NOTES ON AN INEQUALITY

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Abstract: In this note we prove a generalized version of an inequality which was first introduced by A. Q. Ngo, *et al.* and later generalized and proved by W. J. Liu, *et al.* in the paper: "On an open problem concerning an integral inequality", *J. Inequal. Pure & Appl. Math.*, **8**(3) (2007), Art. 74.

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1. Introduction

In [2] the following result was proved: If $f \geq 0$ is a continuous function on $[0,1]$ such that

$$\int_x^1 f(t)dt \geq \int_x^1 t dt, \quad \forall x \in [0,1], \tag{1.1}$$

then

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx, \quad \forall \alpha > 0.$$

The following question was raised in [2]: If f satisfies the above assumptions, under what additional assumptions can one claim that:

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx, \quad \forall \alpha, \beta > 0?$$

It was proved in [1] that if $f \geq 0$ is a continuous function on $[0,1]$ satisfying

$$\int_x^b f^\alpha(t)dt \geq \int_x^b t^\alpha dt, \quad \alpha, b > 0, \forall x \in [0,b],$$

then

$$\int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx, \quad \forall \beta > 0.$$

In this paper, we prove more general results, namely, Theorems 2.4 and 2.5 below.

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2. Results and Proofs

Let us recall the following result:

Lemma 2.1 (Young's inequality). *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y, we have:*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}.$$

Throughout the paper, $[a, b]$ denotes a bounded interval and all functions are real-valued. Let us prove the following lemma:

Lemma 2.2. *Let $f \in L^1[a, b]$, $g \in C^1[a, b]$. Suppose $f \geq 0$, $g > 0$ is nondecreasing. If*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a, b],$$

then $\forall \alpha > 0$ the following inqualities hold

$$\int_a^b g^{\alpha}(x) f(x) dx \geq \int_a^b g^{\alpha+1}(x) dx, \tag{2.1}$$

$$\int_a^b f^{\alpha+1}(x) dx \geq \int_a^b f^{\alpha}(x) g(x) dx, \tag{2.2}$$

$$\int_a^b f^{\alpha+1}(x) dx \geq \int_a^b f(x) g^{\alpha}(x) dx. \tag{2.3}$$

Proof. First, let us prove (2.1). Let A, A^* denote

$$Af(x) := \int_a^x f(t)dt, \quad A^* f(x) := \int_x^b f(t)dt, \quad x \in [a, b],\ f \in L^1[a, b].$$

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Note that these are continuous functions. From the assumption one has

$$A^*f(x) \geq A^*g(x), \quad \forall x \in [a,b].$$

This means

$$(A^*f - A^*g)(x) \geq 0, \quad \forall x \in [a,b].$$

Then $\forall h \in L^1[a,b]$, $h \geq 0$, one obtains

$$\langle A^*f - A^*g, h\rangle := \int_a^b (A^*f - A^*g)(x)h(x)dx \geq 0. \tag{2.4}$$

Note that the left-hand side of (2.4) is finite since A^*f, A^*g are bounded and $h \in L^1[a,b]$. Thus, by Fubini's Theorem, one has

$$\langle f - g, Ah\rangle = \langle A^*f - A^*g, h\rangle \geq 0, \quad \forall h \geq 0, h \in L^1[a,b]. \tag{2.5}$$

Denote $h(x) = \alpha g(x)^{\alpha-1}g'(x)$. One has

$$Ah(x) = \int_a^x h(t)dt = g^\alpha(x) - g^\alpha(a), \quad \forall x \in [a,b].$$

By the assumption,

$$\langle f - g, g^\alpha(a)\rangle = g^\alpha(a)\int_a^b (f(x) - g(x))dx \geq 0. \tag{2.6}$$

Since $h \geq 0$, from (2.5) and (2.6) one gets

$$\langle f - g, g^\alpha\rangle = \langle f - g, Ah\rangle + \langle f - g, g^\alpha(a)\rangle \geq 0, \quad \forall \alpha \geq 0. \tag{2.7}$$

Hence, (2.1) is obtained.

