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GENERALIZATIONS OF THE KY FAN INEQUALITY

AI-JUN LI, XUE-MIN WANG AND CHAO-PING CHEN

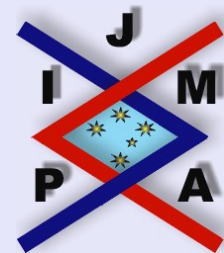
Jiaozuo University
Jiaozuo City, Henan Province
454000, China

EMail: liajun72@163.com

EMail: wangxm881@163.com

College of Mathematics and Informatics
Research Institute of Applied Mathematics
Henan Polytechnic University
Jiaozuo City, Henan 454010, China
EMail: chenchaoping@hpu.edu.cn

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Abstract

In this paper, we extend the Ky Fan inequality to several general integral forms, and obtain the monotonic properties of the function $\frac{L_s(a,b)}{L_s(\alpha-a,\alpha-b)}$ with $\alpha, a, b \in (0, +\infty)$ and $s \in \mathbb{R}$.

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1. Introduction

The following inequality proposed by Ky Fan was recorded in [1, p. 5] : If $0 < x_i \leq \frac{1}{2}$ for $i = 1, 2, \dots, n$, then

$$(1.1) \quad \left(\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1-x_i)} \right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)},$$

unless $x_1 = x_2 = \dots = x_n$.

With the notation

$$(1.2) \quad M_r(x) = \begin{cases} \left(\frac{1}{n} \sum_{i=1}^n x_i^r \right)^{\frac{1}{r}}, & r \neq 0; \\ \left(\prod_{i=1}^n x_i \right)^{\frac{1}{n}}, & r = 0, \end{cases}$$

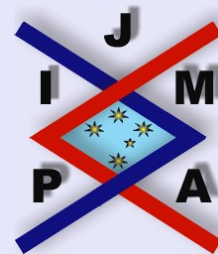
where $M_r(x)$ denotes the r -order power mean of $x_i > 0$ for $i = 1, 2, \dots, n$, the inequality (1.1) can be written as

$$(1.3) \quad \frac{M_0(x)}{M_0(1-x)} \leq \frac{M_1(x)}{M_1(1-x)}.$$

In 1996, Zh. Wang, J. Chen and X. Li [12] found the necessary and sufficient condition for

$$(1.4) \quad \frac{M_r(x)}{M_r(1-x)} \leq \frac{M_s(x)}{M_s(1-x)}$$

when $r < s$. Recently, Ch.-P. Chen proved that the function $\frac{L_r(a,b)}{L_r(1-a,1-b)}$ is strictly increasing for $0 < a < b \leq \frac{1}{2}$ and strictly decreasing for $\frac{1}{2} \leq a < b < 1$,



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where $r \in (-\infty, \infty)$ and $L_r(a, b)$ is the generalized logarithmic mean of two positive numbers a, b , which is a special case of the extended means $E(r, s; x, y)$ defined by Stolarsky [10] in 1975. For more information about the extended means please refer to [4, 6, 8, 11] and references therein.

Moreover, we have,

$$L_r(a, b) = a, \quad a = b;$$

$$L_r(a, b) = \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right)^{\frac{1}{r}}, \quad a \neq b, r \neq -1, 0;$$

$$L_{-1}(a, b) = \frac{b-a}{\ln b - \ln a} = L(a, b);$$

$$L_0(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} = I(a, b),$$

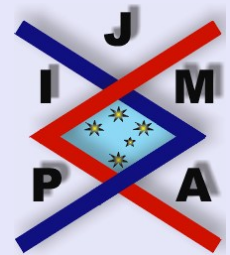
where $L(a, b)$ and $I(a, b)$ are respectively the logarithmic mean and the exponential mean of two positive numbers a and b . When $a \neq b$, $L_r(a, b)$ is a strictly increasing function of r . In particular,

$$\lim_{r \rightarrow -\infty} L_r(a, b) = \min\{a, b\}, \quad \lim_{r \rightarrow +\infty} L_r(a, b) = \max\{a, b\},$$

$$L_1(a, b) = A(a, b), \quad L_{-2}(a, b) = G(a, b),$$

where $A(a, b)$ and $G(a, b)$ are the arithmetic and the geometric means, respectively. For $a \neq b$, the following well known inequality holds:

$$(1.5) \quad G(a, b) < L(a, b) < I(a, b) < A(a, b).$$



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In this paper, motivated by inequality (1.4), we will extend the inequality (1.4) to general integral forms. Some monotonic properties of several related functions will be obtained.

