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ON ONE OF H. ALZER'S PROBLEMS

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ABSTRACT. In this short note, the author solves an inequality problem which was posed by H. Alzer with difference substitution.

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1. THE PROBLEM

In 1993, H. Alzer posed the following inequality problem in [1]. In 2004, Ji-Chang Kuang reposed the problem in his monograph [2].

Problem 1.1. Let $a_1, \dots, a_n$ $(n \in N^*)$ be real numbers with $a_i \in (0, \frac{1}{2}]$, then prove or disprove

$$\prod_{k=1}^{n} \left(\frac{a_k}{1 - a_k} \right) \leq \frac{\sum_{k=1}^{n} a_k^n}{\sum_{k=1}^{n} (1 - a_k)^n} \tag{1.1}$$

where $n = 4$ or $n = 5$.

In 1995, Michael Vowe pointed out that the inequality (1.1) holds when $n \leq 3$ $(n \in N^*)$ and does not hold when $n \geq 6$ $(n \in N^*)$ in [3].

2. SOLUTION OF THE PROBLEM

In this section, we show the reader the proof of the inequality (1.1) while $n = 4$.

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Proof. We set $a_k = \frac{b_k}{u}$ $(k = 1,2,3,4)$, where $u > 0$ and $0 < a_k \le \frac{1}{2}$ $(k = 1,2,3,4)$, then $0 < b_k \le \frac{1}{2}u$ $(k = 1,2,3,4)$. So the inequality (1.1) is equivalent to the inequality as follows.

$$\prod_{k=1}^{4}\left(\frac{b_k}{u-b_k}\right) \le \frac{\sum_{k=1}^{4} b_k^4}{\sum_{k=1}^{4}(u-b_k)^4}. \tag{2.1}$$

Inequality (2.1) is equivalent to the following inequality.

$$\begin{aligned}(2.2)\quad & ({b_3}^4 + {b_4}^4 + {b_2}^4 + {b_1}^4 - 4\,b_1b_2b_3b_4)u^3 + (-b_1{b_3}^4 - {b_1}^5 + 4\,b_1b_2b_3{b_4}^2 - b_2{b_3}^4 - b_4{b_3}^4 \\ & - b_4{b_1}^4 - b_3{b_4}^4 - {b_4}^5 - b_4{b_2}^4 - b_1{b_4}^4 - {b_3}^5 - b_2{b_4}^4 - b_1{b_2}^4 - b_2{b_1}^4 - b_3{b_1}^4 \\ & + 4\,b_1b_2{b_3}^2b_4 - b_3{b_2}^4 - {b_2}^5 + 4\,{b_1}^2b_2b_3b_4 + 4\,b_1{b_2}^2b_3b_4)u^2 + (b_2{b_3}^5 + b_1{b_3}^5 + {b_1}^5b_3 \\ & + b_1{b_4}^5 + {b_1}^5b_4 + {b_2}^5b_3 - 6\,b_1b_2b_3{b_4}^3 - 6\,b_1b_2{b_3}^3b_4 - 6\,b_1{b_2}^3b_3b_4 - 6\,{b_1}^3b_2b_3b_4 \\ & + b_2b_4{b_1}^4 + b_3b_4{b_2}^4 + b_3b_4{b_1}^4 + b_1b_2{b_4}^4 + b_1b_2{b_3}^4 + b_1b_3{b_4}^4 + b_1b_3{b_2}^4 + b_1b_4{b_3}^4 \\ & + b_1b_4{b_2}^4 + b_2b_3{b_4}^4 + b_2b_3{b_1}^4 + b_2b_4{b_3}^4 + b_2{b_4}^5 + {b_2}^5b_4 + b_3{b_4}^5 + {b_3}^5b_4 + {b_1}^5b_2 \\ & + b_1{b_2}^5)u - b_2b_3{b_4}^5 + 3\,b_1b_2b_4{b_3}^4 - b_1b_3{b_4}^5 - b_1{b_3}^5b_4 + 3\,b_1b_3b_4{b_2}^4 + 3\,b_1b_2b_3{b_4}^4 \\ & - {b_1}^5b_3b_4 - b_2{b_3}^5b_4 - {b_2}^5b_3b_4 - b_1{b_2}^5b_4 + 3\,b_2b_3b_4{b_1}^4 - {b_1}^5b_2b_3 - b_1b_2{b_4}^5 \\ & - b_1b_2{b_3}^5 - {b_1}^5b_2b_4 - b_1{b_2}^5b_3 \ge 0.\end{aligned}$$

