.** *Let f and g be two meromorphic functions. If f and g share 1 CM, one of the following three cases holds:*

(i) $T(r, f) \leq N_2(r, \infty, f) + N_2(r, \infty, g) + N_2(r, 0, f) + N_2(r, 0, g) + S(r, f) + S(r, g)$, *the same inequality holding for $T(r, g)$;*

(ii) $f \equiv g$;

(iii) $f \cdot g \equiv 1$.

Lemma 2.9 ([4])**.** *Let f and g be two meromorphic functions, and let l be a positive integer. If $E_{l)}(1, f) = E_{l)}(1, g)$, then one of the following cases must occur:*

(i):

$$\begin{aligned} T(r, f) + T(r, g) \leq{}& N_2(r, \infty; f) + N_2(r, 0; f) + N_2(r, \infty; g) \\ &+ N_2(r, 0; g) + \overline{N}(r, 1; f) + \overline{N}(r, 1; g) - N_{11}(r, 1; f) \\ &+ \overline{N}(r, 1; f| \geq l + 1) + \overline{N}(r, 1; g| \geq l + 1) + S(r, f) + S(r, g); \end{aligned}$$

(ii): $f = \frac{(b+1)g + (a-b-1)}{bg + (a-b)}$, *where $a(\neq 0), b$ are two constants.*

Lemma 2.10 ([4])**.** *Let f and g be two meromorphic functions. If f and g share 1 IM, then one of the following cases must occur:*

(i):

$$\begin{aligned} T(r, f) + T(r, g) \leq{}& 2[N_2(r, \infty; f) + N_2(r, 0; f) + N_2(r, \infty; g) + N_2(r, 0; g)] \\ &+ 3\overline{N}_L(r, 1; f) + 3\overline{N}_L(r, 1; g) + S(r, f) + S(r, g); \end{aligned}$$

(ii): $f = \frac{(b+1)g + (a-b-1)}{bg + (a-b)}$, *where $a(\neq 0), b$ are two constants.*

Lemma 2.11. *Let f and g be two transcendental meromorphic functions, $n > m+6$ be a positive integer, and let $F_1 = \frac{f^nP(f)f'}{z}$ and $G_1 = \frac{g^nP(g)g'}{z}$. If*

$$F_1 = \frac{(b+1)G_1 + (a-b-1)}{bG_1 + (a-b)}, \tag{2.4}$$

where $a(\neq 0)$, b are two constants, then the conclusion of Theorem 1.4 still holds.

Proof. By Lemma 2.1 we know that

$$\begin{aligned} T(r, F_1) &= T\left(r, \frac{f^nP(f)f'}{z}\right) \\ &\le T(r, f^nP(f)) + T(r, f') + \log r \\ &\le (n+m)T(r,f) + 2T(r,f) + \log r + S(r,f) \\ &= (n+m+2)T(r,f) + \log r + S(r,f), \end{aligned} \tag{2.5}$$

$$\begin{aligned} &(n+m)T(r,f) \\ &= T(r, f^nP(f)) + S(r,f) \\ &= N(r, \infty; f^nP(f)) + m(r, f^nP(f)) + S(r,f) \\ &\le N\left(r, \infty; \frac{f^nP(f)f'}{z}\right) - N(r,\infty; f') \\ &\qquad + m\left(r, \frac{f^nP(f)f'}{z}\right) + m\left(r, \frac{1}{f'}\right) + \log r + S(r,f) \\ &\le T\left(r, \frac{f^nP(f)f'}{z}\right) + T(r, f') - N(r,\infty;f') - N(r,0;f') + \log r + S(r,f) \\ &\le T(r, F_1) + T(r,f) - N(r,\infty;f) - N(r,0;f') + \log r + S(r,f). \end{aligned} \tag{2.6}$$

So

$$T(r, F_1) \ge (n+m-1)T(r,f) + N(r,\infty;f) + N(r,0;f') + \log r + S(r,f). \tag{2.7}$$

Thus, by (2.5), (2.7) and $n > m+6$, we get $S(r, F_1) = S(r,f)$. Similarly,

$$T(r, G_1) \ge (n+m-1)T(r,g) + N(r,\infty;g) + N(r,0;g') + \log r + S(r,g). \tag{2.8}$$

Without loss of generality, we suppose that $T(r,f) \le T(r,g)$, $r \in I$, where I is a set with infinite measure. Next, we consider three cases.

