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GENERALIZATIONS OF THE KY FAN INEQUALITY

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Abstract

In this paper, we extend the Ky Fan inequality to several general integral forms, and obtain the monotonic properties of the function $\frac{L_s(a,b)}{L_s(\alpha-a,\alpha-b)}$ with $\alpha, a, b \in (0, +\infty)$ and $s \in \mathbb{R}$.

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1. Introduction

The following inequality proposed by Ky Fan was recorded in [1, p. 5] : If $0 < x_i \leq \frac{1}{2}$ for $i = 1, 2, \dots, n$, then

$$\left(\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1 - x_i)} \right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1 - x_i)}, \tag{1.1}$$

unless $x_1 = x_2 = \cdots = x_n$.

With the notation

$$M_r(x) = \begin{cases} \left(\frac{1}{n} \sum_{i=1}^n x_i^r \right)^{\frac{1}{r}}, & r \neq 0; \\ \left(\prod_{i=1}^n x_i \right)^{\frac{1}{n}}, & r = 0, \end{cases} \tag{1.2}$$

where $M_r(x)$ denotes the r-order power mean of $x_i > 0$ for $i = 1, 2, \dots, n$, the inequality (1.1) can be written as

$$\frac{M_0(x)}{M_0(1 - x)} \leq \frac{M_1(x)}{M_1(1 - x)}. \tag{1.3}$$

In 1996, Zh. Wang, J. Chen and X. Li [12] found the necessary and sufficient condition for

$$\frac{M_r(x)}{M_r(1 - x)} \leq \frac{M_s(x)}{M_s(1 - x)} \tag{1.4}$$

when $r < s$. Recently, Ch.-P. Chen proved that the function $\frac{L_r(a,b)}{L_r(1-a,1-b)}$ is strictly increasing for $0 < a < b \leq \frac{1}{2}$ and strictly decreasing for $\frac{1}{2} \leq a < b < 1$, where $r \in (-\infty, \infty)$ and $L_r(a,b)$ is the generalized logarithmic mean of two positive numbers a, b, which is a special case of the extended means $E(r,s;x,y)$ defined by Stolarsky [10] in 1975. For more information about the extended means please refer to [4, 6, 8, 11] and references therein.

Moreover, we have,

$$\begin{aligned} L_r(a,b) &= a, \qquad a = b; \\ L_r(a,b) &= \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right)^{\frac{1}{r}}, \qquad a \neq b, r \neq -1, 0; \\ L_{-1}(a,b) &= \frac{b-a}{\ln b - \ln a} = L(a,b); \\ L_0(a,b) &= \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} = I(a,b), \end{aligned}$$

where $L(a,b)$ and $I(a,b)$ are respectively the logarithmic mean and the exponential mean of two positive numbers a and b. When $a \neq b$, $L_r(a,b)$ is a strictly increasing function of r. In particular,

$$\begin{aligned} \lim_{r \to -\infty} L_r(a,b) &= \min\{a,b\}, \qquad \lim_{r \to +\infty} L_r(a,b) = \max\{a,b\}, \\ L_1(a,b) &= A(a,b), \qquad L_{-2}(a,b) = G(a,b), \end{aligned}$$

where $A(a,b)$ and $G(a,b)$ are the arithmetic and the geometric means, respectively. For $a \neq b$, the following well known inequality holds:

$$G(a,b) < L(a,b) < I(a,b) < A(a,b). \tag{1.5}$$

In this paper, motivated by inequality (1.4), we will extend the inequality (1.4) to general integral forms. Some monotonic properties of several related functions will be obtained.

Theorem 1.1. *Let*

$$f_\alpha(s) = \left(\frac{\int_a^b x^s \,\mathrm{d}x}{\int_a^b (\alpha - x)^s \,\mathrm{d}x} \right)^{\frac{1}{s}} = \frac{L_s(a,b)}{L_s(\alpha - a, \alpha - b)},$$

$s \in (-\infty, +\infty)$ *and* α *be a positive number. Then* $f_\alpha(s)$ *is a strictly increasing function for* $[a,b] \subseteq (0, \frac{\alpha}{2}]$*, and is a strictly decreasing function for* $[a,b] \subseteq [\frac{\alpha}{2}, \alpha)$.

