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EXPLICIT BOUNDS ON SOME NONLINEAR RETARDED INTEGRAL INEQUALITIES

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ABSTRACT. In this paper some new retarded integral inequalities are established and explicit bounds on the unknown functions are derived. The present results extend some existing ones proved by Lipovan in [A retarded integral inequality and its applications, J. Math. Anal. Appl. 285 (2003) 436-443].

Key words and phrases: Integral inequality; retarded; nonlinear; explicit bound.

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1. INTRODUCTION

During the past decades, studies on integral inequalities have been greatly enriched by the recognition of their potential applications in various applied sciences [1] – [6]. Recently, integral inequalities with delays have received much attention from researchers [7] – [12]. In this paper, we establish some new retarded integral inequalities and derive explicit bounds on unknown functions, the results of which improve some known ones in [9].

2. MAIN RESULTS

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, +\infty)$. $C(M, S)$ denotes the class of all continuous functions from M to S. $C^1(M, S)$ denotes the class of functions with continuous first derivative.

Theorem 2.1. *Suppose that* $p > q \geq 0$ *and* $c \geq 0$ *are constants, and* $u, f, g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$. *Let* $w \in (\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing with* $w(u) > 0$ *on* $(0, \infty)$, *and* $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *be*

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344-07

nondecreasing with $\alpha(t) \le t$ *on* $\mathbb{R}_+$*. Then the following integral inequality*

$$u^p(t) \le c^2 + 2\int_0^{\alpha(t)} \left[f(s)u^q(s) \left(\int_0^s g(\tau)w(u(\tau))d\tau \right) + h(s)u^q(s) \right] ds, \ t \in \mathbb{R}_+ \tag{2.1}$$

implies for $0 \le t \le T$*,*

$$u(t) \le \left\{ G^{-1} \left[G(\xi(t)) + \frac{2(p-q)}{p} \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)d\tau ds \right] \right\}^{\frac{1}{p-q}} \tag{2.2}$$

holds, where

$$\xi(t) = c^{\frac{2(p-q)}{p}} + \frac{2(p-q)}{p} \int_0^{\alpha(t)} h(s)ds, \tag{2.3}$$

$$G(r) = \int_{r_0}^{r} \frac{1}{w\left(s^{\frac{1}{p-q}}\right)} ds, \quad r \ge r_0 > 0, \tag{2.4}$$

G^{-1} *denotes the inverse function of* G*, and* $T \in \mathbb{R}_+$ *is chosen so that*

$$G(\xi(t)) + \frac{2(p-q)}{p} \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)d\tau ds \in \mathit{Dom}\left(G^{-1}\right), \quad \textit{for all } 0 \le t \le T.$$

Proof. The conditions $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha(t) \le t$ imply that $\alpha(0) = 0$. Firstly we assume that $c > 0$. Define the nondeceasing positive function $z(t)$ by

$$z(t) := c^2 + 2\int_0^{\alpha(t)} \left[f(s)u^q(s) \left(\int_0^s g(\tau)w(u(\tau))d\tau \right) + h(s)u^q(s) \right] ds.$$

Then $z(0) = c^2$ and by (2.1) we have

$$u(t) \le [z(t)]^{\frac{1}{p}}, \tag{2.5}$$

and consequently $u(\alpha(t)) \le [z(\alpha(t))]^{\frac{1}{p}} \le [z(t)]^{\frac{1}{p}}$. By differentiation we get

$$\begin{aligned} z'(t) &= 2u^q(\alpha(t)) \left[f(\alpha(t)) \left(\int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau \right) + h(\alpha(t)) \right] \alpha'(t) \\ &\le 2\,[z(t)]^{\frac{q}{p}} \left[f(\alpha(t)) \left(\int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau \right) + h(\alpha(t)) \right] \alpha'(t). \end{aligned}$$

Hence

$$\frac{z'(t)}{[z(t)]^{\frac{q}{p}}} \le 2f(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau + 2h(\alpha(t))\alpha'(t).$$