Case 1. $b \neq 0, -1$, If $a-b-1 \neq 0$, then by (2.4) we know

$$\overline{N}\left(r, -\frac{a-b-1}{b+1}; G_1\right) = \overline{N}(r, 0; F_1).$$

Since

$$N(r,0;g') \le N(r,\infty;g) + N(r,0;g) + S(r,g) \le 2T(r,g) + S(r,g). \tag{2.9}$$

By Nevanlinna's second fundamental theorem and (2.9) we have

$$\begin{aligned} T(r, G_1) &\le \overline{N}(r,\infty;G_1) + \overline{N}(r,0;G_1) + \overline{N}\left(r, -\frac{a-b-1}{b+1}; G_1\right) + S(r, G_1) \\ &\le \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + mT(r,g) + N(r,0;g') + \overline{N}(r,0;f) \\ &\qquad + mT(r,f) + \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + 2\log r + S(r,g) \\ &\le (2m+4)T(r,g) + \overline{N}(r,\infty;g) + N(r,0;g') + 2\log r + S(r,g). \end{aligned}$$

Hence, by $n > m+6$ and (2.8), we know $T(r,g) \leq S(r,g), r \in I$, this is impossible.

If $a-b-1=0$, then by (2.4) we know $F_1 = ((b+1)G_1)/(bG_1+1)$. Obviously,

$$\overline{N}\left(r, -\frac{1}{b}; G_1\right) = \overline{N}(r, \infty; F_1).$$

By the Nevanlinna second fundamental theorem and (2.9) we have

$$\begin{aligned} T(r,G_1) &\leq \overline{N}(r,\infty;G_1) + \overline{N}(r,0;G_1) + \overline{N}\left(r,-\frac{1}{b};G_1\right) + S(r,G_1) \\ &\leq \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + mT(r,g) + N(r,0;g') \\ &\qquad + \overline{N}(r,\infty;f) + 2\log r + S(r,g) \\ &\leq (m+2)T(r,g) + \overline{N}(r,\infty;g) + N(r,0;g') + 2\log r + S(r,g). \end{aligned}$$

Then by $n > m+6$ and (2.8), we know $T(r,g) \leq S(r,g), r \in I$, a contradiction.

Case 2. $b=-1$. Then (2.4) becomes $F_1 = a/(a+1-G_1)$.

If $a+1 \neq 0$, then $\overline{N}(r,a+1;G_1) = \overline{N}(r,\infty;F_1)$. Applying a similar argument to that for **Case 1**, we can again deduce a contradiction.

If $a+1=0$, then $F_1 \cdot G_1 \equiv 1$, that is,

$$f^nP(f)f'g^nP(g)g' \not\equiv z^2.$$

Since $n \geq m+6$, by Lemma 2.7 we get a contradiction.

Case 3. $b=0$. Then (2.4) becomes $F_1 = (G_1+a-1)/a$.

If $a-1 \neq 0$, then $\overline{N}(r,1-a;G_1) = \overline{N}(r,0;F_1)$. Applying a similar argument to that for **Case 1**, we can again deduce a contradiction.

If $a-1=0$, then $F_1 \equiv G_1$, that is

$$f^nP(f)f' \equiv g^nP(g)g'.$$

By Lemma 2.6, we obtain the conclusions of Lemma 2.11.

Thus we complete the proof of Lemma 2.11. □

3. THE PROOFS OF THEOREMS 1.1–1.3

3.1. Proof of Theorem 1.1.

Proof. (i) $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$.

By the assumptions of Theorem 1.1 and Lemma 2.2, we know that either both f and g are transcendental entire functions or both f and g are polynomials.

First, we consider the case when f and g are transcendental entire functions.