Corollary 1.2. *If* $[a,b] \subseteq (0, \frac{\alpha}{2}]$ *and* α *is a positive number, then*

$$\begin{aligned} \frac{a}{\alpha - b} < \frac{G(a,b)}{G(\alpha - a, \alpha - b)} < \frac{L(a,b)}{L(\alpha - a, \alpha - b)} \\ < \frac{I(a,b)}{I(\alpha - a, \alpha - b)} < \frac{A(a,b)}{A(\alpha - a, \alpha - b)} < \frac{b}{\alpha - a}. \end{aligned} \tag{1.6}$$

If $[a,b] \subseteq [\frac{\alpha}{2}, \alpha)$*, the inequalities* (1.6) *is reversed.*

Corollary 1.3. *Let* $h_\alpha(s) = \left(\frac{\int_a^b x^s \,\mathrm{d}x}{\int_{\alpha-b}^{\alpha-a} x^s \,\mathrm{d}x} \right)^{\frac{1}{s}}$*,* $s \in (-\infty, +\infty)$ *and* α *be a positive number. Then* $h_\alpha(s)$ *is a strictly increasing function for* $[a,b] \subseteq (0, \frac{\alpha}{2}]$*, or a strictly decreasing function for* $[a,b] \subseteq [\frac{\alpha}{2}, \alpha)$.

In [13], Feng Qi has proved that the function

$$r \mapsto \left(\frac{\frac{1}{b-a} \int_a^b x^r \, \mathrm{d}x}{\frac{1}{b+\delta-a} \int_a^{b+\delta} x^r \, \mathrm{d}x} \right)^{\frac{1}{r}} = \frac{L_r(a,b)}{L_r(a,b+\delta)}$$

is strictly decreasing with $r \in (-\infty, +\infty)$. Now, we will extend the conclusion in the following theorem.

Theorem 1.4. *Let*

$$f(s) = \left(\frac{\frac{1}{b-a} \int_a^b x^s \, \mathrm{d}x}{\frac{1}{d-c} \int_c^d x^s \, \mathrm{d}x} \right)^{\frac{1}{s}} = \frac{L_s(a,b)}{L_s(c,d)},$$

$s \in (-\infty, +\infty)$ *and* a, b, c, d *be positive numbers. Then* $f(s)$ *is a strictly increasing function for* $ad < bc$*, or a strictly decreasing function for* $ad > bc$*.*

Corollary 1.5. *Let*

$$h(s) = \left(\frac{\frac{1}{b-a} \int_a^b x^s \, \mathrm{d}x}{\frac{1}{d-a} \int_a^d x^s \, \mathrm{d}x} \right)^{\frac{1}{s}} = \frac{L_s(a,b)}{L_s(a,d)},$$

$s \in (-\infty, +\infty)$ *and* a, b, d *are positive numbers. Then* $h(s)$ *is a strictly increasing function for* $d < b$*, or a strictly decreasing function for* $d > b$*.*

2. Proofs of Theorems

In order to prove Theorem 1.1, we make use of the following elementary lemma which can be found in [3, p. 395].

Lemma 2.1 ([3, p. 395]). *Let the second derivative of $\phi(x)$ be continuous with $x \in (-\infty, \infty)$ and $\phi(0) = 0$. Define*

$$g(x) = \begin{cases} \dfrac{\phi(x)}{x}, & x \neq 0; \\ \phi'(0), & x = 0. \end{cases} \tag{2.1}$$

Then $\phi(x)$ is strictly convex (concave) if and only if $g(x)$ is strictly increasing (decreasing) with $x \in (-\infty, \infty)$.