Theorem 1.1. *Let*

$$f_{\alpha}(s) = \left(\frac{\int_a^b x^s dx}{\int_a^b (\alpha - x)^s dx} \right)^{\frac{1}{s}} = \frac{L_s(a, b)}{L_s(\alpha - a, \alpha - b)},$$

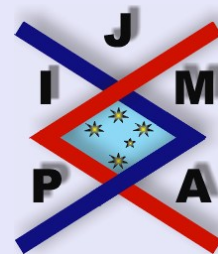
$s \in (-\infty, +\infty)$ and α be a positive number. Then $f_{\alpha}(s)$ is a strictly increasing function for $[a, b] \subseteq (0, \frac{\alpha}{2}]$, and is a strictly decreasing function for $[a, b] \subseteq [\frac{\alpha}{2}, \alpha)$.

Corollary 1.2. *If $[a, b] \subseteq (0, \frac{\alpha}{2}]$ and α is a positive number, then*

$$(1.6) \quad \frac{a}{\alpha - b} < \frac{G(a, b)}{G(\alpha - a, \alpha - b)} < \frac{L(a, b)}{L(\alpha - a, \alpha - b)} < \frac{I(a, b)}{I(\alpha - a, \alpha - b)} < \frac{A(a, b)}{A(\alpha - a, \alpha - b)} < \frac{b}{\alpha - a}.$$

If $[a, b] \subseteq [\frac{\alpha}{2}, \alpha)$, the inequalities (1.6) is reversed.

Corollary 1.3. *Let $h_{\alpha}(s) = \left(\frac{\int_a^b x^s dx}{\int_{\alpha-b}^{\alpha-a} x^s dx} \right)^{\frac{1}{s}}$, $s \in (-\infty, +\infty)$ and α be a positive number. Then $h_{\alpha}(s)$ is a strictly increasing function for $[a, b] \subseteq (0, \frac{\alpha}{2}]$, or a strictly decreasing function for $[a, b] \subseteq [\frac{\alpha}{2}, \alpha)$.*



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In [13], Feng Qi has proved that the function

$$r \mapsto \left(\frac{\frac{1}{b-a} \int_a^b x^r dx}{\frac{1}{b+\delta-a} \int_a^{b+\delta} x^r dx} \right)^{\frac{1}{r}} = \frac{L_r(a, b)}{L_r(a, b + \delta)}$$

is strictly decreasing with $r \in (-\infty, +\infty)$. Now, we will extend the conclusion in the following theorem.

Theorem 1.4. *Let*

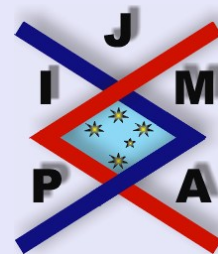
$$f(s) = \left(\frac{\frac{1}{b-a} \int_a^b x^s dx}{\frac{1}{d-c} \int_c^d x^s dx} \right)^{\frac{1}{s}} = \frac{L_s(a, b)}{L_s(c, d)},$$

$s \in (-\infty, +\infty)$ and a, b, c, d be positive numbers. Then $f(s)$ is a strictly increasing function for $ad < bc$, or a strictly decreasing function for $ad > bc$.

Corollary 1.5. *Let*

$$h(s) = \left(\frac{\frac{1}{b-a} \int_a^b x^s dx}{\frac{1}{d-a} \int_a^d x^s dx} \right)^{\frac{1}{s}} = \frac{L_s(a, b)}{L_s(a, d)},$$

$s \in (-\infty, +\infty)$ and a, b, d are positive numbers. Then $h(s)$ is a strictly increasing function for $d < b$, or a strictly decreasing function for $d > b$.



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2. Proofs of Theorems

In order to prove Theorem 1.1, we make use of the following elementary lemma which can be found in [3, p. 395].

Lemma 2.1 ([3, p. 395]). *Let the second derivative of $\phi(x)$ be continuous with $x \in (-\infty, \infty)$ and $\phi(0) = 0$. Define*

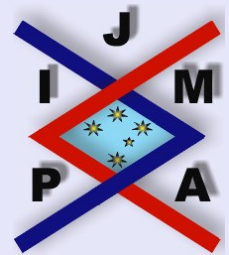
$$(2.1) \quad g(x) = \begin{cases} \frac{\phi(x)}{x}, & x \neq 0; \\ \phi'(0), & x = 0. \end{cases}$$

Then $\phi(x)$ is strictly convex (concave) if and only if $g(x)$ is strictly increasing (decreasing) with $x \in (-\infty, \infty)$.