Let $F = f^nP(f)$ and $G = g^nP(g)$, from the condition of Theorem 1.1, we know that F, G share $(1,0)$.

By Lemma 2.1 we can easily get

$$\begin{aligned} \Theta(0,F) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,0;F)}{T(r,F)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,0;f^nP(f))}{(n+m)T(r,f)} \\ &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,0;f) + \overline{N}(r,0;P(f))}{(n+m)T(r,f)}, \end{aligned}$$

$i.e.$

$$\Theta(0,F) \geq 1 - \frac{m+1}{n+m} = \frac{n-1}{n+m}. \tag{3.1}$$

Similarly, we have

$$\Theta(0,G) \geq \frac{n-1}{n+m}. \tag{3.2}$$

Next, by the definition of $N_k(r,a;f)$ we have

$$\delta_{k+1}(0,f) = 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r,0;f)}{T(r,f)} \geq 1 - \limsup_{r\to\infty} \frac{(k+1)\overline{N}(r,0;f)}{T(r,f)},$$

$$\delta_{k+1}(0,F) = 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r,0;f^nP(f))}{T(r,F)}.$$

Therefore

$$\delta_{k+1}(0,F) \geq 1 - \limsup_{r\to\infty} \frac{(m+k+1)T(r,f)}{(n+m)T(r,f)} = \frac{n-k-1}{n+m}. \tag{3.3}$$

Similarly we get

$$\delta_{k+1}(0,G) \geq \frac{n-k-1}{n+m} \tag{3.4}$$

and

$$\delta_{k+2}(0,F) \geq \frac{n-k-2}{n+m}, \qquad \delta_{k+2}(0,G) \geq \frac{n-k-2}{n+m}. \tag{3.5}$$

From (3.1) – (3.5) and F,G share $(1,0)$, we can get

$$\begin{aligned}\Theta(0,f) &+ \delta_k(0,f) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) + \delta_{k+2}(0,f) + \delta_{k+2}(0,g)\\ &\geq \frac{n-1}{n+m} + \frac{n-k}{n+m} + 2\frac{n-k-1}{n+m} + 2\frac{n-k-2}{n+m} = \frac{6n-5k-7}{n+m}.\end{aligned}$$

By $n > 5m+5k+7$, we have

$$\Theta(0,f) + \delta_k(0,f) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) + \delta_{k+2}(0,f) + \delta_{k+2}(0,g) > 5.$$

Therefore, by Lemma 2.4, we deduce either $F^{(k)} \cdot G^{(k)} \equiv 1$ or $F \equiv G$.

If $F^{(k)} \cdot G^{(k)} \equiv 1$, that is

$$[f^n(a_mf^m + a_{m-1}f^{m-1} + \cdots + a_0)]^{(k)}[g^n(a_mg^m + a_{m-1}g^{m-1} + \cdots + a_0)]^{(k)} \equiv 1, \tag{3.6}$$

then by the assumptions of Theorem 1.1 and Lemma 2.3 we can get a contradiction. Hence, we deduce that $F \equiv G$, that is

$$f^n(a_mf^m + a_{m-1}f^{m-1} + \cdots + a_0) = g^n(a_mg^m + a_{m-1}g^{m-1} + \cdots + a_0). \tag{3.7}$$

Let $h = f/g$. If h is a constant, then substituting $f = gh$ into (3.7) we deduce

$$a_mg^{n+m}(h^{n+m}-1) + a_{m-1}g^{n+m-1}(h^{n+m-1}-1) + \cdots + a_0g^n(h^n-1) = 0,$$

which implies $h^d = 1$, where $d = (n+m,\ldots,n+m-i,\ldots,n)$,$a_{m-1} \neq 0$ for some $i = 0,1,\ldots,m$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m,\ldots,n+m-i,\ldots,n)$,$a_{m-i} \neq 0$ for some $i = 0,1,\ldots,m$.

If h is not a constant, then we know by (3.7) that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_1\omega_1 + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_1\omega_2 + a_0)$. This proves (i) of Theorem 1.1.