Remark 1. *A general conclusion was given in [7, p. 18]: A function ϕ is convex on $[a, b]$ if and only if $\frac{\phi(x)-\phi(x_0)}{x-x_0}$ is nondecreasing on $[a, b]$ for every point $x_0 \in [a, b]$.*

Proof of Theorem 1.1. It is obvious that

$$f_\alpha(s) = \left(\frac{\int_a^b x^s \,\mathrm{d}x}{\int_a^b (\alpha - x)^s \,\mathrm{d}x} \right)^{\frac{1}{s}} = \left(\frac{b^{s+1} - a^{s+1}}{(\alpha - a)^{s+1} - (\alpha - b)^{s+1}} \right)^{\frac{1}{s}}$$
$$= \frac{L_s(a, b)}{L_s(\alpha - a, \alpha - b)}.$$

Define for $s \in (-\infty, \infty)$,

$$\varphi(s) = \begin{cases} \ln\left(\dfrac{b^{s+1} - a^{s+1}}{(\alpha - a)^{s+1} - (\alpha - b)^{s+1}}\right), & s \neq -1; \\ \ln\left(\dfrac{\ln(b/a)}{\ln[(\alpha - a)/(\alpha - b)]}\right), & s = -1. \end{cases} \tag{2.2}$$

Then

$$\ln f_\alpha(s) = \begin{cases} \dfrac{\varphi(s)}{s}, & s \neq 0; \\ \varphi'(0), & s = 0. \end{cases} \tag{2.3}$$

In order to prove that $\ln f_\alpha$ is strictly increasing (decreasing), it suffices to show that φ is strictly convex (concave) on $(-\infty, \infty)$. A simple calculation reveals that

$$\varphi(-1 - s) = \varphi(-1 + s) + s \ln \frac{(\alpha - a)(\alpha - b)}{ab}, \tag{2.4}$$

which implies that $\varphi''(-1 - s) = \varphi''(-1 + s)$, and φ has the same convexity (concavity) on both $(-\infty, -1)$ and $(-1, \infty)$. Hence, it is sufficient to prove that φ is strictly convex (concave) on $(-1, \infty)$.

A computation yields

$$\varphi'(s) = \frac{b^{s+1} \ln b - a^{s+1} \ln a}{b^{s+1} - a^{s+1}} - \frac{(\alpha - b)^{s+1} \ln(\alpha - b) - (\alpha - a)^{s+1} \ln(\alpha - a)}{(\alpha - b)^{s+1} - (\alpha - a)^{s+1}},$$

$$(2.5)\quad (s+1)^2\varphi''(s) = (s+1)^2\left[-\frac{a^{s+1}b^{s+1}(\ln\frac{a}{b})^2}{(b^{s+1}-a^{s+1})^2}+\frac{(\alpha-a)^{s+1}(\alpha-b)^{s+1}(\ln\frac{\alpha-b}{\alpha-a})^2}{[(\alpha-a)^{s+1}-(\alpha-b)^{s+1}]^2}\right] = -\frac{(\frac{a}{b})^{s+1}[\ln(\frac{a}{b})^{s+1}]^2}{[1-(\frac{a}{b})^{s+1}]^2}+\frac{(\frac{\alpha-b}{\alpha-a})^{s+1}[\ln(\frac{\alpha-b}{\alpha-a})^{s+1}]^2}{[1-(\frac{\alpha-b}{\alpha-a})^{s+1}]^2}.$$

Define for $0 < t < 1$,

$$(2.6)\qquad \omega(t) = \frac{t(\ln t)^2}{(1-t)^2}.$$

Differentiation yields

$$(2.7)\qquad (1-t)t\ln t\frac{\omega'(t)}{\omega(t)} = (1+t)\ln t + 2(1-t) = -\sum_{n=2}^{\infty}\frac{n-1}{n(n+1)}(1-t)^{n+1} < 0,$$

which implies that $\omega'(t) > 0$ for $0 < t < 1$. It is easy to see that

$$(2.8)\qquad 0 < \left(\frac{a}{b}\right)^{s+1} < \left(\frac{\alpha-b}{\alpha-a}\right)^{s+1} < 1 \quad \text{for} \quad [a,b]\subseteq\left(0,\frac{\alpha}{2}\right],\ s > -1,$$

$$(2.9)\qquad 0 < \left(\frac{\alpha-b}{\alpha-a}\right)^{s+1} < \left(\frac{a}{b}\right)^{s+1} < 1 \quad \text{for} \quad [a,b]\subseteq\left[\frac{\alpha}{2},\alpha\right),\ s > -1,$$

and therefore $\varphi''(s) > 0$ for $[a,b] \subseteq (0, \frac{\alpha}{2}]$ and $s > -1$, $\varphi''(s) < 0$ for $[a,b] \subseteq [\frac{\alpha}{2}, \alpha)$ and $s > -1$. Then φ is strictly convex (concave) on $(-1, \infty)$ for $[a,b] \subseteq (0, \frac{\alpha}{2}]$ $([a,b] \subseteq [\frac{\alpha}{2}, \alpha))$ respectively. By Lemma 2.1 above, Theorem 1.1 holds. □