Remark 1. *A general conclusion was given in [7, p. 18]: A function ϕ is convex on $[a, b]$ if and only if $\frac{\phi(x) - \phi(x_0)}{x - x_0}$ is nondecreasing on $[a, b]$ for every point $x_0 \in [a, b]$.*

Proof of Theorem 1.1. It is obvious that

$$\begin{aligned} f_\alpha(s) &= \left(\frac{\int_a^b x^s dx}{\int_a^b (\alpha - x)^s dx} \right)^{\frac{1}{s}} = \left(\frac{b^{s+1} - a^{s+1}}{(\alpha - a)^{s+1} - (\alpha - b)^{s+1}} \right)^{\frac{1}{s}} \\ &= \frac{L_s(a, b)}{L_s(\alpha - a, \alpha - b)}. \end{aligned}$$



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Define for $s \in (-\infty, \infty)$,

$$(2.2) \quad \varphi(s) = \begin{cases} \ln \left(\frac{b^{s+1} - a^{s+1}}{(\alpha - a)^{s+1} - (\alpha - b)^{s+1}} \right), & s \neq -1; \\ \ln \left(\frac{\ln(b/a)}{\ln[(\alpha - a)/(\alpha - b)]} \right), & s = -1. \end{cases}$$

Then

$$(2.3) \quad \ln f_\alpha(s) = \begin{cases} \frac{\varphi(s)}{s}, & s \neq 0; \\ \varphi'(0), & s = 0. \end{cases}$$

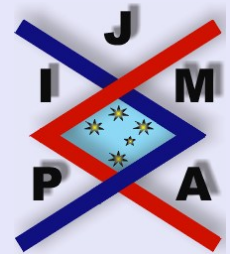
In order to prove that $\ln f_\alpha$ is strictly increasing (decreasing), it suffices to show that φ is strictly convex (concave) on $(-\infty, \infty)$. A simple calculation reveals that

$$(2.4) \quad \varphi(-1 - s) = \varphi(-1 + s) + s \ln \frac{(\alpha - a)(\alpha - b)}{ab},$$

which implies that $\varphi''(-1 - s) = \varphi''(-1 + s)$, and φ has the same convexity (concavity) on both $(-\infty, -1)$ and $(-1, \infty)$. Hence, it is sufficient to prove that φ is strictly convex (concave) on $(-1, \infty)$.

A computation yields

$$\varphi'(s) = \frac{b^{s+1} \ln b - a^{s+1} \ln a}{b^{s+1} - a^{s+1}} - \frac{(\alpha - b)^{s+1} \ln(\alpha - b) - (\alpha - a)^{s+1} \ln(\alpha - a)}{(\alpha - b)^{s+1} - (\alpha - a)^{s+1}},$$



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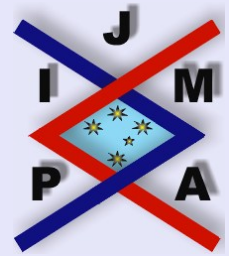


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$$\begin{aligned}
 (2.5) \quad & (s+1)^2 \varphi''(s) \\
 &= (s+1)^2 \left[-\frac{a^{s+1} b^{s+1} (\ln \frac{a}{b})^2}{(b^{s+1} - a^{s+1})^2} + \frac{(\alpha - a)^{s+1} (\alpha - b)^{s+1} (\ln \frac{\alpha - b}{\alpha - a})^2}{[(\alpha - a)^{s+1} - (\alpha - b)^{s+1}]^2} \right] \\
 &= -\frac{(\frac{a}{b})^{s+1} [\ln(\frac{a}{b})^{s+1}]^2}{[1 - (\frac{a}{b})^{s+1}]^2} + \frac{(\frac{\alpha - b}{\alpha - a})^{s+1} [\ln(\frac{\alpha - b}{\alpha - a})^{s+1}]^2}{[1 - (\frac{\alpha - b}{\alpha - a})^{s+1}]^2}.
 \end{aligned}$$

Define for $0 < t < 1$,

$$(2.6) \quad \omega(t) = \frac{t(\ln t)^2}{(1-t)^2}.$$

Differentiation yields

$$\begin{aligned}
 (2.7) \quad & (1-t)t \ln t \frac{\omega'(t)}{\omega(t)} = (1+t) \ln t + 2(1-t) \\
 &= -\sum_{n=2}^{\infty} \frac{n-1}{n(n+1)} (1-t)^{n+1} < 0,
 \end{aligned}$$

which implies that $\omega'(t) > 0$ for $0 < t < 1$. It is easy to see that

$$(2.8) \quad 0 < \left(\frac{a}{b}\right)^{s+1} < \left(\frac{\alpha - b}{\alpha - a}\right)^{s+1} < 1 \quad \text{for} \quad [a, b] \subseteq \left(0, \frac{\alpha}{2}\right], \quad s > -1,$$

$$(2.9) \quad 0 < \left(\frac{\alpha - b}{\alpha - a}\right)^{s+1} < \left(\frac{a}{b}\right)^{s+1} < 1 \quad \text{for} \quad [a, b] \subseteq \left[\frac{\alpha}{2}, \alpha\right), \quad s > -1,$$

and therefore $\varphi''(s) > 0$ for $[a, b] \subseteq (0, \frac{\alpha}{2}]$ and $s > -1$, $\varphi''(s) < 0$ for $[a, b] \subseteq [\frac{\alpha}{2}, \alpha)$ and $s > -1$. Then φ is strictly convex (concave) on $(-1, \infty)$ for $[a, b] \subseteq (0, \frac{\alpha}{2}]$ ($[a, b] \subseteq [\frac{\alpha}{2}, \alpha)$) respectively. By Lemma 2.1 above, Theorem 1.1 holds. \square

Since $f_\alpha(s)$ is a strictly increasing (decreasing) function for $[a, b] \subseteq (0, \frac{\alpha}{2}]$ ($[a, b] \subseteq [\frac{\alpha}{2}, \alpha)$), put $s = -2, -1, 0, 1$ respectively. The inequalities (1.6) are deduced.