Integrating both sides of last relation on $[0, t]$ yields

$$\frac{p}{p-q}[z(t)]^{\frac{p-q}{p}} \le \frac{p}{p-q}[z(0)]^{\frac{p-q}{p}} + 2\int_0^{\alpha(t)} h(s)ds + 2\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)w(u(\tau))d\tau ds,$$

which can be rewritten as

$$\begin{aligned} [z(t)]^{\frac{p-q}{p}} \le c^{\frac{2(p-q)}{p}} &+ \frac{2(p-q)}{p} \int_0^{\alpha(t)} h(s)ds \\ &+ \frac{2(p-q)}{p} \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)w(u(\tau))d\tau ds. \end{aligned} \tag{2.6}$$

Let $T_1(\leq T)$ be an arbitrary number. For $0 \leq t \leq T_1$, from (2.3) and (2.6) we have

$$[z(t)]^{\frac{p-q}{p}} \leq \xi(T_1) + \frac{2(p-q)}{p} \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) w(u(\tau)) d\tau ds. \tag{2.7}$$

Denoting the right-hand side of (2.7) by $m(t)$, we know $u(t) \leq [z(t)]^{\frac{1}{p}} \leq [m(t)]^{\frac{1}{p-q}}$. Since w is nondecreasing, we obtain

$$w[u(\tau)] \leq w\left[(z(\tau))^{\frac{1}{p}}\right] \leq w\left[(z(\alpha(t)))^{\frac{1}{p}}\right] \leq w\left[(z(t))^{\frac{1}{p}}\right], \quad \text{for} \quad \tau \in [0, \alpha(t)].$$

Hence

$$\begin{aligned} m'(t) &= \frac{2(p-q)}{p} f(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(\tau) w(u(\tau)) d\tau \\ &\leq \frac{2(p-q)}{p} w\left[(z(t))^{\frac{1}{p}}\right] f(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(\tau) d\tau \\ &\leq \frac{2(p-q)}{p} w\left[(m(t))^{\frac{1}{p-q}}\right] f(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(\tau) d\tau. \end{aligned}$$

That is

$$\frac{m'(t)}{w[(m(t))^{\frac{1}{p-q}}]} \leq \frac{2(p-q)}{p} f(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(\tau) d\tau. \tag{2.8}$$

Integrating both sides of the last inequality on $[0, t]$ and using the definition (2.4), we get

$$G(m(t)) - G(m(0)) \leq \frac{2(p-q)}{p} \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) d\tau ds. \tag{2.9}$$

Taking $t = T_1$ in inequality (2.9) and using $u(t) \leq [m(t)]^{\frac{1}{p-q}}$, we have

$$u(T_1) \leq \left\{ G^{-1}\left[G\left[\xi(T_1)\right] + \frac{2(p-q)}{p} \int_0^{\alpha(T_1)} f(s) \int_0^s g(\tau) d\tau ds \right] \right\}^{\frac{1}{p-q}}.$$

Since $T_1(\leq T)$ is arbitrary, we have proved the desired inequality (2.2).

The case $c = 0$ can be handled by repeating the above procedure with $\varepsilon > 0$ instead of c and subsequently letting $\varepsilon \to 0$. This completes the proof. □

Remark 1. If $c = 0$ and $h(t) \equiv 0$ hold, $G(\xi(t)) = G(0)$ in (2.4) is not defined. In such a case, the upper bound on solutions of the integral inequality (2.1) can be calculated as

$$u(t) \leq \lim_{\varepsilon \to 0+} \left\{ G^{-1}\left[G(\varepsilon) + \frac{2(p-q)}{p} \int_0^{\alpha(t)} f(s) \int_0^s g(\tau) d\tau ds \right] \right\}^{\frac{1}{p-q}}.$$

From Theorem 2.1, we can easily derive the following corollaries.

Corollary 2.2. *Suppose that $u, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $c \geq 0$ is a constant. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \leq t$ on $\mathbb{R}_+$. Then the following inequality*

$$u^2(t) \leq c^2 + 2\int_0^{\alpha(t)} h(s) u(s) ds,$$

implies

$$u(t) \leq c + \int_0^{\alpha(t)} h(s) ds.$$

Remark 2. If $\alpha(t) \equiv t$, from Corollary 2.2 we get the Ou-Iang inequality.