Since $f_\alpha(s)$ is a strictly increasing (decreasing) function for $[a,b] \subseteq (0, \frac{\alpha}{2}]$ $([a,b] \subseteq [\frac{\alpha}{2}, \alpha))$, put $s = -2, -1, 0, 1$ respectively. The inequalities (1.6) are deduced.

Then, let $(\alpha - x) = t$ and apply it to the function $\left(\frac{\int_a^b x^s \,\mathrm{d}x}{\int_a^b (\alpha-x)^s \,\mathrm{d}x}\right)^{\frac{1}{s}}$. We get Corollary 1.3.

Proof of Theorem 1.4. Using an analogous method of proof to that of Theorem 1.1, we get

$$f(s) = \left(\frac{\frac{1}{b-a}\int_a^b x^s \,\mathrm{d}x}{\frac{1}{d-c}\int_c^d x^s \,\mathrm{d}x}\right)^{\frac{1}{s}} = \left[\frac{\frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}}{\frac{d^{s+1}-c^{s+1}}{(s+1)(d-c)}}\right]^{\frac{1}{s}}$$
$$= \left[\frac{(d-c)}{(b-a)}\frac{(b^{s+1}-a^{s+1})}{(d^{s+1}-c^{s+1})}\right]^{\frac{1}{s}} = \frac{L_s(a,b)}{L_s(c,d)}.$$

Let $M = \frac{(d-c)}{(b-a)}$, and define for $s \in (-\infty, \infty)$,

$$\varphi(s) = \begin{cases} \ln\left(M\dfrac{b^{s+1}-a^{s+1}}{d^{s+1}-c^{s+1}}\right), & s \neq -1; \\ \ln\left[M\dfrac{\ln(b/a)}{\ln(d/c)}\right], & s = -1. \end{cases} \tag{2.10}$$

Then

$$\ln f(s) = \begin{cases} \dfrac{\varphi(s)}{s}, & s \neq 0; \\ \varphi'(0), & s = 0, \end{cases} \tag{2.11}$$

and φ has the same convexity (concavity) on both $(-\infty, -1)$ and $(-1, \infty)$.

A computation yields

$$(s+1)^2\varphi''(s) = -\frac{(\frac{a}{b})^{s+1}[\ln(\frac{a}{b})^{s+1}]^2}{[1-(\frac{a}{b})^{s+1}]^2} + \frac{(\frac{c}{d})^{s+1}[\ln(\frac{c}{d})^{s+1}]^2}{[1-(\frac{c}{d})^{s+1}]^2}.$$

Define for $0 < t < 1$,

$$\omega(t) = \frac{t(\ln t)^2}{(1-t)^2}. \tag{2.12}$$

Differentiation yields $\omega'(t) > 0$ for $0 < t < 1$. It is easy to see that

$$0 < \left(\frac{a}{b}\right)^{s+1} < \left(\frac{c}{d}\right)^{s+1} < 1 \quad \text{for} \quad ad < bc,\ s > -1, \tag{2.13}$$

$$0 < \left(\frac{c}{d}\right)^{s+1} < \left(\frac{a}{b}\right)^{s+1} < 1 \quad \text{for} \quad ad > bc,\ s > -1, \tag{2.14}$$

and therefore $\varphi''(s) > 0$ for $ad < bc$ and $s > -1$, $\varphi''(s) < 0$ for $ad > bc$ and $s > -1$ Then φ is strictly convex (concave) on $(-1, \infty)$ for $ad < bc$ $(ad > bc)$ respectively. The proof is complete. □

In Theorem 1.4, let $a = c$. Then $f(s)$ is a strictly increasing function for $d < b$, or a strictly decreasing function for $d > b$. Thus Corollary 1.5 holds.

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