ON A GENERALIZATION OF THE HERMITE-HADAMARD INEQUALITY II

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Abstract: Generalized form of Hermite-Hadamard inequality for (2n)-convex Lebesgue

integrable functions are obtained through generalization of Taylor's Formula.

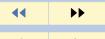


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The classical Hermite-Hadamard inequality gives us an estimate, from below and from above, of the mean value of a convex function $f : [a, b] \to \mathbb{R}$ (see [1, pp. 137]):

(HH)
$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \, dx \le \frac{f(a)+f(b)}{2}.$$

In [2] the first author with Sabir Hussain proved the following two theorems

Theorem 1. Assume that f is Lebesgue integrable and convex on (a, b). Then

$$\frac{1}{b-a} \int_{a}^{b} f(y)dy + f'_{+}(x) \left(x - \frac{a+b}{2} \right) - f(x)$$

$$\geq \left| \frac{1}{b-a} \int_{a}^{b} |f(y) - f(x)| \, dy - \left| f'_{+}(x) \right| \frac{(x-a)^{2} + (b-x)^{2}}{2(b-a)} \right|$$

for all $x \in (a, b)$.

Theorem 2. Assume that $f:[a,b] \to \mathbb{R}$ is a convex function. Then

$$\frac{1}{2} \left[f(x) + \frac{f(b)(b-x) + f(a)(x-a)}{b-a} \right] - \frac{1}{b-a} \int_{a}^{b} f(y) dy$$

$$\geq \frac{1}{2} \left| \frac{1}{b-a} \int_{a}^{b} |f(x) - f(y)| dy - \frac{1}{b-a} \int_{a}^{b} |x-y| |f'(y)| dy \right|$$

for all $x \in (a, b)$.

Remark 1. For $x = \frac{a+b}{2}$ in Theorem 1 and x = a or x = b in Theorem 2, we obtain improvements of inequality (HH).

In this paper we will prove further generalizations of these results.



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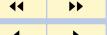
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Theorem 3. Assume that $f:[a,b] \to \mathbb{R}$ is a (2n-1)-times differentiable and (2n)-convex function. Then

$$\frac{1}{(b-a)} \int_{a}^{b} f(y)dy - (b-a)f(x) - \sum_{1}^{2n-1} \frac{(b-x)^{k+1} - (a-x)^{k+1}}{(k+1)!(b-a)} f^{(k)}(x) \\
\ge \left| \frac{1}{(b-a)} \int_{a}^{b} \left| f(y) - f(x) - \sum_{1}^{2n-2} \frac{(y-x)^{k}}{k!} f^{(k)}(x) \right| dy \\
- \left| f^{(2n-1)}(x) \frac{(b-x)^{2n} - (a-x)^{2n}}{(2n)!(b-a)} \right| \right|$$

for all $x \in (a, b)$.

Proof. It is well known that a continuous (2n)—convex function can be uniformly approximated by a (2n)—convex polynomial. So we can suppose that we have (2n)—derivatives of f. By Taylor's formula,

$$f(y) = f(x) + (y - x)f'(x) + \frac{(y - x)^2}{2!}f''(x) + \cdots + \frac{(y - x)^{2n-1}}{2n - 1!}f^{(2n-1)}(x) + \frac{(y - x)^{2n}}{2n!}f^{(2n)}(\xi),$$

for $x, y \in [a, b]$, $\xi \in (a, b)$. Since f is (2n)—convex, we have $f^{(2n)}(x) \ge 0$. So

$$f(y) \ge f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2!}f''(x) + \dots + \frac{(y-x)^{2n-1}}{(2n-1)!}f^{(2n-1)}(x)$$

and we can write

$$f(y) - f(x) - (y - x)f'(x) - \frac{(y - x)^2}{2!}f''(x) - \dots - \frac{(y - x)^{2n - 1}}{2n - 1!}f^{(2n - 1)}(x) \ge 0,$$



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i.e.,

$$f(y) - f(x) - (y - x)f'(x) - \frac{(y - x)^2}{2!}f''(x) - \dots - \frac{(y - x)^{2n-1}}{(2n-1)!}f^{(2n-1)}(x)$$

$$= \left| f(y) - f(x) - (y - x)f'(x) - \frac{(y - x)^2}{2!}f''(x) - \dots - \frac{(y - x)^{2n-1}}{(2n-1)!}f^{(2n-1)}(x) \right|.$$