Corollary 2.3. *Suppose that $u, f, g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $c \geq 0$ is a constant. Let $w \in (\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $w(u) > 0$ on $(0, \infty)$, and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \leq t$ on $\mathbb{R}_+$. Then the following inequality*

$$u^2(t) \leq c^2 + 2\int_0^{\alpha(t)} \left[f(s)u(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right) + h(s)u(s)\right] ds$$

implies

$$u(t) \leq \xi(t) \exp\left(\int_0^{\alpha(t)} f(s)\left(\int_0^s g(\tau)d\tau\right) ds\right)$$

where $\xi(t) = c + \int_0^{\alpha(t)} h(s)ds$.

Theorem 2.4. *Suppose that $p > q \geq 0$ and $c \geq 0$ are constants, and $u, f, g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$. Let $w \in (\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $w(u) > 0$ on $(0, \infty)$, and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \leq t$ on $\mathbb{R}_+$. Then the following integral inequality*

$$\begin{aligned} u^p(t) \leq c^2 + 2\int_0^{\alpha(t)} \Bigg[f(s)u^q(s)\Bigg(& w(u(s)) \\ & + \int_0^s g(\tau)w(u(\tau))d\tau\Bigg) + h(s)u^q(s)\Bigg] ds, \qquad t \in \mathbb{R}_+ \end{aligned} \tag{2.10}$$

implies for $0 \leq t \leq T$

$$u(t) \leq \left\{ G^{-1}\left[G(\xi(t)) + \frac{2(p-q)}{p}\int_0^{\alpha(t)} f(s)\left(1 + \int_0^s g(\tau)d\tau\right) ds\right]\right\}^{\frac{1}{p-q}}, \tag{2.11}$$

where $\xi(t)$ and $G(r)$ are defined by (2.3) and (2.4), respectively, and $T \in \mathbb{R}_+$ is chosen so that

$$G(\xi(t)) + \frac{2(p-q)}{p}\int_0^{\alpha(t)} f(s)\left(1 + \int_0^s g(\tau)d\tau\right) ds \in \textit{Dom}\left(G^{-1}\right), \textit{ for all } 0 \leq t \leq T.$$

Proof. Firstly we assume that $c > 0$. Define the nondeceasing positive function by

$$z(t) := c^2 + 2\int_0^{\alpha(t)} \left[f(s)u^q(s)\left(w(u(s)) + \int_0^s g(\tau)w(u(\tau))d\tau\right) + h(s)u^q(s)\right] ds,$$

then $z(0) = c^2$ and by (2.10) we have

$$u(t) \leq [z(t)]^{\frac{1}{p}}, \tag{2.12}$$

and

$$\begin{aligned} z'(t) &= 2u^q(\alpha(t))\left[f(\alpha(t))\left(w(u(\alpha(t))) + \int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau\right) + h(\alpha(t))\right]\alpha'(t) \\ &\leq 2\left[z(t)\right]^{\frac{q}{p}}\left[f(\alpha(t))\left(w(u(\alpha(t))) + \int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau\right) + h(\alpha(t))\right]\alpha'(t). \end{aligned}$$

Hence

$$\frac{z'(t)}{[z(t)]^{\frac{q}{p}}} \leq 2h(\alpha(t))\alpha'(t) + 2f(\alpha(t))\alpha'(t)\left(w(u(\alpha(t)) + \int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau\right).$$

Integrating both sides of the last inequality on $[0,t]$, we get

$$\frac{p}{p-q}[z(t)]^{\frac{p-q}{p}} \le \frac{p}{p-q}[z(0)]^{\frac{p-q}{p}} + 2\int_0^{\alpha(t)} h(s)ds + 2\int_0^{\alpha(t)} f(s)\left(w(u(s)) + \int_0^s g(\tau)w(u(\tau))d\tau\right)ds.$$