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Since

$$(f(x) - g(x))(f^\alpha(x) - g^\alpha(x)) \geq 0, \quad \forall x \in [a,b], \quad \forall \alpha \geq 0,$$

one gets

$$\langle f - g, f^\alpha - g^\alpha \rangle \geq 0, \quad \forall \alpha \geq 0. \tag{2.8}$$

Inequalities (2.7) and (2.8) imply

$$\langle f - g, f^\alpha \rangle = \langle f - g, f^\alpha - g^\alpha \rangle + \langle f - g, g^\alpha \rangle \geq 0, \quad \forall \alpha > 0.$$

Thus, (2.2) holds.

By Lemma 2.1,

$$\frac{1}{\alpha+1} f^{\alpha+1}(x) + \frac{\alpha}{\alpha+1} g^{\alpha+1}(x) \geq g^\alpha(x) f(x), \quad \forall x \in [a,b].$$

Thus,

$$\begin{aligned} \frac{1}{\alpha+1}\int_a^b f^{\alpha+1}(x)dx + \frac{\alpha}{\alpha+1}\int_a^b g^{\alpha+1}(x)dx \\ \geq \int_a^b g^\alpha(x) f(x) dx, \quad \forall \alpha > 0. \end{aligned} \tag{2.9}$$

From (2.1) and (2.9) one obtains

$$\int_a^b f^{\alpha+1}(x)dx \geq \int_a^b g^\alpha(x) f(x) dx, \quad \forall \alpha \geq 0.$$

The proof is complete. □

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In particular, one has the following result

Corollary 2.3. *Suppose* $f \in L^1[a,b]$, $g \in C^1[a,b]$ $f, g \geq 0$, g *is nondecreasing. If*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a,b]$$

then the following inequality holds

$$\int_a^b f^\beta(x)dx \geq \int_a^b g^\beta(x)dx, \quad \forall \beta \geq 1. \tag{2.10}$$

Proof. Denote $f_\epsilon := f + \epsilon$, $g_\epsilon := g + \epsilon$ where $\epsilon > 0$. It is clear that $g_\epsilon > 0$ and

$$\int_x^b f_\epsilon(t)dt \geq \int_x^b g_\epsilon(t)dt, \quad \forall x \in [a,b].$$

By (2.1) and (2.3) in Lemma 2.2 one has

$$\int_a^b f_\epsilon^\beta(x)dx \geq \int_a^b g_\epsilon^\beta(x)dx, \quad \forall \beta \geq 1. \tag{2.11}$$

Inequality (2.10) is obtained from (2.11) by letting $\epsilon \to 0$. □

Theorem 2.4. *Suppose* $f \in L^1[a,b]$, $g \in C^1[a,b]$, $f, g \geq 0$, g *is nondecreasing. If*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a,b],$$

then $\forall \alpha, \beta \geq 0$, $\alpha + \beta \geq 1$, *the following inequality holds*

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^\alpha(x)g^\beta(x)dx. \tag{2.12}$$

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Proof. Lemma 2.1 shows that

$$\frac{\alpha}{\alpha+\beta} f(x)^{\alpha+\beta} + \frac{\beta}{\alpha+\beta} g(x)^{\alpha+\beta} \ge f^{\alpha}(x) g^{\beta}(x), \quad \forall x \in [a,b],\ \forall \alpha, \beta > 0.$$

Therefore, $\forall \alpha, \beta > 0$ one has

$$\frac{\alpha}{\alpha+\beta} \int_a^b f(x)^{\alpha+\beta} dx + \frac{\beta}{\alpha+\beta} \int_a^b g(x)^{\alpha+\beta} dx \ge \int_a^b f^{\alpha}(x) g^{\beta}(x) dx. \tag{2.13}$$

Corollary 2.3 implies

$$\int_a^b f(x)^{\alpha+\beta} dx \ge \int_a^b g(x)^{\alpha+\beta} dx, \quad \forall \alpha, \beta \ge 0,\ \alpha+\beta \ge 1. \tag{2.14}$$