Now by using the triangle inequality

(1)
$$f(y) - f(x) - (y - x)f'(x) - \frac{(y - x)^2}{2!}f''(x) - \dots - \frac{(y - x)^{2n-1}}{2n - 1!}f^{(2n-1)}(x)$$

$$\geq \left| \left| f(y) - f(x) - (y - x)f'(x) - \dots - \frac{(y - x)^{2n-2}}{2n - 2!}f^{(2n-2)}(x) \right| - \left| \frac{(y - x)^{2n-1}}{2n - 1!}f^{(2n-1)}(x) \right| \right|.$$

Now integrating the last inequality with respect to y and using the triangle inequality for integrals, we get

$$\int_{a}^{b} f(y)dy - (b-a)f(x) - \sum_{1}^{2n-1} \frac{(b-x)^{k+1} - (a-x)^{k+1}}{(k+1)!} f^{(k)}(x)$$

$$\geq \left| \int_{a}^{b} \left| f(y) - f(x) - \sum_{1}^{2n-2} \frac{(y-x)^{k}}{k!} f^{(k)}(x) \right| dy$$

$$- \left| f^{(2n-1)}(x) \frac{(b-x)^{2n} - (a-x)^{2n}}{(2n)!} \right| \right|.$$



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Theorem 4. Assume that $f:[a,b] \to \mathbb{R}$ is a (2n-1)-times differentiable and (2n)-convex function. Then

$$f(x) - \frac{2n}{(b-a)} \int_{a}^{b} f(y)dy - \sum_{1}^{2n-1} \frac{2n-k}{k!(b-a)} [(x-b)^{k} f^{(k-1)}(b) - (x-a)^{k} f^{(k-1)}(a)]$$

$$\geq \left| \frac{1}{b-a} \int_{a}^{b} \left| f(x) - f(y) - \sum_{1}^{2n-2} \frac{(x-y)^{k}}{k!} f^{(k)}(y) \right| dy$$

$$- \frac{1}{b-a} \int_{a}^{b} \left| \frac{(x-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right| dy \right|.$$

Proof. Integrating (1) with respect to x and by using the triangle inequality for integrals, we get

$$(2) \quad (b-a)f(y) - \int_{a}^{b} f(x)dx - \int_{a}^{b} \sum_{1}^{2n-1} \frac{(y-x)^{k}}{k!} f^{(k)}(x)dx$$

$$\geq \left| \int_{a}^{b} \left| f(y) - f(x) - \sum_{1}^{2n-2} \frac{(y-x)^{k}}{k!} f^{(k)}(x) \right| dx - \int_{a}^{b} \left| \frac{(y-x)^{2n-1}}{(2n-1)!} f^{(2n-1)}(x) \right| dx \right|.$$

By replacing x and y we obtain the required result.

Corollary 5. Suppose that $f:[a,b] \to \mathbb{R}$ is a (2n-1)-times differentiable and (2n)-convex function. Then

$$\frac{1}{(b-a)} \int_a^b f(y) dy - (b-a) f\left(\frac{a+b}{2}\right) - \sum_1^{2n-1} \frac{\left(\frac{b-a}{2}\right)^{k+1} - \left(\frac{a-b}{2}\right)^{k+1}}{(k+1)!(b-a)} f^{(k)}\left(\frac{a+b}{2}\right)$$



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$$\geq \left| \frac{1}{(b-a)} \int_{a}^{b} \left| f(y) - f\left(\frac{a+b}{2}\right) - \sum_{1}^{2n-2} \frac{\left(y - \frac{a+b}{2}\right)^{k}}{k!} f^{(k)}\left(\frac{a+b}{2}\right) \right| dy \\ - \left| f^{(2n-1)}\left(\frac{a+b}{2}\right) \frac{(b-a)^{2n} - (a-b)^{2n}}{(2n)!(b-a)2^{2n}} \right| \right|.$$

Proof. Set $x = \frac{a+b}{2}$ in Theorem 3.

Corollary 6. Suppose that $f:[a,b] \to \mathbb{R}$ is a (2n-1)-times differentiable and (2n)-convex function. Then

$$f(a) - \frac{2n}{(b-a)} \int_{a}^{b} f(y)dy - \sum_{1}^{2n-1} \frac{2n-k}{k!(b-a)} [(a-b)^{k} f^{(k-1)}(b)]$$

$$\geq \left| \frac{1}{b-a} \int_{a}^{b} \left| f(a) - f(y) - \sum_{1}^{2n-2} \frac{(a-y)^{k}}{k!} f^{(k)}(y) \right| dy$$

$$- \frac{1}{b-a} \int_{a}^{b} \left| \frac{(a-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right| dy \right|.$$

Proof. Set x = a in Theorem 4.



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