Using (2.3), we get

$$[z(t)]^{\frac{p-q}{p}} \le \xi(t) + \frac{2(p-q)}{p}\int_0^{\alpha(t)} f(s)\left(w(u(s)) + \int_0^s g(\tau)w(u(\tau))d\tau\right)ds.$$

Let $T_1(\le T)$ be an arbitrary number. From last inequality we know the following relation holds for $t \in [0,T_1]$,

$$[z(t)]^{\frac{p-q}{p}} \le \xi(T_1) + \frac{2(p-q)}{p}\int_0^{\alpha(t)} f(s)\left(w(u(s)) + \int_0^s g(\tau)w(u(\tau))d\tau\right)ds.$$

Letting

$$m(t) = \xi(T_1) + \frac{2(p-q)}{p}\int_0^{\alpha(t)} f(s)\left(w(u(s)) + \int_0^s g(\tau)w(u(\tau))d\tau\right)ds, \tag{2.13}$$

we get $[z(t)]^{\frac{p-q}{p}} \le m(t)$. Since w is nondecreasing, we have

$$w[u(\alpha(t))] \le w\left[(z(\alpha(t)))^{\frac{1}{p}}\right] \le w\left[(z(t))^{\frac{1}{p}}\right] \le w\left[(m(t))^{\frac{1}{p-q}}\right]$$

and

$$w[u(\tau)] \le w\left[(z(\tau))^{\frac{1}{p}}\right] \le w\left[(z(\alpha(t)))^{\frac{1}{p}}\right] \le w\left[(z(t))^{\frac{1}{p}}\right], \quad \text{for} \quad \tau \in [0,\alpha(t)].$$

From (2.13), by differentiation we obtain

$$\begin{aligned} m'(t) &= \frac{2(p-q)}{p}f(\alpha(t))\left(w(u(\alpha(t))) + \int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau\right)\alpha'(t) \\ &\le \frac{2(p-q)}{p}f(\alpha(t))\left\{w\left([m(t)]^{\frac{1}{p-q}}\right) + \int_0^{\alpha(t)} g(\tau)w\left([m(t)]^{\frac{1}{p-q}}\right)d\tau\right\}\alpha'(t) \\ &= w\left([m(t)]^{\frac{1}{p-q}}\right)\frac{2(p-q)}{p}f(\alpha(t))\left(1 + \int_0^{\alpha(t)} g(\tau)d\tau\right)\alpha'(t). \end{aligned}$$

Hence

$$\frac{m'(t)}{w\left([m(t)]^{\frac{1}{p-q}}\right)} \le \frac{2(p-q)}{p}f(\alpha(t))\left(1 + \int_0^{\alpha(t)} g(\tau)d\tau\right)\alpha'(t).$$

Integrating both sides of the last inequality on $[0,t]$, from (2.4) we get

$$G(m(t)) \le G(m(0)) + \frac{2(p-q)}{p}\int_0^{\alpha(t)} f(s)\left(1 + \int_0^s g(\tau)d\tau\right)ds.$$

Hence

$$m(t) \le G^{-1}\left[G(\xi(T_1)) + \frac{2(p-q)}{p}\int_0^{\alpha(t)} f(s)\left(1 + \int_0^s g(\tau)d\tau\right)ds\right]. \tag{2.14}$$

Taking $t = T_1$ in inequality (2.14) and using $u(t) \le [m(t)]^{\frac{1}{p-q}}$, we have

$$u(T_1) \le \left\{ G^{-1} \left[G(\xi(T_1)) + \frac{2(p-q)}{p} \int_0^{\alpha(T_1)} f(s) \left(1 + \int_0^s g(\tau)d\tau\right) ds \right] \right\}^{\frac{1}{p-q}}.$$

Since $T_1 (\le T)$ is arbitrary we have proved the desired inequality (2.11).

If $c = 0$, the result can be proved by repeating the above procedure with $\varepsilon > 0$ instead of c and subsequently letting $\varepsilon \to 0$. This completes the proof. $\square$

Remark 3. Theorem 2.1 of Lipovan in [9] is special case of above Theorem 2.4, under the assumptions that $p = 2$, $q = 1$ and $g(t) \equiv 0$.