Now we consider the case when f and g are two polynomials. From F, G share (1,0), we have

$$[f^n(a_m f^m + a_{m-1}f^{m-1} + \cdots + a_0)]^{(k)} - 1 \\ = c\{[g^n(a_m g^m + a_{m-1}g^{m-1} + \cdots + a_0)]^{(k)} - 1\}, \tag{3.8}$$

where c is a nonzero constant. Let $\deg f = l$. Then by (3.8) we know that $\deg g = l$. Differentiating the two sides of (3.8), we can get

$$f^{n-k-1}q_1 = g^{n-k-1}q_2, \tag{3.9}$$

where q_1, q_2 are two polynomials with $\deg q_1 = \deg q_2 = (m + k + 1)l - (k + 1)$. By $n \geq 4m + 4k + 8$, we have $\deg g^{n-k-1} = (n - k - 1)l > \deg q_2$. Therefore by (3.9) we know that there exists z_0 such that $f(z_0) = g(z_0) = 0$. Hence, by (3.8) and $f(z_0) = g(z_0) = 0$, we deduce that $c = 1$, i.e.,

$$[f^n(a_m f^m + a_{m-1}f^{m-1} + \cdots + a_0)]^{(k)} \equiv [g^n(a_m g^m + a_{m-1}g^{m-1} + \cdots + a_0)]^{(k)}. \tag{3.10}$$

Then we have

$$f^n(a_m f^m + a_{m-1}f^{m-1} + \cdots + a_0) - g^n(a_m g^m + a_{m-1}g^{m-1} + \cdots + a_0) = p(z), \tag{3.11}$$

where $p(z)$ is a polynomial of degree at most $k - 1$. Next, we prove $p(z) = 0$ by rewriting (3.10) as

$$f^{n-k}p_1 = g^{n-k}p_2, \tag{3.12}$$

where p_1, p_2 are two polynomials with $\deg p_1 = \deg p_2 = (m + k)l - k$ and $\deg f = l$.

Therefore, the total number of the common zeros of f^{n-k} and g^{n-k} is at least k. Then by (3.11) we deduce that $p(z) \equiv 0$, i.e.,

$$f^n(a_m f^m + a_{m-1}f^{m-1} + \cdots + a_1 f + a_0) \equiv g^n(a_m g^m + a_{m-1}g^{m-1} + \cdots + a_1 g + a_0).$$

Then using the same argument of (3.7), we can also get the case (i) of Theorem 1.1.

(ii) $P(z) \equiv c_0$. From Theorem B, we can easily see that the case (ii) of Theorem 1.1 holds.

Thus, we complete the proof of Theorem 1.1. □

3.2. **Proof of Theorem 1.2.**

Proof. From the condition of Theorem 1.2 and Lemma 2.4(ii), using the same argument of Theorem 1.1, Theorem 1.2 can be easily proved . □

3.3. **Proof of Theorem 1.3.**

Proof. From the condition of Theorem 1.3 and Lemma 2.4(iii), using the same argument of Theorem 1.1, Theorem 1.3 can be easily proved . □

4. THE PROOFS OF THEOREMS 1.4–1.6

Let F_1 and G_1 be defined as in Lemma 2.11 and

$$F^* = \frac{a_m f^{n+m+1}}{n+m+1} + \frac{a_{m-1}f^{n+m-1}}{n+m} + \cdots + \frac{a_0 f^{n+1}}{n+1}$$

and

$$G^* = \frac{a_m g^{n+m+1}}{n+m+1} + \frac{a_{m-1}g^{n+m-1}}{n+m} + \cdots + \frac{a_0 g^{n+1}}{n+1}.$$

4.1. **Proof of Theorem 1.4.**

Proof. From the condition of Theorem 1.4, then F_1 and G_1 share $z\ CM$.