Inequality (2.2) is symmetrical for b_k $(1 \le k \le 4,\ k \in N^*)$, so there is no harm in supposing that $b_1 \le b_2 \le b_3 \le b_4$. Then we can set

$$\begin{cases} b_2 = b_1 + c_1; \\ b_3 = b_1 + c_1 + c_2; \\ b_4 = b_1 + c_1 + c_2 + c_3; \\ u = 2(b_1 + c_1 + c_2 + c_3) + c_4, \end{cases} \tag{2.3}$$

where $c_i \ge 0$ $(1 \le i \le 4, i \in N^*)$.
The substitution (2.3) was called a difference substitution in [4] (see also [5]). Substituting (2.3) in (2.2), we obtain the result (3.1) (see Appendix). Since every monomial on the left of (3.1) is nonnegative, the last inequality obviously holds, then the inequality (1.1) holds when $n = 4$.

Thus, the proof of the inequality (1.1) $(n = 4)$ is completed. □

3. REMARKS

Remark 3.1. In the same manner, we can also prove the inequality (1.1) holds when $n = 5$.

Remark 3.2. The operations in this paper were implemented using mathematics software Maple 9.0.

APPENDIX

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\begin{aligned}
(3.1)\quad & 30b_1{c_4}^2{c_1}^4+10\,b_1{c_1}^6+120\,c_4{b_1}^3c_1c_2c_3+720\,c_4b_1{c_1}^2c_2{c_3}^2+9\,c_4{c_1}^6\\
&+824\,c_4b_1{c_1}^2{c_2}^2c_3+10\,c_4{c_2}^6+498\,c_4b_1{c_1}^3c_2c_3+13\,{b_1}^2{c_1}^5+51\,{b_1}^2{c_3}^5\\
&+44\,{b_1}^2{c_2}^5+8\,{b_1}^3{c_1}^4+40\,{b_1}^3{c_3}^4+32\,{b_1}^3{c_2}^4+120\,{c_4}^2{b_1}^2c_1c_2c_3+10\,{b_1}^4{c_3}^3\\
&+20\,{b_1}^4c_1c_2c_3+152\,{b_1}^3{c_1}^2c_2c_3+208\,{b_1}^3c_1{c_2}^2c_3+200\,{b_1}^3c_1c_2{c_3}^2\\
&+346\,{b_1}^2{c_1}^3c_2c_3+624\,{b_1}^2{c_1}^2{c_2}^2c_3+600\,{b_1}^2{c_1}^2c_2{c_3}^2+550\,{b_1}^2c_1c_2{c_3}^3\\