Theorem 2.5. *Suppose that $p > q \ge 0$ and $c \ge 0$ are constants, and $u, f, g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$. Let $w \in (\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $w(u) > 0$ on $(0, \infty)$, and $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha(t) \le t$, $\beta(t) \le t$ on $\mathbb{R}_+$. Then the following integral inequality*

$$(2.15)\quad u^p(t) \le c^2 + 2\int_0^{\alpha(t)} \left[f(s)u^q(s) \left(w(u(s)) + \int_0^s g(\tau)w(u(\tau))d\tau \right) \right] ds + 2\int_0^{\beta(t)} h(s)u^q(s)\, w(u(s))\, ds, \quad t \in \mathbb{R}_+$$

implies for $0 \le t \le T$

$$(2.16)\quad u(t) \le \left\{ G^{-1} \left[G(c^{\frac{2(p-q)}{p}}) + \frac{2(p-q)}{p} \int_0^{\alpha(t)} f(s) \left(1 + \int_0^s g(\tau)d\tau\right) ds + \frac{2(p-q)}{p} \int_0^{\beta(t)} h(s)ds \right] \right\}^{\frac{1}{p-q}},$$

where $G(r)$ is defined by (2.4) and $T \in \mathbb{R}_+$ is chosen so that

$$G\left(c^{\frac{2(p-q)}{p}}\right) + \frac{2(p-q)}{p} \int_0^{\alpha(t)} f(s) \left(1 + \int_0^s g(\tau)d\tau\right) ds + \frac{2(p-q)}{p} \int_0^{\beta(t)} h(s)ds \in \mathit{Dom}\left(G^{-1}\right), \quad \textit{for all} \quad 0 \le t \le T.$$

Proof. The conditions that $\alpha, \beta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing with $\alpha(t) \le t$, $\beta(t) \le t$ imply that $\alpha(0) = 0$ and $\beta(0) = 0$.

Let us first assume that $c > 0$. Denoting the right-hand side of (2.15) by $z(t)$, we know $z(t)$ is nondecreasing, $z(0) = c^2$ and $u(t) \le [z(t)]^{\frac{1}{p}}$. Consequently we have

$$u(\alpha(t)) \le [z(\alpha(t))]^{\frac{1}{p}} \le [z(t)]^{\frac{1}{p}} \quad \text{and} \quad u(\beta(t)) \le [z(\beta(t))]^{\frac{1}{p}} \le [z(t)]^{\frac{1}{p}}.$$

Since w is nondecreasing, we obtain

$$\begin{aligned} z'(t) &= 2f(\alpha(t))u^q(\alpha(t)) \left(w(u(\alpha(t))) + \int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau \right) \alpha'(t) \\ &\quad + 2h(\beta(t))\, u^q(\beta(t))\, w(u(\beta(t)))\, \beta'(t) \\ &\le 2[z(t)]^{\frac{q}{p}} \left[f(\alpha(t)) \left(w(u(\alpha(t))) + \int_0^{\alpha(t)} g(\tau)\, w(u(\tau))\, d\tau \right) \alpha'(t) \right. \\ &\quad \left. + h(\beta(t))\, w(u(\beta(t)))\, \beta'(t) \right]. \end{aligned}$$

Hence

$$\frac{z'(t)}{[z(t)]^{\frac{q}{p}}} \le 2f(\alpha(t))\left(w(u(\alpha(t))) + \int_0^{\alpha(t)} g(\tau)\,w(u(\tau))d\tau\right)\alpha'(t) + 2h(\beta(t))\,w(u(\beta(t)))\,\beta'(t).$$