By Lemma 2.1, we have

$$T(r,F^*)=(n+m+1)T(r,f)+S(r,f),\qquad T(r,G^*)=(n+m+1)T(r,g)+S(r,g). \tag{4.1}$$

Since $(F^*)'=F_1z$, we deduce

$$m\left(r,\frac{1}{F^*}\right)\le m\left(r,\frac{1}{zF_1}\right)+S(r,f)\le m\left(r,\frac{1}{F_1}\right)+\log r+S(r,f),$$

and by Nevanlinna's first fundamental theorem

$$\begin{aligned}T(r,F^*)&\le T(r,F_1)+N(r,0;F^*)-N(r,0;F_1)+\log r+S(r,f)\\&\le T(r,F_1)+N(r,0;f)+N(r,b_1;f)+\cdots+N(r,b_m;f)-N(r,c_1;f)\\&\quad-\cdots-N(r,c_m;f)-N(r,0;f')+\log r+S(r,f),\end{aligned} \tag{4.2}$$

where $b_1,b_2,\dots,b_m$ are roots of the algebraic equation

$$\frac{a_mz^m}{n+m+1}+\frac{a_{m-1}z^{m-1}}{n+m}+\cdots+\frac{a_0}{n+1}=0,$$

and $c_1,c_2,\dots,c_m$ are roots of the algebraic equation

$$a_mz^m+a_{m-1}z^{m-1}+\cdots+a_1z+a_0=0.$$

By the definition of F_1,G_1, we have

$$\begin{aligned}N_2(r,0;F_1)+N_2(r,\infty;F_1)&\le 2\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+N(r,c_1;f)\\&\quad+\cdots+N(r,c_m;f)+N(r,0;f')+2\log r.\end{aligned} \tag{4.3}$$

Similarly, we obtain

$$\begin{aligned}N_2(r,0;G_1)+N_2(r,\infty;G_1)&\le 2\overline{N}(r,\infty;g)+2\overline{N}(r,0;g)+N(r,c_1;g)\\&\quad+\cdots+N(r,c_m;g)+N(r,0;g')+2\log r.\end{aligned} \tag{4.4}$$

If Lemma 2.8(i) holds, from (4.2) – (4.4) we have

$$T(r,F_1)\le(m+5)T(r,f)+(m+6)T(r,g)+4\log r+S(r,f)+S(r,g). \tag{4.5}$$

Similarly, we obtain

$$T(r,G_1)\le(m+5)T(r,g)+(m+6)T(r,f)+4\log r+S(r,f)+S(r,g). \tag{4.6}$$

By (4.1), (4.5), (4.6) and $n>m+10$, we can obtain a contradiction.

If Lemma 2.8(ii) holds, then $F_1\equiv G_1$. By Lemma 2.6, we can get the conclusion of Theorem 1.4.

If Lemma 2.8(iii) holds, then $F_1\cdot G_1\equiv 1$. By Lemma 2.7 and $n>m+10$, we can get a contradiction.

Therefore, we complete the proof of Theorem 1.4. □

4.2. **Proof of Theorem 1.5.**

Proof. Suppose that (i) in Lemma 2.10 holds. Since

$$\begin{aligned}\overline{N}_L(r,1;F_1) &\leq N\left(r,\infty;\frac{F_1}{F_1'}\right) \\ &= N\left(r,\infty;\frac{F_1'}{F_1}\right)+S(r,f) \\ &\leq \overline{N}(r,\infty;F_1)+\overline{N}(r,0;F_1)+S(r,f) \\ &\leq (m+4)T(r,f)+2\log r+S(r,f),\end{aligned} \tag{4.7}$$

similarly, we have

$$\overline{N}_L(r,1;G_1) \leq (m+4)T(r,g)+2\log r+S(r,g). \tag{4.8}$$

By (4.2) – (4.4), (4.7) – (4.8) and Lemma 2.10(i), we have

$$(n-4m-22)[T(r,f)+T(r,g)] \leq 20\log r+S(r,f)+S(r,g). \tag{4.9}$$

By $n>4m+22$, we get a contradiction. Hence F_1 and G_1 satisfy (ii) in Lemma 2.10. By Lemma 2.11, we can get the conclusion of Theorem 1.5.