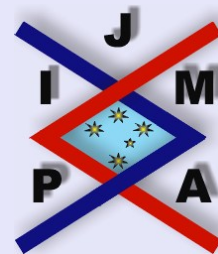
Then, let $(\alpha - x) = t$ and apply it to the function $\left(\frac{\int_a^b x^s dx}{\int_a^b (\alpha-x)^s dx}\right)^{\frac{1}{s}}$. We get Corollary 1.3.

Proof of Theorem 1.4. Using an analogous method of proof to that of Theorem 1.1, we get

$$\begin{aligned} f(s) &= \left(\frac{\frac{1}{b-a} \int_a^b x^s dx}{\frac{1}{d-c} \int_c^d x^s dx}\right)^{\frac{1}{s}} = \left[\frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right]^{\frac{1}{s}} \\ &= \left[\frac{d^{s+1}-c^{s+1}}{(s+1)(d-c)}\right]^{\frac{1}{s}} = \frac{L_s(a, b)}{L_s(c, d)}. \end{aligned}$$

Let $M = \frac{(d-c)}{(b-a)}$, and define for $s \in (-\infty, \infty)$,

$$(2.10) \quad \varphi(s) = \begin{cases} \ln \left(M \frac{b^{s+1} - a^{s+1}}{d^{s+1} - c^{s+1}} \right), & s \neq -1; \\ \ln \left[M \frac{\ln(b/a)}{\ln(d/c)} \right], & s = -1. \end{cases}$$



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Then

$$(2.11) \quad \ln f(s) = \begin{cases} \frac{\varphi(s)}{s}, & s \neq 0; \\ \varphi'(0), & s = 0, \end{cases}$$

and φ has the same convexity (concavity) on both $(-\infty, -1)$ and $(-1, \infty)$.

A computation yields

$$(s+1)^2 \varphi''(s) = -\frac{\left(\frac{a}{b}\right)^{s+1} [\ln\left(\frac{a}{b}\right)^{s+1}]^2}{\left[1 - \left(\frac{a}{b}\right)^{s+1}\right]^2} + \frac{\left(\frac{c}{d}\right)^{s+1} [\ln\left(\frac{c}{d}\right)^{s+1}]^2}{\left[1 - \left(\frac{c}{d}\right)^{s+1}\right]^2}.$$

Define for $0 < t < 1$,

$$(2.12) \quad \omega(t) = \frac{t(\ln t)^2}{(1-t)^2}.$$

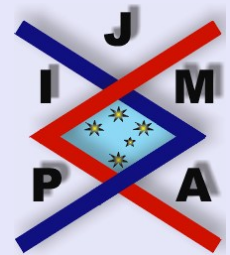
Differentiation yields $\omega'(t) > 0$ for $0 < t < 1$. It is easy to see that

$$(2.13) \quad 0 < \left(\frac{a}{b}\right)^{s+1} < \left(\frac{c}{d}\right)^{s+1} < 1 \quad \text{for } ad < bc, s > -1,$$

$$(2.14) \quad 0 < \left(\frac{c}{d}\right)^{s+1} < \left(\frac{a}{b}\right)^{s+1} < 1 \quad \text{for } ad > bc, s > -1,$$

and therefore $\varphi''(s) > 0$ for $ad < bc$ and $s > -1$, $\varphi''(s) < 0$ for $ad > bc$ and $s > -1$. Then φ is strictly convex (concave) on $(-1, \infty)$ for $ad < bc$ ($ad > bc$) respectively. The proof is complete. \square

In Theorem 1.4, let $a = c$. Then $f(s)$ is a strictly increasing function for $d < b$, or a strictly decreasing function for $d > b$. Thus Corollary 1.5 holds.



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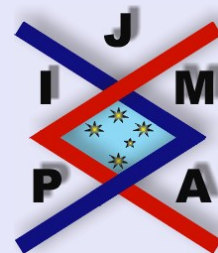
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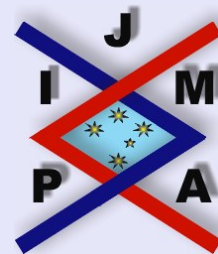
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