Integrating both sides on $[0,t]$, we get

$$\frac{p}{p-q}[z(t)]^{\frac{p-q}{p}} \le \frac{p}{p-q}[z(0)]^{\frac{p-q}{p}} + 2\int_0^{\alpha(t)} f(s)\left(w(u(s)) + \int_0^s g(\tau)w(u(\tau))d\tau\right)ds + 2\int_0^{\beta(t)} h(s)w(u(s))\,ds,$$

which can be rewritten as

$$[z(t)]^{\frac{p-q}{p}} \le c^{\frac{2(p-q)}{p}} + \frac{2(p-q)}{p}\int_0^{\alpha(t)} f(s)\left(w(u(s)) + \int_0^s g(\tau)w(u(\tau))d\tau\right)ds + \frac{2(p-q)}{p}\int_0^{\beta(t)} h(s)w(u(s))\,ds. \tag{2.17}$$

Denoting the right-hand side of (2.17) by $m(t)$, we know $[z(t)]^{\frac{p-q}{p}} \le m(t)$ and

$$\begin{aligned}
m'(t) &= \frac{2(p-q)}{p}f(\alpha(t))\left(w(u(\alpha(t))) + \int_0^{\alpha(t)} g(\tau)w(u(\tau))d\tau\right)\alpha'(t) \\
&\quad + \frac{2(p-q)}{p}h(\beta(t))\,w(u(\beta(t)))\,\beta'(t) \\
&\le \frac{2(p-q)}{p}f(\alpha(t))\left(w\left(z^{\frac{1}{p}}(\alpha(t))\right) + \int_0^{\alpha(t)} g(\tau)w\left(z^{\frac{1}{p}}(\tau)\right)d\tau\right)\alpha'(t) \\
&\quad + \frac{2(p-q)}{p}h(\beta(t))\,w\left(z^{\frac{1}{p}}(\beta(t))\right)\beta'(t) \\
&\le w\left(z^{\frac{1}{p}}(t)\right)\frac{2(p-q)}{p}\left[f(\alpha(t))\left(1 + \int_0^{\alpha(t)} g(\tau)d\tau\right)\alpha'(t) + h(\beta(t))\,\beta'(t)\right] \\
&\le w\left(m^{\frac{1}{p-q}}(t)\right)\frac{2(p-q)}{p}\left[f(\alpha(t))\left(1 + \int_0^{\alpha(t)} g(\tau)d\tau\right)\alpha'(t) + h(\beta(t))\,\beta'(t)\right].
\end{aligned}$$

The above relation gives

$$\frac{m'(t)}{w\left(m^{\frac{1}{p-q}}(t)\right)} \le \frac{2(p-q)}{p}\left[f(\alpha(t))\left(1 + \int_0^{\alpha(t)} g(\tau)d\tau\right)\alpha'(t) + h(\beta(t))\,\beta'(t)\right].$$

Integrating both sides on $[0,t]$ and using definition (2.4) we get

$$\begin{aligned}
G(m(t)) &\le G(m(0)) + \frac{2(p-q)}{p}\left[\int_0^{\alpha(t)} f(s)\left(1 + \int_0^s g(\tau)\,d\tau\right)ds + \int_0^{\beta(t)} h(s)\,ds\right] \\
&\le G\left(c^{\frac{2(p-q)}{p}}\right) + \frac{2(p-q)}{p}\left[\int_0^{\alpha(t)} f(s)\left(1 + \int_0^s g(\tau)\,d\tau\right)ds + \int_0^{\beta(t)} h(s)\,ds\right].
\end{aligned}$$

Using the relation $u(t) \le [z(t)]^{\frac{1}{p}} \le [m(t)]^{\frac{1}{p-q}}$, we get the desired inequality (2.16).

If $c = 0$, the result can be proved by repeating the above procedure with $\varepsilon > 0$ instead of c and subsequently letting $\varepsilon \to 0$. This completes the proof. □

Remark 4. Theorem 2 of Lipovan in [9] is a special case of Theorem 2.5 above, under the assumptions that $p = 2$, $q = 1$, $g(t) \equiv 0$ and $\beta(t) \equiv t$.