&+556\,{b_1}^2c_1{c_2}^3c_3+780\,{b_1}^2c_1{c_2}^2{c_3}^2+904\,b_1c_1{c_2}^2{c_3}^3+318\,b_1{c_1}^4c_2c_3\\
&+696\,b_1{c_1}^3{c_2}^2c_3+652\,b_1{c_1}^3c_2{c_3}^2+756\,b_1{c_1}^2c_2{c_3}^3+824\,b_1{c_1}^2{c_2}^3c_3\\
&+1128\,b_1{c_1}^2{c_2}^2{c_3}^2+490\,b_1c_1c_2{c_3}^4+504\,b_1c_1{c_2}^4c_3+912\,b_1c_1{c_2}^3{c_3}^2\\
&+2\,{b_1}^4{c_1}^3+8\,{b_1}^4{c_2}^3+74\,{b_1}^2{c_1}^4c_2+40\,{b_1}^3{c_1}^3c_2+8\,{b_1}^4{c_1}^2c_2\\
&+10\,{b_1}^4{c_1}^2c_3+12\,{b_1}^4c_1{c_2}^2+10\,{b_1}^4c_1{c_3}^2+20\,{b_1}^4{c_2}^2c_3+20\,{b_1}^4c_2{c_3}^2\\
&+48\,{b_1}^3{c_1}^3c_3+80\,{b_1}^3{c_1}^2{c_2}^2+80\,{b_1}^3{c_1}^2{c_3}^2+80\,{b_1}^3c_1{c_2}^3\\
&+80\,{b_1}^3c_1{c_3}^3+120\,{b_1}^3c_2{c_3}^3+112\,{b_1}^3{c_2}^3c_3+160\,{b_1}^3{c_2}^2{c_3}^2\\
&+158\,{b_1}^2c_1{c_2}^4+165\,{b_1}^2c_1{c_3}^4+83\,{b_1}^2{c_1}^4c_3+178\,{b_1}^2{c_1}^3{c_2}^2\\
&+180\,{b_1}^2{c_1}^3{c_3}^2+232\,{b_1}^2{c_1}^2{c_2}^3+220\,{b_1}^2{c_1}^2{c_3}^3+210\,{b_1}^2c_2{c_3}^4\\
&+370\,{b_1}^2{c_2}^2{c_3}^3+194\,{b_1}^2{c_2}^4c_3+360\,{b_1}^2{c_2}^3{c_3}^2+112\,b_1c_1{c_3}^5\\
&+116\,b_1c_1{c_2}^5+236\,b_1{c_1}^2{c_2}^4+210\,b_1{c_1}^2{c_3}^4+62\,b_1{c_1}^5c_2+64\,b_1{c_1}^5c_3\\
&+168\,b_1{c_1}^4{c_2}^2+158\,b_1{c_1}^4{c_3}^2+260\,b_1{c_1}^3{c_2}^3+224\,b_1{c_1}^3{c_3}^3\\
&+122\,b_1c_2{c_3}^5+3\,{c_1}^7+4\,{c_2}^7+356\,b_1{c_2}^3{c_3}^3+124\,b_1{c_2}^5c_3+276\,b_1{c_2}^4{c_3}^2\\
&+280\,{c_1}^2c_2{c_3}^4+534\,{c_1}^2{c_2}^2{c_3}^3+24\,b_1{c_2}^6+72\,{c_1}^2{c_2}^5+61\,{c_1}^2{c_3}^5+20\,{c_1}^6c_2\\
&+26\,c_1{c_2}^6+59\,{c_2}^2{c_1}^5+72\,{c_2}^2{c_3}^5+110\,{c_1}^3{c_2}^4+85\,{c_1}^3{c_3}^4+19\,{c_1}^6c_3\\