Thus, we complete the proof of Theorem 1.5. □

4.3. **Proof of Theorem 1.6.**

Proof. (i) If $l=1$. Since

$$\begin{aligned}\overline{N}(r,1;F_1)+&\overline{N}(r,1;G_1)-N_{11}(r,1;F_1) \\ &\leq \frac{1}{2}N(r,1;F_1)+\frac{1}{2}N(r,1;G_1) \\ &\leq \frac{1}{2}T(r,F_1)+\frac{1}{2}T(r,G_1)+S(r,f)+S(r,g).\end{aligned}$$

Suppose that (i) in Lemma 2.9 holds, then we have

$$\begin{aligned}T(r,F_1)+T(r,G_1) \leq 2[&N_2(r,0;F_1)+N_2(r,\infty;F_1)+N_2(r,0;G_1) \\ &+N_2(r,\infty;G_1)+\overline{N}(r,1;F_1|\geq 2)+\overline{N}(r,1;G_1|\geq 2)]+S(r,f)+S(r,g).\end{aligned} \tag{4.10}$$

Since

$$\begin{aligned}\overline{N}\,(r,1;F_1|\geq 2) &\leq N\left(r,\infty;\frac{F_1}{F_1'}\right) \\ &= N\left(r,\infty;\frac{F_1'}{F_1}\right)+S(r,f) \\ &\leq \overline{N}(r,\infty;F_1)+\overline{N}(r,0;F_1)+S(r,f) \\ &\leq (m+4)T(r,f)+2\log r+S(r,f),\end{aligned} \tag{4.11}$$

similarly, we have

$$\overline{N}(r,1;G_1|\geq 2) \leq (m+4)T(r,g)+2\log r+S(r,g). \tag{4.12}$$

By (4.2) – (4.4), (4.10) – (4.12) and Lemma 2.9(i), we have

$$(n-3m-18)[T(r,f)+T(r,g)] \leq 16\log r+S(r,f)+S(r,g). \tag{4.13}$$

By $n>3m+18$, we get a contradiction. Hence F_1 and G_1 satisfy (ii) in Lemma 2.9. By Lemma 2.11, we can get the conclusion of Theorem 1.6(i).

(ii) If $l = 2$. Since

$$\overline{N}(r,1;F_1) + \overline{N}(r,1;G_1) - N_{11}(r,1;F_1) + \frac{1}{2}\overline{N}\,(r,1;F_1| \geq 3) + \frac{1}{2}\overline{N}\,(r,1;G_1| \geq 3) \leq \frac{1}{2}N(r,1;F_1) + \frac{1}{2}N(r,1;G_1) \leq \frac{1}{2}T(r,F_1) + \frac{1}{2}T(r,G_1) + S(r,f) + S(r,g).$$

Suppose that (i) in Lemma 2.9 holds, then we have

$$T(r,F_1) + T(r,G_1) \leq 2[N_2(r,0;F_1) + N_2(r,\infty;F_1) + N_2(r,0;G_1) + N_2(r,\infty;G_1) + \overline{N}(r,1;F_1| \geq 3) + \overline{N}(r,1;G_1| \geq 3)] + S(r,f) + S(r,g). \tag{4.14}$$

Since

$$\begin{aligned}\overline{N}\,(r,1;F_1| \geq 3) &\leq \frac{1}{2}N\left(r,\infty;\frac{F_1}{F_1'}\right) \\ &= \frac{1}{2}N\left(r,\infty;\frac{F_1'}{F_1}\right) + S(r,f) \\ &\leq \frac{1}{2}(m+4)T(r,f) + \log r + S(r,f),\end{aligned} \tag{4.15}$$

similarly, we have

$$\overline{N}(r,1;G_1| \geq 3) \leq \frac{1}{2}(m+4)T(r,g) + \log r + S(r,g). \tag{4.16}$$

By (4.2) – (4.4), (4.14) – (4.16) and Lemma 2.9(i), we have

$$\left(n - \frac{3}{2}m - 12\right)[T(r,f) + T(r,g)] \leq 16\log r + S(r,f) + S(r,g). \tag{4.17}$$