Inequality (2.12) is obtained from (2.13) and (2.14). □

Remark 1. Theorem 2.4 is not true if we drop the assumption $\alpha + \beta \ge 1$. Indeed, take $g \equiv 1$, $[a,b] = [0,1]$, and define

$$f(x) = c(1-x)^{c-1}, \quad 0 \le x \le 1,$$

where $c \in (0,1)$. One has

$$(1-x)^c = \int_x^1 f(t) dt \ge \int_x^1 g(t) dt = (1-x), \quad \forall x \in [0,1],\ c \in (0,1),$$

but

$$\frac{2\sqrt{c}}{c+1} = \int_0^1 \sqrt{f(t)} dt < \int_0^1 \sqrt{g(t)} dt = 1, \quad \forall c \in (0,1).$$

Assuming that the condition $g \in C^1[a,b]$ can be dropped and replaced by $g \in L^1[a,b]$, we have the following result:

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Theorem 2.5. *Suppose* $f, g \in L^1[a,b]$, $f, g \geq 0$, g *is nondecreasing. If*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \quad \forall x \in [a,b], \tag{2.15}$$

then

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^{\alpha}(x)g^{\beta}(x)dx, \quad \forall \alpha, \beta \geq 0,\ \alpha+\beta \geq 1. \tag{2.16}$$

Proof. Since $C^1[a,b]$ is dense in L^1, there exists a sequence $(g_n)_{n=1}^{\infty} \in C^1[a,b]$ such that g_n is nondecreasing, $g_n \nearrow g$ a.e. Since $g_n \nearrow g$ a.e.,

$$\int_x^b g(t)dt \geq \int_x^b g_n(t)dt, \quad \forall x \in [a,b],\ \forall n. \tag{2.17}$$

Inequalities (2.15), (2.17) and Theorem 2.4 imply

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^{\alpha}(x)g_n^{\beta}(x)dx, \quad \forall n,\ \forall \alpha, \beta \geq 0,\ \alpha+\beta \geq 1. \tag{2.18}$$

Since $f^{\alpha}g_n^{\beta} \nearrow f^{\alpha}g^{\beta}$ a.e., $f^{\alpha}g_n^{\beta} \geq 0$ is measurable satisfying (2.18), by the Monotone convergence theorem (see [3, 4]) $\|f^{\alpha}g_n^{\beta} \to f^{\alpha}g^{\beta}\|_{L^1} \to 0$ as $n \to \infty$. Hence,

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^{\alpha}(x)g^{\beta}(x)dx, \quad \forall \alpha, \beta \geq 0,\ \alpha+\beta \geq 1.$$

The proof is complete. □

Remark 2. One may wish to extend Theorem 2.5 to the case where $[a,b]$ is unbounded. Note that the case $b = \infty$ is not meaningful. It is because if $g \neq 0$ a.e.,

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then both sides of (2.15) are infinite. If $b < \infty$ and $a = -\infty$ and inequality (2.15) holds for $a = \infty$, then it holds as well for all finite $a < 0$. Hence, inequality (2.16) holds for all $a < 0$. Thus, by letting $a \to -\infty$ in Theorem 2.5, one gets the result of Theorem 2.5 in the case $a = -\infty$.

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References

[1] W.J. LIU, C.C. LI AND J.W. DONG, On an open problem concerning an integral inequality, *J. Inequal. Pure & Appl. Math.*, **8**(3) (2007), Art. 74. [ONLINE: http://jipam.vu.edu.au/article.php?sid=882].

[2] Q.A. NGO, D.D. THANG, T.T. DAT AND D.A. TUAN, Notes on an integral inequality, *J. Inequal. Pure & Appl. Math.*, **7**(4) (2006), Art. 120. [ONLINE: http://jipam.vu.edu.au/article.php?sid=737].

[3] M. REED AND B. SIMON, *Methods of Modern Mathematicals Physics, Functional Analysis I*, Academic Press, Revised and enlarged edition, (1980).

[4] W. RUDIN, *Real and Complex Analysis*, McGraw-Hill Series in Higher Mathematics, Second edition, (1974).

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