&+50\,{c_1}^5{c_3}^2+102\,{c_1}^4{c_2}^3+78\,{c_1}^4{c_3}^3+110\,{c_2}^3{c_3}^4+104\,{c_2}^4{c_3}^3+24\,{c_2}^6c_3\\
&+64\,{c_2}^5{c_3}^2+22\,b_1{c_3}^6+24\,c_1{c_3}^6+26\,c_2{c_3}^6+107\,{c_1}^5c_2c_3+268\,{c_1}^4{c_2}^2c_3\\
&+242\,{c_1}^4c_2{c_3}^2+133\,c_1c_2{c_3}^5+380\,{c_1}^3{c_2}^3c_3+508\,{c_1}^3{c_2}^2{c_3}^2+310\,{c_2}^4{c_1}^2c_3\\
&+552\,{c_2}^3{c_1}^2{c_3}^2+280\,{c_2}^2b_1{c_3}^4+46\,c_4{b_1}^2{c_1}^4+84\,c_4{b_1}^2{c_2}^4+78\,c_4{b_1}^2{c_3}^4\\
&+6\,c_4{b_1}^4{c_1}^2+8\,c_4{b_1}^4{c_2}^2+6\,c_4{b_1}^4{c_3}^2+56\,c_4c_1{c_2}^5+42\,c_4c_1{c_3}^5+28\,c_4{c_1}^3{b_1}^3\\
&+48\,c_4c_2{c_1}^5+45\,c_4c_2{c_3}^5+192\,c_4c_1{b_1}^2{c_3}^3+326\,{c_1}^3c_2{c_3}^3+305\,c_1{c_2}^2{c_3}^4\\
&+386\,c_1{c_2}^3{c_3}^3+134\,c_1{c_2}^5c_3+298\,c_1{c_2}^4{c_3}^2+44\,c_4{b_1}^3{c_3}^3+33\,c_4b_1{c_1}^5\\
&+52\,c_4b_1{c_2}^5+39\,c_4b_1{c_3}^5+168\,c_4{b_1}^2{c_1}^3c_2+8\,c_4{b_1}^4c_1c_2+4\,c_4{b_1}^4c_1c_3\\
&+8\,c_4{b_1}^4c_2c_3+72\,c_4{b_1}^3{c_1}^2c_2+60\,c_4{b_1}^3{c_1}^2c_3+88\,c_4{b_1}^3c_1{c_2}^2\\
&+60\,c_4{b_1}^3c_1{c_3}^2+104\,c_4{b_1}^3{c_2}^2c_3+96\,c_4{b_1}^3c_2{c_3}^2+150\,c_4{c_1}^4b_1c_2\\
&+152\,c_4{c_1}^3{b_1}^2c_3+276\,c_4{c_1}^2{b_1}^2{c_2}^2+208\,c_4{c_1}^2{b_1}^2{c_3}^2+187\,c_4{c_1}^4c_2c_3\\
&+376\,c_4{c_1}^3{c_2}^2c_3+320\,c_4{c_1}^3c_2{c_3}^2+214\,c_4c_1b_1{c_2}^4+169\,c_4c_1b_1{c_3}^4\\
&+252\,c_4c_2{b_1}^2{c_3}^3+306\,c_4{c_2}^2b_1{c_1}^3+240\,c_4{c_2}^3{b_1}^2c_1+264\,c_4{c_2}^3{b_1}^2c_3\\
&+352\,c_4{c_2}^2{b_1}^2{c_3}^2+404\,c_4{c_2}^3{c_1}^2c_3+516\,c_4{c_2}^2{c_1}^2{c_3}^2+182\,c_4c_2b_1{c_3}^4
\end{aligned}
$$