3. APPLICATION

Example 3.1. Consider the delay integral equation

$$x^5(t) = x_0^2 + 2\int_0^{\alpha(t)} \left[x^3(s)M\left(s, x(s), \int_0^s N(s,\tau,w(|x(\tau)|))d\tau\right) + h(s)x^3(s)\right]ds. \tag{3.1}$$

Assume that

$$|M(s,t,v)| \le f(s)\,|v|\,, \quad |N(s,t,v)| \le g(t)\,|v|\,, \tag{3.2}$$

where f, g, h, α and w are as defined in Theorem 2.1. From (3.1) and (3.2) we obtain

$$|x(t)|^5 \le x_0^2 + 2\int_0^{\alpha(t)} \left[|x(s)|^3 f(s)\int_0^s g(\tau)w(|x(\tau)|)d\tau + h(s)\,|x(s)|^3\right]ds.$$

Applying Theorem 2.1 to the last relation, we get an explicit bound on an unknown function

$$|x(t)| \le \left\{G^{-1}\left[G(\xi(t)) + \frac{4}{5}\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau ds\right]\right\}^{\frac{1}{2}}, \tag{3.3}$$

where

$$\xi(t) = \left|\sqrt[5]{x_0^4}\right| + \frac{4}{5}\int_0^{\alpha(t)} h(s)ds.$$

In particular, if $\omega(t) \equiv t$ holds in (3.1), from (2.4) we derive

$$G(t) = \int_0^t \frac{1}{\omega\left(s^{\frac{1}{p-q}}\right)}ds = \int_0^t \frac{1}{s^{\frac{1}{p-q}}}ds = \int_0^t s^{-\frac{1}{2}}ds = 2\sqrt{t} \tag{3.4}$$

and

$$G^{-1}(t) = \frac{1}{4}t^2. \tag{3.5}$$

Substituting (3.4) and (3.5) into inequality (3.3), we get

$$|x(t)| \le \sqrt{\xi(t)} + \frac{2}{5}\int_0^{\alpha(t)} f(s)\int_0^s g(\tau)d\tau.$$

Example 3.2. Consider the following equation

$$\begin{aligned} x^8(t) = x_0^2 + 2\int_0^{\alpha(t)} \Bigg[x^4(s)\Bigg(& M(s, x(s), w(|x(s)|)) \\ &+ \int_0^s N(s,\tau,w(|x(\tau)|))d\tau\Bigg)\Bigg]ds + 2\int_0^{\alpha(t)} \left[h(s)x^4(s)\right]ds. \end{aligned} \tag{3.6}$$

Assume that

$$|M(s,t,v)| \le f(s)\,|v|\,, \quad |N(s,t,v)| \le f(s)g(t)\,|v|\,, \tag{3.7}$$

where f, g, h, α and w are as defined in Theorem 2.4. From (3.6) and (3.7) we obtain

$$|x(t)|^8 \le x_0^2 + 2\int_0^{\alpha(t)} \left[|x(s)|^4 f(s) \left(w(|x(s)|) + \int_0^s g(\tau)w(|x(\tau)|)d\tau \right) + h(s)\,|x(s)|^4 \right] ds.$$

By Theorem 2.4 we get an explicit bound on an unknown function

$$|x(t)| \le \left\{ G^{-1}\left[G\left(\xi(t)\right) + \int_0^{\alpha(t)} f(s)\left(1 + \int_0^s g(\tau)d\tau\right) ds \right] \right\}^{\frac{1}{4}}, \tag{3.8}$$

where

$$\xi(t) = |x_0| + \int_0^{\alpha(t)} h(s)ds.$$

In particular, if $\omega\,(t) \equiv t^3$ holds in (3.6), from (2.4) we obtain

$$G(t) = \int_0^t \frac{1}{\omega\left(s^{\frac{1}{p-q}}\right)} ds = \int_0^t \frac{1}{s^{\frac{3}{p-q}}} ds = \int_0^t s^{-\frac{3}{4}} ds = 4t^{\frac{1}{4}} \tag{3.9}$$

and

$$G^{-1}\,(t) = \frac{1}{256} t^4. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8) we get

$$|x(t)| \le \left[\xi\,(t)\right]^{\frac{1}{4}} + \frac{1}{4}\int_0^{\alpha(t)} f\,(s)\left(1 + \int_0^s g\,(\tau)\,d\tau\right) ds.$$

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