By $n > \frac{3}{2}m + 12$, we get a contradiction. Hence F_1 and G_1 satisfy (ii) in Lemma 2.9. By Lemma 2.11, we can get the conclusion of Theorem 1.6(ii). □

5. REMARKS

It follows from the proof of Theorem 1.4(1.5) that if the condition $f^nP(f)f'$ and $g^nP(g)g'$ share z CM (IM) are replaced by the condition $f^nP(f)f'$ and $g^nP(g)g'$ share $a(z)$ CM (IM), where $a(z)$ is a meromorphic function such that $a(z) \not\equiv 0, \infty$ and $T(r,f) = o\{T(r,f), T(r,g)\}$, the conclusion of Theorem 1.4(1.5) still holds. Similarly, if the condition $E_{l)}(z, f^nP(f)f') = E_{l)}(z, g^nP(g)g')(l = 1,2)$ is replaced by the condition $E_{l)}(a(z),$ $f^nP(f)f') = E_{l)}(a(z), g^nP(g)g')(l = 1,2)$ respectively, then the conclusion of Theorem 1.6 still holds.

REFERENCES

[1] A. BANERJEE, Weighted sharing of a small function by a meromorphic function and its derivative, *Comput. Math. Appl.*, **53** (2007), 1750–1761.

[2] S.S. BHOOSNURMATH AND R.S. DYAVANAL, Uniqueness and value-sharing of meromorphic functions, *Comput. Math. Appl.*, **53** (2007), 1191–1205.

[3] M.-L. FANG AND X.-H. HUA, Entire functions that share one value, *J. Nanjing Univ. Math.*, Biquarterly **13** (1996), 44–48.

[4] C.-Y. FANG AND M.-L. FANG, Uniqueness of meromorphic functions and differential polynomials, *Comput. Math. Appl.*, **44** (2002) 607–617.

[5] M.-L. FANG, Uniqueness and value-sharing of entire functions, *Comput. Math. Appl.*, **44** (2002), 823–831.

[6] W.K. HAYMAN, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[7] W.-C. LIN AND H.-X. YI, Uniqueness theorems for meromorphic function, *Indian J. Pure Appl. Math.*, **35**(2) (2004), 121–132.

[8] W.-C. LIN AND H.-X. YI, Uniqueness theorems for meormorphic functions concerning fixed-points, *Complex Variables Theory Appl.*, **49**(11) (2004), 793–806.

[9] I. LAHIRI, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math.J.*, **16**(1) (2001), 193–206.

[10] I. LAHIRI, Weighted value sharing and uniqueness of meromorphic functions, *Complex Variables Theory Appl.*, **46**(3) (2001), 241–253.

[11] H.-Y. XU, Uniqueness of Transcendental Meromorphic Functions with Their Nonlinear Differential Polynomials Sharing the Small Function , *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 84961, 14 pages, 2007. doi:10.1155/2007/84961

[12] H.-Y. XU, C.-F. YI AND T.-B. CAO, Uniqueness of meromorphic functions and differential polynomials sharing one value with finite weight, *Ann. Polon. Math.*, **95** (2009), 51–66.

[13] L. YANG, *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.

[14] C.-C. YANG AND X.-H. HUA, Uniqueness and value-sharing of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.*, **22**(2) (1997), 395–406.

[15] C.-C. YANG AND H.-X. YI, Uniqueness theory of meromorphic functions, Science Press/Kluwer Academic Publishers, Beijing/New York, (1995/2003).

[16] H.-X. YI, Meromorphic functions that share one or two values II, *Kodai Math. J.*, **22**(2) (1999), 264–272.

[17] X.-Y. ZHANG AND W.-C. LIN, Uniqueness and value-sharing of entire functions, *J. Math. Anal. Appl.*, **43**(2) (2008), 938–950.

[18] X.-Y. ZHANG, J.-F. CHEN AND W.C. LIN, Entire or meromorphic functions sharing one value, *Comput. Math. Appl.*, **56** (2008), 1876–1883.