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\begin{aligned}
&+436\,c_4c_1^2b_1^2c_2c_3+552\,c_4c_1b_1^2c_2^2c_3+496\,c_4c_1b_1^2c_2c_3^2+48\,c_4c_2^3b_1^3\\
&+130\,c_4c_1^2c_2^4+91\,c_4c_1^2c_3^4+42\,c_4c_1^5c_3+115\,c_4c_1^4c_2^2+82\,c_4c_1^4c_3^2\\
&+160\,c_4c_1^3c_2^3+104\,c_4c_1^3c_3^3+105\,c_4c_2^2c_3^4+130\,c_4c_2^3c_3^3+46\,c_4c_2^5c_3\\
&+98\,c_4c_2^4c_3^2+135\,c_4b_1c_1^4c_3+228\,c_4b_1c_1^3c_3^2+352\,c_4b_1c_1^2c_2^3\\
&+252\,c_4b_1c_1^2c_3^3+338\,c_4b_1c_2^2c_3^3+202\,c_4b_1c_2^4c_3+344\,c_4b_1c_2^3c_3^2\\
&+196\,c_4c_1c_2c_3^4+364\,c_4c_1c_2^2c_3^3+216\,c_4c_1c_2^4c_3+368\,c_4c_1c_2^3c_3^2\\
&+338\,c_4c_1^2c_2c_3^3+590\,c_4b_1c_1c_2c_3^3+668\,c_4b_1c_1c_2^3c_3+868\,c_4b_1c_1c_2^2c_3^2\\
&+16\,c_4^2b_1^3c_1c_2+8\,c_4^2b_1^3c_1c_3+80\,c_4^2b_1^2c_1^2c_2+58\,c_4^2b_1^2c_1^2c_3\\
&+96\,c_4^2b_1^2c_1c_2^2+62\,c_4^2b_1^2c_1c_3^2+16\,c_4^2b_1^3c_2c_3+96\,c_4^2b_1^2c_2^2c_3\\
&+88\,c_4^2b_1^2c_2c_3^2+12\,c_4^2b_1^3c_1^2+34\,c_4^2b_1^2c_1^3+16\,c_4^2b_1^3c_2^2\\
&+48\,c_4^2b_1^2c_2^3+12\,c_4^2b_1^3c_3^2+96\,b_1c_4^2c_1^3c_2+148\,b_1c_4^2c_1^2c_2^2\\
&+120\,b_1c_4^2c_1c_2^3+72\,b_1c_4^2c_1^3c_3+88\,b_1c_4^2c_1^2c_3^2+196\,b_1c_4^2c_1^2c_2c_3\\
&+240\,b_1c_4^2c_1c_2^2c_3+208\,b_1c_4^2c_1c_2c_3^2+9\,c_4^2c_1^5+8\,c_4^2c_2^5\\
&+36\,c_4^2b_1c_2^4+38\,c_4^2c_1c_2^4+36\,c_4^2c_1^4c_2+68\,c_4^2c_1^3c_2^2\\
&+72\,c_4^2c_1^2c_2^3+8\,c_3^6c_4+5\,c_4^2c_3^5+4\,c_3^7+38\,c_4^2b_1^2c_3^3\\
&+22\,c_4^2b_1c_3^4+92\,c_4^2c_1^3c_2c_3+144\,c_4^2c_1^2c_2^2c_3+120\,c_4^2c_1^2c_2c_3^2\\
&+23\,c_4^2c_1c_3^4+27\,c_4^2c_1^4c_3+38\,c_4^2c_1^3c_3^2+42\,c_4^2c_1^2c_3^3+24\,c_4^2c_2c_3^4\\
&+46\,c_4^2c_2^2c_3^3+26\,c_4^2c_2^4c_3+44\,c_4^2c_2^3c_3^2+80\,c_4^2b_1c_1c_3^3\\
&+84\,c_4^2b_1c_2c_3^3+96\,c_4^2b_1c_2^3c_3+120\,c_4^2b_1c_2^2c_3^2+88\,c_4^2c_1c_2c_3^3\\
&+100\,c_4^2c_1c_2^3c_3+126\,c_4^2c_1c_2^2c_3^2+3\,c_4^3c_1^4+6\,c_4^3b_1^2c_1^2\\
&+8\,c_4^3b_1c_1^3+2\,c_4^3c_2^4+c_4^3c_3^4+8\,c_4^3b_1^2c_1c_2+4\,c_4^3b_1^2c_1c_3\\
&+8\,c_4^3b_1^2c_2c_3+12\,c_4^3c_1^2c_2c_3+16\,c_4^3b_1c_1^2c_2+8\,c_4^3b_1c_1^2c_3\\
&+20\,c_4^3b_1c_1c_2^2+12\,c_4^3b_1c_1c_3^2+12\,c_4^3b_1c_2^2c_3+12\,c_4^3b_1c_2c_3^2\\
&+12\,c_4^3c_1c_2^2c_3+12\,c_4^3c_1c_2c_3^2+20\,c_4^3b_1c_1c_2c_3+8\,c_4^3b_1^2c_2^2\\
&+6\,c_4^3b_1^2c_3^2+8\,c_4^3b_1c_2^3+4\,c_4^3b_1c_3^3+8\,c_4^3c_1^3c_2\\
&+4\,c_4^3c_1^3c_3+12\,c_4^3c_1^2c_2^2+6\,c_4^3c_1^2c_3^2+8\,c_4^3c_1c_2^3\\
&+4\,c_4^3c_1c_3^3+4\,c_4^3c_2c_3^3+4\,c_4^3c_2^3c_3+6\,c_4^3c_2^2c_3^2\ge 0.
\end{aligned}
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